# AN INITIAL VALUE METHOD FOR SOLVING SINGULAR PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS

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الخلاصـــة :

نصف طريقة خوارزمية لحل مسائل القيم الحدية غير الخطية الـمُـرَحَّلة أُحادياً . وسوف نستعمل إحدى طرق ( رانج – كـتا – فلبرج ) متغيرة الخطوة . وطريقتنا تظهر تحسيناً ملموساً فى الدقة مقارنة بطريقة ( رانج – كـتا ) رباعية الرتبة .

## ABSTRACT

We describe an algorithm for solving nonlinear singular perturbed two-point boundary value problems. We use a variable-step Runge-Kutta-Fehlberg method. Our method shows a substantial improvement in accuracy over the classical fourth-order Runge-Kutta method.

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## 1. INTRODUCTION

Consider the nonlinear singular perturbed two-point boundary value problem

$$\varepsilon u''(x) + (p(x)u(x))' + q(x, u(x)) = f(x), \quad a \le x \le b, \tag{1}$$

$$u(a) = \alpha, \quad u(b) = \beta, \tag{2}$$

where

$$\varepsilon > 0, \ (p(x)u(x)), \ f \in C^2(\mathbf{R}), \ q(x, u(x)) \in C^1(\mathbf{R}^2)$$

and

$$\partial q(x, u(x))/\partial u \le \delta < 0$$

hold on $[a, b] \times \mathbf{R}$. Under these assumptions the problem (1), (2) admits a unique solution that displays a boundary layer at the left end of the interval $[a, b]$ for small values of $\varepsilon$ [1, 2]. Relevant work, but using the fourth-order Runge–Kutta method, can be found in reference [2].

In this paper we present a technique for solving nonlinear singular two-point boundary value problems, which are of great importance in fields such as fluid mechanics, electrical networks, chemical reactions, quantum mechanics, aerodynamics, and elasticity. A numerical example is also provided that compares favorably with earlier results [2]. Our results were derived using MATH VAX and Fortran.

## 2. INITIAL VALUE METHOD FOR SINGULAR PERTURBED NONLINEAR PROBLEMS

In the following Section we describe an algorithm for solving this class of problems. We are following a scheme suggested in [2] with different implementation to improve the results. The second-order problem (1), (2) is replaced by an equivalent first-order problem and is solved as an initial-value problem following these steps.

(*i*) Set $\varepsilon = 0$; then (1) becomes:

$$(p(x)u(x))' + q(x, u(x)) = f(x), \tag{3}$$

and its solution is denoted by $\tilde{U}(x)$.

(*ii*) Approximate Equation (1) by the following boundary value problem:

$$\varepsilon u''(x) + (p(x)u(x))' + q(x, \tilde{U}(x)) = f(x), \tag{4}$$

$$u(a) = \alpha, \quad u(b) = \beta,$$

in which we replace $u(x)$ by $\tilde{U}(x)$ in the term $q(x, u(x))$. If the Equation (1) does not contain this term, we skip steps (*i*), (*ii*) and integrate the problem directly.

(*iii*) In Equation (4) we set

$$V'(x) = q(x, \tilde{U}(x)). \tag{5}$$

Substituting (5) into (4) we get:

$$\varepsilon u''(x) + (p(x)u(x))' + V'(x) = f(x). \tag{6}$$

(*iv*) We integrate (6) to get:

$$\varepsilon u'(x) + (p(x)u(x))' + V'(x) = F(x) + C \tag{7}$$

where $C$ is the constant of integration to be determined and

$$F(x) = \int f(x)\, \mathrm{d}x. \tag{8}$$

In order to determine $C$, we require that Equation (7) with $\varepsilon = 0$ satisfy $u(b) = \beta$, that is

$$C = p(b)\beta + V(b) - F(b). \tag{9}$$

(*v*) Finally, we solve the following initial-value problem

$$\varepsilon u'(x) + p(x)u(x) + V(x) = F(x) + p(b)\beta + V(b) - F(b), \tag{10}$$

$$u(a) = \alpha. \tag{11}$$

In order to solve the initial-value problem (10), with the condition (11) we make use of a variable-step Runge–Kutta–Fehlberg (RKF) method [3]. As far as we know we are the first to use this method with Runge–Kutta–Fehlberg for this class of problems. The use of RKF is justified since the refinement of the mesh size is desirable especially near the boundary layer region. Our results, which we present in the next section, show substantial improvement in accuracy over the classical fourth-order Runge–Kutta method used in [2].

## 3. NUMERICAL RESULTS

As a test example, we have considered the following equation:

$$\varepsilon u''(x) = -2u'(x) - e^u, \quad 0 \le x \le 1, \tag{12}$$

$$u(0) = 0, \quad u(1) = 0. \tag{13}$$

For comparison, we used Bender's [4] uniformly valid approximate solution given by

$$u(x) = \log[2/(1+x)] - \exp(-2x/\varepsilon)\log(2). \tag{14}$$

To implement the method described above we set $\varepsilon = 0$; then the reduced problem is given by:

$$2u'(x) + e^{u} = 0, \quad 0 \le x \le 1, \tag{15}$$

$$u(1) = 0,$$

whose solution is denoted by:

$$\tilde{U}(x) = \log[2/(1+x)].$$

Using Step (*ii*) we approximate (12) by the following:

$$\varepsilon u''(x) = -2u'(x) - [2/(1+x)]. \tag{16}$$

Integrating (16) we get:

$$\varepsilon u'(x) = -2u(x) - 2\log(1+x) + C. \tag{17}$$

Following Step (*iv*) we use Equations (8) and (9) to determine $C$, that is:

$$C = 2\log(2).$$

As a result of substituting the value $C$ in (17) we get the initial value problem

$$\begin{cases} \varepsilon u'(x) = 2[-u(x) + \log(2/(1+x))], & 0 \le x \le 1, \\ u(0) = 0. \end{cases} \tag{18}$$

**Table 1. Numerical Results for $\varepsilon = 10^{-2}$.**

| $x$ | $h$ | Approximate Solution $u(x)$ | Uniform Solution |
|---|---|---|---|
| 0.0 | 0.0100 | 0.0 | 0.0 |
| 0.0005 | 0.0005 | 0.0701 | 0.0696 |
| 0.0021 | 0.0005 | 0.2455 | 0.2437 |
| 0.0066 | 0.0006 | 0.5074 | 0.5037 |
| 0.0105 | 0.0008 | 0.6029 | 0.5985 |
| 0.0467 | 0.0039 | 0.6521 | 0.6473 |
| 0.0846 | 0.0066 | 0.6165 | 0.6118 |
| 0.2059 | 0.0069 | 0.5100 | 0.5058 |
| 0.4068 | 0.0076 | 0.3553 | 0.3518 |
| 0.6053 | 0.0076 | 0.2229 | 0.2197 |
| 0.7092 | 0.0080 | 0.1599 | 0.1570 |
| 0.9014 | 0.0086 | 0.0531 | 0.0505 |
| 0.9966 | 0.0092 | 0.0042 | 0.0017 |

We have solved Equation (18) using the Runge–Kutta–Fehlberg scheme, the computational results are shown in Tables 1 and 2 for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$. The numerical results show that this method is accurate and easy to implement.

**Table 2. Numerical Results for $\varepsilon = 10^{-3}$.**

| $x$ | $h$ | Approximate Solution $u(x)$ | Uniform Solution |
|---|---|---|---|
| 0.0 | 0.01000 | 0.0 | 0.0 |
| 0.00003 | 0.00003 | 0.04039 | 0.04036 |
| 0.00008 | 0.00002 | 0.10814 | 0.10806 |
| 0.00021 | 0.00003 | 0.24013 | 0.24001 |
| 0.00042 | 0.00003 | 0.39421 | 0.39392 |
| 0.00061 | 0.00004 | 0.48934 | 0.48899 |
| 0.00082 | 0.00004 | 0.55899 | 0.55858 |
| 0.00100 | 0.00004 | 0.59969 | 0.59925 |
| 0.00309 | 0.00013 | 0.68913 | 0.68863 |
| 0.00574 | 0.00036 | 0.68790 | 0.68740 |
| 0.00680 | 0.00052 | 0.68685 | 0.68636 |
| 0.00822 | 0.00067 | 0.68545 | 0.68495 |
| 0.01066 | 0.00084 | 0.68303 | 0.68254 |
| 0.03055 | 0.00095 | 0.66353 | 0.66304 |
| 0.05236 | 0.00096 | 0.64258 | 0.64210 |
| 0.07236 | 0.00079 | 0.63374 | 0.62328 |
| 0.09106 | 0.00094 | 0.60645 | 0.60599 |
| 0.20011 | 0.00098 | 0.51114 | 0.51072 |
| 0.40037 | 0.00111 | 0.35676 | 0.35640 |
| 0.60386 | 0.00109 | 0.22103 | 0.22072 |
| 0.90033 | 0.00106 | 0.05137 | 0.05111 |
| 0.99999 | 0.00086 | 0.00025 | 0.00000 |

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