ON AN INFINITELY STRATIFIED SCATTERER IN THE PRESENCE OF A LOW-FREQUENCY ELECTROMAGNETIC PLANE WAVE

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الخلاصـــة :

تصف هذه الورقة موجة مستوية كهرومغناطيسية تـتشتت بواسطة مشتت لانهائي الطبقات ؛ حيث ستُختَزل مسألة التشتت إلى متسلسلة متكررة من مسائل الكوامن للمعاملات ذات التردد المنخفض . وباستخدام الحدود المسبقة نستخلص تمثيلاً تكاملياً للمجال الكهربي الخارجي الكلّي .

ABSTRACT

An incident plane electromagnetic wave is scattered by an infinitely stratified scatterer. The scattering problem is reduced to an iterative sequence of potential problems for the low-frequency coefficients. Using *a priori* bounds, we derive an integral representation for the total exterior electric field.

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INTRODUCTION

One of the present authors has previously studied the theory of scattering of a low-frequency electromagnetic wave, by a scatterer consisting of a finite number of layers with a perfect conducting core [1]. The present work extends these results in the case of an infinitely stratified scatterer.

The method of reducing electromagnetic scattering problems to a series of problems in potential theory was first investigated by Lord Rayleigh [2]. A significant contribution to electromagnetic low-frequency scattering by a dielectric was made by Stevenson [3]. Werner [4, 5] established the validity of the series expansions for the electric and the magnetic fields. *A posteriori* bounds for the error, and various properties of the solution of the corresponding potential problem, were derived by Jones [6, 7]; he evaluated the zeroth-order approximation, for the general case of scattering by inhomogeneous dielectric particles. Results for a prolate spheroid scatterer in the case of Dirichlet and Neumann problems, are given in [8] and [9]. In [10], the zeroth and the first order low-frequency electromagnetic fields are evaluated for a dielectric ellipsoidal scatterer. Further results for low-frequency scattering can be found in [11] and [12]. The most complicated homogeneous case tackled so far, for Rayleigh scattering for bianisotropic scatterers in a biisotropic medium, is that studied by Lakhtakia in [13].

The purpose of this work is to establish integral representations for the exterior electric field for the scattering by an infinitely stratified scatterer. In Section 1, we give the more general form for the integral representations of the solutions of Maxwell's curl equations as in [14], in terms of the free space Green's dyadic. In Section 2 we describe precisely the scatterer; we formulate the scattering problem at low-frequencies; and present in brief the necessary results from [1], without any quantitative analysis. In Section 3 we construct bounds for the low-frequency coefficients, using the well known *a priori* bounds for the solutions of Poisson's equations in bounded domains; and prove the convergence of the series for the low-frequency coefficients of the total exterior electric field. Finally, we remark on the relevance of our problem to the approximation of a continuously varying inhomogeneous body by discrete layers.

1. PRELIMINARIES

Let a spherical region of radius a be occupied by a general linear homogeneous medium; consider this region embedded in free space (ε_o, μ_o), and being centred at the origin. In the absence of any impressed sources, Maxwell's curl equations, satisfied by time-harmonic electromagnetic fields everywhere can be concisely expressed as

$$\nabla\times\mathbf{H}(\mathbf{r})+i\omega\varepsilon_o\mathbf{E}(\mathbf{r})=\mathbf{J}(\mathbf{r}),$$
$$\nabla\times\mathbf{E}(\mathbf{r})-i\omega\mu_o\mathbf{H}(\mathbf{r})=-\mathbf{K}(\mathbf{r}) \qquad (1.1)$$

where ω is the angular frequency, ε_o the dielectric constant, μ_o the permeability and

$$\mathbf{J}(\mathbf{r})=\begin{cases}0 & ,\ \mathbf{r}\notin V\\ -i\omega\varepsilon_o[(\tilde{\varepsilon}-\tilde{I})\cdot\mathbf{E}(\mathbf{r})+\tilde{\alpha}\cdot\mathbf{H}(\mathbf{r})] & ,\ \mathbf{r}\in V\end{cases},$$

$$\mathbf{K}(\mathbf{r})=\begin{cases}0 & ,\ \mathbf{r}\notin V\\ -i\omega\mu_o[\tilde{\beta}\cdot\mathbf{E}(\mathbf{r})+(\tilde{\mu}-\tilde{I})\cdot\mathbf{H}(\mathbf{r})] & ,\ \mathbf{r}\in V\end{cases} \qquad (1.2)$$

where $\tilde{\mu}$ is the relative permeability dyadic, $\tilde{\varepsilon}$ the relative permittivity dyadic, $\tilde{\alpha}$, $\tilde{\beta}$ the magnetoelectric dyadics and $\tilde{I}$ the idempotent.

The solution of (1.1) is given in [14], in a very general form, by

$$\mathbf{E}^{sc}(\mathbf{r}) = \frac{1}{3i\omega\varepsilon_o}\mathbf{J}(\mathbf{r})+P_{val}\Big\{i\omega\mu_o\int_V \mathbf{J}(\mathbf{r}')\cdot\tilde{G}_o(\mathbf{r},\mathbf{r}')dV(\mathbf{r}') - \frac{1}{3i\omega\varepsilon_o}\mathbf{J}(\mathbf{r}) - \int_V \mathbf{K}(\mathbf{r}')\cdot[\nabla_{r'}\times\tilde{G}_o(\mathbf{r},\mathbf{r}')]dV(\mathbf{r}')\Big\},\ \mathbf{r}\in V \qquad (1.3)$$

$$\mathbf{H}^{sc}(\mathbf{r}) = \frac{1}{3i\omega\mu_o}\mathbf{K}(\mathbf{r})+P_{val}\Big\{i\omega\varepsilon_o\int_V \mathbf{K}(\mathbf{r}')\cdot\tilde{G}_o(\mathbf{r},\mathbf{r}')dV(\mathbf{r}') - \frac{1}{3i\omega\mu_o}\mathbf{K}(\mathbf{r}) + \int_V \mathbf{J}(\mathbf{r}')\cdot[\nabla_{r'}\times\tilde{G}_o(\mathbf{r},\mathbf{r}')]dV(\mathbf{r}')\Big\},\ \mathbf{r}\in V \qquad (1.4)$$

where P_{val} denotes the principal value, and the free space Green's dyadic, [15], is given by

$$\tilde{G}_o(\mathbf{r},\mathbf{r}') = \left(\tilde{I} + \frac{\nabla\nabla}{k_o^2}\right)\frac{\exp(ik_o\|\mathbf{r}-\mathbf{r}'\|)}{4\pi\|\mathbf{r}-\mathbf{r}'\|} \tag{1.5}$$

where $k_o = \omega(\mu_o\varepsilon_o)^{1/2}$ is the wave number in V. $\mathbf{E}^{sc}(\mathbf{r})$, $\mathbf{H}^{sc}(\mathbf{r})$ denote the scattered electric and magnetic field respectively.

In our problem we consider a dielectric scatterer containing a perfectly conducting core with surface S. It is well known [16, 17] that Maxwell's equations and the boundary conditions on a dielectric are invariant under the substitution $\mathbf{E}\rightarrow\mathbf{H}$, $\mathbf{H}\rightarrow -\mathbf{E}$, $\varepsilon \leftrightarrow \mu$. This invariance property does not hold on S, since the boundary conditions, there, are given by:

$$\hat{\mathbf{n}}\times\mathbf{E}(\mathbf{r}) = \mathbf{0}, \quad \hat{\mathbf{n}}\cdot\mathbf{H}(\mathbf{r}) = 0. \tag{1.6}$$

Hence, it is more convenient to use the following integral representation from [1]

$$\mathbf{E}^{sc}(\mathbf{r}) = \frac{1}{4\pi}\int_S \Big\{(\nabla\times\mathbf{E}(\mathbf{r}'))\cdot(\hat{\mathbf{n}}\times\tilde{G}_o(\mathbf{r},\mathbf{r}')) - (\hat{\mathbf{n}}\times\mathbf{E}(\mathbf{r}'))\cdot(\nabla_{r'}\times\tilde{G}_o(\mathbf{r},\mathbf{r}'))\Big\}\, dS(\mathbf{r}') \tag{1.7}$$

without replacing $\nabla\times\mathbf{E}(\mathbf{r})$ by a multiple of $\mathbf{H}(\mathbf{r})$. The main reason for this, is that in (1.7) we can incorporate the boundary conditions (1.6), and construct a boundary value problem for the electric field only.

2. FORMULATION OF THE PROBLEM

The infinitely stratified scatterer is a bounded, convex and closed subset of $\mathbb{R}^3$. We assume that a perfectly conducting region V_c, with a closed and smooth boundary S_c, lies entirely within the scatterer. Let V_o denote the exterior of the scatterer, which is the propagation space of the incident plane wave. The scatterer consists of an infinite number of (annuli-like) layers V_j, $j = 1, 2, \ldots$, enclosing V_c, each of which has very thin width, and such that:

$$\inf\{\operatorname{dist}(\mathbf{r}, V_c) : \mathbf{r}\in V_{j+1}\} < \inf\{\operatorname{dist}(\mathbf{r}, V_c) : \mathbf{r}\in V_j\}.$$

The boundary of the layer V_j, $j = 1, 2, \ldots$, is

$$\partial V_j = S_{j-1} U S_j$$

where S_{j-1} and S_j are closed and smooth surfaces. The space V_j is a filled homogeneous, isotropic medium, having dielectric constant ε_j and permeability μ_j, $j = 0, 1, 2, \ldots$.

From physical considerations we may, and shall, assume that there exist ε, μ, and $\mu^* > 0$, such that

$$\varepsilon : = \sup\{|\varepsilon_j| : j = 0, 1, 2, \ldots\},$$

$$\mu : = \sup\{|\mu_j| : j = 0, 1, 2, \ldots\},$$

and

$$\mu^* : = \inf\{|\mu_j| : j = 0, 1, 2, \ldots\}.$$

An incident plane wave $\mathbf{E}^{inc}$, $\mathbf{H}^{inc}$ propagates in the homogeneous, isotropic medium V_o, with a harmonic time dependence $\exp(-i\omega t)$:

$$\mathbf{E}^{inc}(\mathbf{r}) = \hat{\mathbf{b}}\exp(ik_o\hat{\mathbf{k}}\cdot\mathbf{r}) \tag{2.1}$$

$$\mathbf{H}^{inc}(\mathbf{r}) = (\hat{\mathbf{k}}\times\hat{\mathbf{b}})\left(\frac{\varepsilon_o}{\mu_o}\right)^{1/2}\exp(ik_o\hat{\mathbf{k}}\cdot\mathbf{r}) \tag{2.2}$$

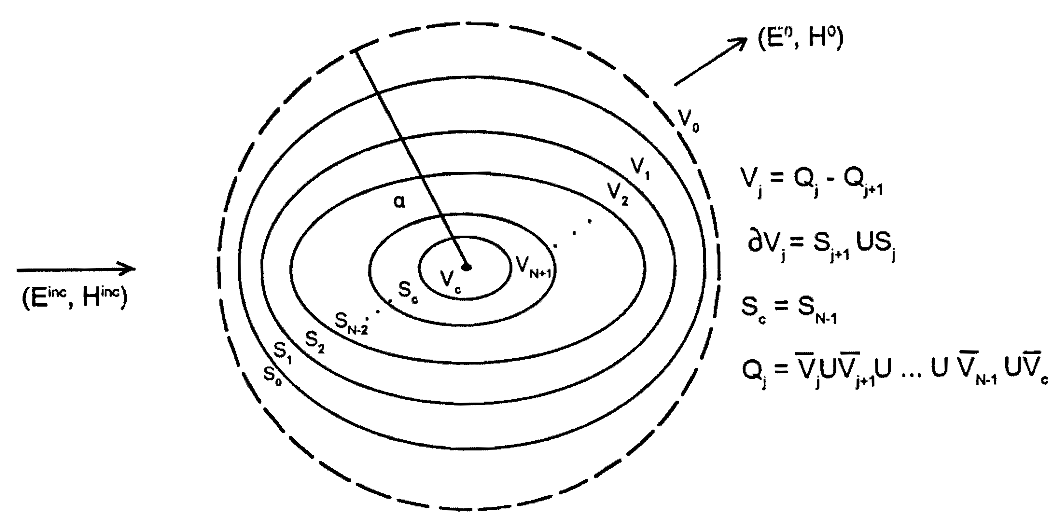

Figure 1.

where $\hat{\mathbf{k}}$ is the unit vector in the direction of propagation, $\hat{\mathbf{b}}$ is the unit polarization vector for the electric field with $\hat{\mathbf{b}}\cdot\hat{\mathbf{k}}=0$, and $\hat{\mathbf{k}}\times\hat{\mathbf{b}}$ is the polarization vector for the magnetic field.

Let $\mathbf{E}^{sc}(\mathbf{r})$, $\mathbf{H}^{sc}(\mathbf{r})$ denote the scattered electromagnetic field, and $\mathbf{E}_j(\mathbf{r})$, $\mathbf{H}_j(\mathbf{r})$ denote the electromagnetic field in V_j. Each of the vectors $\mathbf{E}^{inc}$, $\mathbf{H}^{inc}$, $\mathbf{E}^{sc}$, $\mathbf{H}^{sc}$, $\mathbf{E}_j$, $\mathbf{H}_j$, must satisfy the equations:

$$\left.\begin{aligned}&\nabla\times\nabla\times\mathbf{w}(\mathbf{r})-k_j^2\,\mathbf{w}(\mathbf{r})=\mathbf{0}\\&\nabla\cdot\mathbf{w}(\mathbf{r})=0\end{aligned}\right\},\ \mathbf{r}\in V_j,\quad j=0,1,2,\ldots \tag{2.3}$$

where the wave number $k_j=\omega(\mu_j\varepsilon_j)^{1/2}$ in V_j, satisfies the relation

$$k_j^2=\frac{\mu_j\varepsilon_j}{\mu_o\varepsilon_o}\,k_o^2 \tag{2.4}$$

On the surface S_j of the dielectric V_j, the electric fields satisfy the following boundary conditions

$$\left.\begin{aligned}&\hat{\mathbf{n}}\times\mathbf{E}_j(\mathbf{r})=\hat{\mathbf{n}}\times\mathbf{E}_{j+1}(\mathbf{r})\\&\hat{\mathbf{n}}\cdot\mathbf{E}_j(\mathbf{r})=\frac{\varepsilon_{j+1}}{\varepsilon_j}\,\hat{\mathbf{n}}\cdot\mathbf{E}_{j+1}(\mathbf{r})\end{aligned}\right\},\ \mathbf{r}\in S_j,\quad j=0,1,2,\ldots \tag{2.5}$$

while, on the surface of the perfect conductor, the boundary conditions are given by (1.6).

It is well-known that $\mathbf{E}^{sc}(\mathbf{r})$, $\mathbf{H}^{sc}(\mathbf{r})$ must satisfy, some appropriate, [18], radiation condition.

In the subsequent part of this work, we shall restrict ourselves to the study of the electric field. The corresponding analysis for the magnetic field can, for the reasons explained above, be performed analogously.

In [1], it is proved that the total electric field $\mathbf{E}_o(\mathbf{r})$ in V_o has the following integral representation:

$$\begin{aligned}\mathbf{E}_o(\mathbf{r})&=\mathbf{E}^{inc}(\mathbf{r})\\&+\frac{1}{4\pi}\cdot\frac{\mu_o}{\mu_N}\cdot\int_{S_N}(\nabla\times\mathbf{E}_N(\mathbf{r}'))\cdot(\hat{\mathbf{n}}\times\tilde{G}_o(\mathbf{r},\mathbf{r}'))\,dS(\mathbf{r}')\\&-\frac{1}{4\pi}k_o^2\sum_{j=1}^{N}\left(\frac{\varepsilon_j}{\varepsilon_o}-1\right)\int_{V_j}\mathbf{E}_j(\mathbf{r}')\cdot\tilde{G}_o(\mathbf{r},\mathbf{r}')\,dV(\mathbf{r}')\\&-\frac{1}{4\pi}\sum_{j=1}^{N}\left(1-\frac{\mu_o}{\mu_j}\right)\int_{V_j}(\nabla\times\mathbf{E}_j(\mathbf{r}'))\cdot(\nabla_{r'}\times\tilde{G}(\mathbf{r},\mathbf{r}'))\,dV(\mathbf{r}')\end{aligned} \tag{2.6}$$

in the case of a scatterer with layers V_j, $j=1,2,\ldots,N$.

It can easily be seen that (2.3) give a (vector) Helmholtz equation, and since its solutions — considered as functions of the wave number — are analytic in a neighborhood of zero, [4], [5], we can expand them in convergent power series.

Therefore, we have

$$\mathbf{E}_j(\mathbf{r})=\sum_{n=o}^{\infty}\frac{(ik_o)^n}{n!}\,\boldsymbol{\phi}_n^{(j)}(\mathbf{r}),\quad\mathbf{r}\in V_j,\quad j=1,2,\ldots \tag{2.7}$$

for the electric fields, and

$$\mathbf{H}_j(\mathbf{r})=\sum_{n=o}^{\infty}\frac{(ik_o)^n}{n!}\,\boldsymbol{\psi}_n^{(j)}(\mathbf{r}),\quad\mathbf{r}\in V_j,\quad j=1,2,\ldots \tag{2.8}$$

for the magnetic fields, where $\boldsymbol{\phi}_n^{(j)}(\mathbf{r})$ and $\boldsymbol{\psi}_n^{(j)}(\mathbf{r})$ are independent of k_o.

From Maxwell's equations and the series expansions (2.7), (2.8), we conclude that

$$\nabla\times\boldsymbol{\phi}_n^{(j)}(\mathbf{r})=\frac{\mu_j}{(\mu_o\varepsilon_o)^{1/2}}\,n\,\boldsymbol{\psi}_{n-1}^{(j)}(\mathbf{r})\quad\mathbf{r}\in V_j,\quad j=0,1,2,\ldots \tag{2.9}$$

A substitution of the low-frequency expansions (2.7), (2.8) into (2.3), (2.5), (1.6), reduces the above scattering problem to a sequence of potential problems that can be solved iteratively.

Therefore, the coefficients $\boldsymbol{\phi}_n^{(j)}(\mathbf{r})$ and $\boldsymbol{\psi}_n^{(j)}(\mathbf{r})$ of the expansions (2.7), (2.8) satisfy the equations, for all $n=0,1,2,\ldots$.

$$\left.\begin{aligned}&\nabla\times\nabla\times\mathbf{u}_n^{(j)}(\mathbf{r})+n(n-1)\,\frac{\mu_j\varepsilon_j}{\mu_o\varepsilon_o}\,\mathbf{u}_{n-2}^{(j)}(\mathbf{r})=\mathbf{0}\\&\nabla\cdot\mathbf{u}_n^{(j)}(\mathbf{r})=0\end{aligned}\right\},\mathbf{r}\in V_j, \tag{2.10}$$

and the boundary conditions

$$\left.\begin{aligned}&\hat{\mathbf{n}}\times\mathbf{u}_n^{(j)}(\mathbf{r})=\hat{\mathbf{n}}\times\mathbf{u}_n^{(j+1)}(\mathbf{r})\\&\hat{\mathbf{n}}\cdot\mathbf{u}_n^{(j)}(\mathbf{r})=\frac{\sigma_{j+1}}{\sigma_j}\,\hat{\mathbf{n}}\cdot\mathbf{u}_n^{(j+1)}(\mathbf{r})\end{aligned}\right\},\ \mathbf{r}\in S_j, \tag{2.11}$$

where

$$\sigma_j=\begin{cases}\varepsilon_j, & \text{when } \mathbf{u}_n^{(j)}=\boldsymbol{\phi}_n^{(j)}\\ \mu_j, & \text{when } \mathbf{u}_n^{(j)}=\boldsymbol{\psi}_n^{(j)}\end{cases}$$

and

$$\left.\begin{array}{l}\hat{\mathbf{n}}\times\boldsymbol{\phi}_n^{\circ}(\mathbf{r})=\mathbf{0}\\ \hat{\mathbf{n}}\cdot\boldsymbol{\psi}_n^{\circ}(\mathbf{r})=0\end{array}\right\},\ \mathbf{r}\in S_c, \tag{2.12}$$

In [1], it is shown that the low-frequency coefficient of the total electric field $\boldsymbol{\phi}_n^{(o)}$, is given in terms of the coefficients $\boldsymbol{\phi}_o^{(j)}, \boldsymbol{\phi}_1^{(j)}, \boldsymbol{\phi}_2^{(j)}, \ldots, \boldsymbol{\phi}_n^{(j)}$, by the relation:

$$\begin{aligned}\boldsymbol{\phi}_n^{(o)}(\mathbf{r}) &= \hat{\mathbf{b}}(\hat{\mathbf{k}}\cdot\mathbf{r})^n + \frac{1}{4\pi}\frac{\mu_o}{\mu_N}\sum_{\rho=o}^{n}\binom{n}{\rho}\int_{S_C}(\nabla\times\boldsymbol{\phi}_\rho^{(c)}(\mathbf{r}'))\cdot(\hat{\mathbf{n}}\times\tilde{\boldsymbol{\gamma}}_{n-\rho}(\mathbf{r},\mathbf{r}'))\,dS(\mathbf{r}')\\ &+\frac{1}{4\pi}\sum_{\rho=o}^{n}\sum_{j=1}^{N}\binom{n}{\rho}\left(\frac{\varepsilon_j}{\varepsilon_o}-1\right)\rho(\rho-1)\int_{V_j}\boldsymbol{\phi}_{\rho-2}^{(j)}(\mathbf{r}')\cdot\tilde{\boldsymbol{\gamma}}_{n-\rho}(\mathbf{r},\mathbf{r}')\,dV(\mathbf{r}')\\ &-\frac{1}{4\pi}\sum_{\rho=o}^{n}\sum_{j=1}^{N}\binom{n}{\rho}\left(1-\frac{\mu_o}{\mu_j}\right)\int_{V_j}(\nabla\times\boldsymbol{\phi}_\rho^{(j)}(\mathbf{r}'))\cdot\tilde{\boldsymbol{\delta}}_{n-\rho}(\mathbf{r},\mathbf{r}')\,dV(\mathbf{r}'),\end{aligned} \tag{2.13}$$

where the dyadics $\tilde{\boldsymbol{\gamma}}_n(\mathbf{r},\mathbf{r}')$ and $\tilde{\boldsymbol{\delta}}_n(\mathbf{r},\mathbf{r}')$ are given by

$$\tilde{\boldsymbol{\gamma}}_n(\mathbf{r},\mathbf{r}') = -\frac{\|\mathbf{r}-\mathbf{r}'\|^{n-1}}{n+2}\left\{(n+1)\tilde{I} - (n-1)\frac{(\mathbf{r}-\mathbf{r}')\otimes(\mathbf{r}-\mathbf{r}')}{\|\mathbf{r}-\mathbf{r}'\|^2}\right\} \tag{2.14}$$

and

$$\tilde{\boldsymbol{\delta}}_n(\mathbf{r},\mathbf{r}') = (n-1)\|\mathbf{r}-\mathbf{r}'\|^{n-3}(\mathbf{r}-\mathbf{r}')\times\tilde{I} \tag{2.15}$$

3. LOW-FREQUENCY COEFFICIENTS WHEN $N\to\infty$

We shall study the existence of $\boldsymbol{\phi}_n^{\circ}(\mathbf{r})$ in the case where the finite sums with respect to j in (2.17) become series.

From (2.13), we get

$$\nabla^2\mathbf{u}_n^{(j)}(\mathbf{r}) = n(n-1)\frac{\mu_j\varepsilon_j}{\mu_o\varepsilon_o}\mathbf{u}_{n-2}^{(j)}(\mathbf{r}). \tag{3.1}$$

Letting

$$T_j := \frac{\mu_j\varepsilon_j}{\mu_o\varepsilon_o} \tag{3.2}$$

and, repeatedly, with respect to n, using the well known *a priori* bounds for the solutions of Poisson's equation [19], we get the result that:

$$\sup_{V_j}\|\mathbf{u}_{2l}^{(j)}(\mathbf{r})\| \le \sum_{i=o}^{l}\frac{(2l)!}{(2i)!}[T_j(e^{2a}-1)]^{l-1}\sup_{\partial V_j}\|\mathbf{u}_{2l}^{(j)}(\mathbf{r})\|,$$

and

$$\sup_{V_j}\|\mathbf{u}_{2l+1}^{(j)}(\mathbf{r})\| \le \sum_{i=o}^{l}\frac{(2l+1)!}{(2i+1)!}[T_j(e^{2a}-1)]^{l-1}\sup_{\partial V_j}\|\mathbf{u}_{2l+1}^{(j)}(\mathbf{r})\|, \tag{3.3}$$

where a is the characteristic dimension of the scatterer, *i.e.* the radius of the smallest sphere, circumscribable around the scatterer.

By (2.11), (2.12) we have that there exist for each j, the

$$\sup_{\partial V_j}\|\mathbf{u}_n^{(j)}(\mathbf{r})\| := \xi_{n,j},\quad n=0,1,2,\ldots \tag{3.4}$$

and we, therefore, conclude that

$$\|\mathbf{u}_n^{(j)}(\mathbf{r})\| \le m_{n,j} \tag{3.5}$$

for all $n=0,1,2,\ldots$, and each $j=1,2\ldots$, where

$$m_{n,j} := \begin{cases}\displaystyle\sum_{i=o}^{n/2}\frac{n!}{(2i)!}[T_j(e^{2a}-1)]^{\frac{n-2i}{2}}\xi_{2i,j}, & n:\text{even}\\ \displaystyle\sum_{i=o}^{(n-1)/2}\frac{n!}{(2i+1)!}[T_j(e^{2a}-1)]^{\frac{n-1-2i}{2}}\xi_{2i+1,j}, & n:\text{odd}\end{cases} \tag{3.6}$$

Now, let

$$\xi_{n,j} = \begin{cases}\xi_{n,j}^E, & \text{when } \mathbf{u}_n^{(j)}(\mathbf{r}) = \boldsymbol{\phi}_n^{(j)}(\mathbf{r})\\ \xi_{n,j}^H, & \text{when } \mathbf{u}_n^{(j)}(\mathbf{r}) = \boldsymbol{\psi}_n^{(j)}(\mathbf{r})\end{cases}$$

and

$$m_{n,j} = \begin{cases}m_{n,j}^E, & \text{when } \mathbf{u}_n^{(j)}(\mathbf{r}) = \boldsymbol{\phi}_n^{(j)}(\mathbf{r})\\ m_{n,j}^H, & \text{when } \mathbf{u}_n^{(j)}(\mathbf{r}) = \boldsymbol{\psi}_n^{(j)}(\mathbf{r}).\end{cases}$$

If, in addition we assume that:

$$\sup_j\{\xi_{n,j}^E\} = \xi_n^E < +\infty \text{ and } \sup_j\{\xi_{n,j}^H\} = \xi_n^H < +\infty,$$

conditions that are physically meaningful, we have from (3.6), (3.2) and the existence of $\sup_j|\varepsilon_j|$ and $\sup_j|\mu_j|$, that

$$m_{n,j}^E \le c_E \quad\text{and}\quad m_{n,j}^H \le c_H, \tag{3.7}$$

where c_E and c_H are constants with respect to j.

Now we consider the two terms appearing in (2.13), when the finite sums with respect to j become series:

$$\sum_{j=1}^{\infty}\left(\frac{\varepsilon_j}{\varepsilon_o}-1\right)\int_{V_j}\boldsymbol{\phi}_{\rho-2}^{(j)}(\mathbf{r}')\cdot\tilde{\gamma}_{n-\rho}(\mathbf{r},\mathbf{r}')\,dV(\mathbf{r}') \tag{3.8}$$

and

$$\sum_{j=1}^{\infty}\left(1-\frac{\mu_o}{\mu_j}\right)\int_{V_j}(\nabla\times\boldsymbol{\phi}_{\rho-2}^{(j)}(\mathbf{r}'))\cdot\tilde{\delta}_{n-\rho}(\mathbf{r},\mathbf{r}')\,dV(\mathbf{r}') \tag{3.9}$$

After tedious calculations, and taking into account (2.14), (2.15), and (2.9) for (3.9), we arrive at

$$\|\boldsymbol{\phi}_{\rho-2}^{(j)}(\mathbf{r}')\cdot\tilde{\gamma}_{n-\rho}(\mathbf{r},\mathbf{r}')\| \leq \frac{2^{n-\rho}(n-\rho)}{n-\rho+2}\,r^{n-\rho-1}\,\|\boldsymbol{\phi}_{\rho-2}^{(j)}(\mathbf{r}')\| \tag{3.10}$$

and

$$\|(\nabla\times\boldsymbol{\phi}_{\rho-2}^{(j)}(\mathbf{r}')\cdot\tilde{\delta}_{n-\rho}(\mathbf{r},\mathbf{r}')\| \leq 2^{n-\rho-1}(n-\rho-1)\,r^{n-\rho-2}\,\|\boldsymbol{\psi}_{\rho-1}^{(j)}(\mathbf{r}')\|. \tag{3.11}$$

Therefore, we have

$$\left\|\left(\frac{\varepsilon_j}{\varepsilon_o}-1\right)\int_{V_j}\boldsymbol{\phi}_{\rho-2}^{(j)}(\mathbf{r}')\cdot\tilde{\gamma}_{n-\rho}(\mathbf{r},\mathbf{r}')dV(\mathbf{r}')\right\| \leq \frac{2^{n-\rho}(n-\rho)}{n-\rho+2}\;r^{n-\rho-1}\,m_{\rho,j}^{E}\left|\frac{\varepsilon_j}{\varepsilon_o}-1\right||V_j| \leq \frac{2^{n-\rho}(n-\rho)}{n-\rho+2}\;r^{n-\rho-1}\,c_E\left(1+\left|\frac{\varepsilon_j}{\varepsilon_o}\right|\right)|V_j| \tag{3.12}$$

and, similarly,

$$\left\|\left(1-\frac{\mu_o}{\mu_j}\right)\int_{V_j}(\nabla\times\boldsymbol{\phi}_{\rho}^{(j)}(\mathbf{r}'))\cdot\tilde{\delta}_{n-\rho}(\mathbf{r},\mathbf{r}')dV(\mathbf{r}')\right\| \leq 2^{n-\rho-1}(n-\rho-1)\,r^{n-\rho-2}\,c_H\left(1+\left|\frac{\mu_o}{\mu^*}\right|\right)|V_j| \tag{3.13}$$

where $|V_j|$ denotes the volume of V_j.

Let Q_j be the volume of the solid body with boundary $S_c U S_{j-1}$; therefore $\lim_{j\to\infty} Q_j = V_c$.

Hence,

$$\sum_{j=1}^{\infty}|V_j| = \sum_{j=1}^{\infty}(Q_j - Q_{j+1}) = Q_1 - |V_c|$$

Using the Weierstrass M-test, we conclude that (3.8), (3.9) converge. Therefore, we have the following expression for the low-frequency coefficient in the exterior region V_o.

$$\begin{aligned}\boldsymbol{\phi}_n^{(o)}(\mathbf{r}) &= \hat{\mathbf{b}}(\hat{\mathbf{k}}\cdot\mathbf{r})^n \\ &+ \frac{1}{4\pi}\frac{\mu_o}{\mu_c}\sum_{\rho=o}^{n}\binom{n}{\rho}\int_{S_c}(\nabla\times\boldsymbol{\phi}_{\rho}^{(c)}(\mathbf{r}')) \cdot(\hat{\mathbf{n}}\times\tilde{\gamma}_{n-\rho}(\mathbf{r},\mathbf{r}'))\,ds(\mathbf{r}') \\ &+ \frac{1}{4\pi}\sum_{\rho=o}^{n}\binom{n}{\rho}\rho(\rho-1)\,\mathbf{Z}_{\rho}^{E}(\mathbf{r}) \\ &- \frac{1}{4\pi}\sum_{\rho=o}^{n}\binom{n}{\rho}\mathbf{Z}_{\rho}^{H}(\mathbf{r}),\end{aligned} \tag{3.14}$$

where $\mathbf{Z}_{\rho}^{E}(\mathbf{r})$ and $\mathbf{Z}_{\rho}^{H}(\mathbf{r})$ are the sums of (3.8) and (3.9) respectively. Since

$$\tilde{\gamma}_o(\mathbf{r},\mathbf{r}') = 0\left(\frac{1}{r}\right), \qquad \tilde{\delta}_o(\mathbf{r},\mathbf{r}') = 0\left(\frac{1}{r}\right), \qquad r\to\infty \tag{3.15}$$

the asymptotic representation for the n-th order coefficient can be derived from the integral relation (3.14), if the n-th term, which is of order $1/r$, is omitted.

The term $\hat{\mathbf{b}}(\hat{\mathbf{k}}\cdot\mathbf{r})^n$ represents the main contribution of the incident wave to the corresponding approximation.

In the special case of a non-homogeneous ellipsoidal dielectric scatterer with permittivity $\varepsilon(\mathbf{r})$ and permeability $\mu(\mathbf{r})$, containing a perfectly conducting confocal ellipsoidal core, we can approximate the scattered field by a "stratification" as described in Section 2. Each layer V_j is contained between two consecutive ellipsoidal surfaces S_{j-1}, S_j given in ellipsoidal coordinates by $\rho = \alpha_{j-1}$ and $\rho = \alpha_j$, α_j being the largest semiaxis of the corresponding ellipsoid. We suppose that $\alpha_j = \alpha_1 - (j-1)N^{-1}(\alpha_1 - \alpha_c)$, $j = 1, 2, \ldots, N+1$, where α_c and α_1 are, respectively, the largest semiaxes of the core, and the exterior surface of the scatterer. We assume that in V_j, we have constant permittivity ε_j and permeability μ_j, *i.e.* that $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ are approximated by piecewise constant functions ([18], p. 89). Then, taking (3.14) into account, the low-frequency coefficients of the stratified scatterer when $N\to\infty$, can produce an approximation for the scattered field of the non-homogeneous scatterer.

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