ON THE CONJUGACY THEOREM AND THE CONJUGACY CLASSES FOR GROUPS ACTING ON TREES WITH INVERSIONS

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الخلاصـــة :

نوسّـع في هذا البحث نظرية الترافق للضرب الحرّ بـين الـزُّمر مع الدمج ومجموعات HNN لمجموعات تعمل على شجرات مع إمكانية التحويل أو التعاكس . والبحث يشمل نظرية الترافق للضرب الـشَّجري بـين المجموعات ومجموعات HNN الشجرية . ونبرهن أيضاً نتيجة تتعلق بترافق صفوف المجموعات المؤثرة على الشجرات .

ABSTRACT

In this paper we extend the conjugacy theorem for free products of groups with amalgamation and HNN groups to groups acting on trees in which inversions are possible. This will include the conjugacy theorem for free products of groups, and treed-HNN groups. Also we prove a result concerning conjugacy classes of groups acting on trees.

ON THE CONJUGACY THEOREM AND THE CONJUGACY CLASSES FOR GROUPS ACTING ON TREES WITH INVERSIONS

1. INTRODUCTION

The conjugacy theorem for free products of groups with amalgamation, known as "Solitar's Theorem" was formulated by Magnus, Karrass, and Solitar ([1], Theorem 4.6, p. 212), and the conjugacy theorem for HNN groups, known as "Collin's Lemma" was formulated by Collins ([2], general Lemma 3, p. 123). See also Lyndon and Schupp ([3], Theorem 25, p. 185).

Free products of groups with amalgamation and HNN groups are special cases of groups acting on trees. In this paper we formulate the conjugacy theorem for groups acting on trees in general, to include the conjugacy theorems for tree products of groups and treed-HNN groups.

2. PRELIMINARY DEFINITIONS AND NOTATION

We begin by giving preliminary definitions. By a graph X we understand a pair of disjoint sets $V(X)$ and $E(X)$, with $V(X)$ non-empty, together with a mapping $E(X)\rightarrow V(X)\times V(X)$, $y\rightarrow(o(y),t(y))$, and a mapping $E(X)\rightarrow E(X)$, $y\rightarrow\bar{y}$ satisfying $\bar{\bar{y}}=y$ and $o(\bar{y})=t(y)$, for all $y\in E(X)$. The case $\bar{y}=y$ is possible for some $y\in E(X)$.

A path in a graph X is defined to be either a single vertex $v\in V(X)$ (a trivial path), or a finite sequence of edges $y_1,y_2,\ldots,y_n$, $n\geq 1$ such that $t(y_i)=o(y_{i+1})$ for $i=1,2,\ldots,n-1$.

A path $y_1,y_2,\ldots,y_n$ is reduced if $y_{i+1}\neq\bar{y}_i$, for $i=1,2,\ldots,n-1$. A graph X is connected, if for every pair of vertices u and v of $V(X)$ there is a path $y_1,y_2,\ldots,y_n$ in X such that $o(y_1)=u$ and $t(y_n)=v$.

A graph X is called a tree if for every pair of vertices of $V(X)$ there is a unique reduced path in X joining them.

A subgraph Y of a graph X consists of sets $V(Y)\subseteq V(X)$ and $E(Y)\subseteq E(X)$ such that if $y\in E(Y)$, then $\bar{y}\in E(Y)$, $o(y)$ and $t(y)$ are in $V(Y)$. We write $Y\subseteq X$. We take any vertex to be a subtree without edges.

A reduced path $y_1,y_2,\ldots,y_n$ is called a circuit if $o(y_1)=t(y_n)$, and $o(y_i)\neq o(y_j)$ when $i\neq j$. It is clear that a graph X is a tree if X is connected and contains no circuits.

If X_1 and X_2 are two graphs, then the map $f\colon X_1\rightarrow X_2$ is called a morphism, if f takes vertices to vertices and edges such that:

$$f(\bar{y}) = \overline{f(y)}$$
$$f(o(y)) = o(f(y))$$
$$\text{and}\qquad f(t(y)) = t(f(y))\qquad\text{for all } y\in E(X_1);$$

f is called an isomorphism if it is one-to-one and onto, and is called an automorphism if it is an isomorphism and $X_1=X_2$. The automorphisms of X form a group under composition of maps, denoted by $\mathrm{Aut}(X)$.

We say that a group G acts on a graph X, if there is a group homomorphism $\phi\colon G\rightarrow\mathrm{Aut}(X)$. If $x\in X$ is a vertex or an edge, we write $g(x)$ for $\phi(g)(x)$. If $y\in E(X)$, then $g(\bar{y})=\overline{g(y)}$, $g(o(y))=o(g(y))$, and $g(t(y))=t(g(y))$. The case $g(y)=\bar{y}$ for some $y\in E(X)$ and $g\in G$ may occur, *i.e.* G acts with inversions on X. If $y\in X$ (vertex or edge), we define $G(y)=\{g(y)\colon g\in G\}$ and this set is called an orbit.

If $x,y\in X$, we define $G(x,y)=\{g\in G\colon g(y)=x\}$, and $G(x,x)=G_x$, the stabilizer of x. Thus, $G(x,y)\neq\varnothing$ if and only if x and y are in the same G-orbit. It is clear that if $v\in V(X)$, $y\in E(X)$ and $u\in\{o(y),\ t(y)\}$, then $G(v,y)=\varnothing$, $G_{\bar{y}}=G_y$ and G_y is a subgroup of G_u.

As a result of the action of a group G on a graph X we have the graph: $X/G=\{G(x)\colon x\in X\}$, called the quotient graph defined as follows:

$$V(X/G)=\{G(v)\colon v\in V(X)\},$$
$$E(X/G)=\{G(y)\colon y\in E(X)\},$$

and for $y\in E(X)$ we have

$$\overline{G(y)}=G(\bar{y}),\ t(G(y))=G(t(y)),$$
$$\text{and}\qquad o(G(y))=G(o(y)).$$

It is clear that there is obvious morphism $p\colon X\rightarrow X/G$ given by $p(x)=G(x)$, which is called the projection.

It can be easily shown that if X is connected, then X/G is connected.

Definition 2.1. Let G be a group acting on a connected graph X. A subtree T of X is called a tree of representatives for the action of G on X if T contains exactly one vertex from each G-vertex orbit.

A subtree Y of X containing a tree of representatives, T(say), is called a fundamental domain for the action of G on X if each edge in Y has at least one end point in T and Y contains exactly one edge, y(say), from each G-edge orbit such that $G(\bar{y}, y) = \varnothing$ and exactly one pair x and $\bar{x}$ from each G-edge orbit such that $G(\bar{x}, x) \neq \varnothing$.

For the existence of T and Y see Khanfar and Mahmud [4].

Properties of T and Y:

(1) If $u,\ v \in V(T)$ such that $G(u, v) \neq \varnothing$, then $u = v$.

(2) If $v \in V(X)$, then $G(v) \cap T$ consists of exactly one vertex.

(3) $G(\bar{y}, y) = \varnothing$, for all $y \in E(T)$.

(4) $V(T)$ is in one-to-one correspondence with $V(X/G)$ under the map $v \to G(v)$.

(5) If $y_1, y_2 \in E(Y)$ such that $G(y_1, y_2) \neq \varnothing$, then $y_1 \in \{y_2, \bar{y}_2\}$.

(6) If G acts without inversions on X, then Y is in one-to-one correspondence with X/G under the map $y \to G(y)$.

(7) If $u \in V(X)$, then there exists an element $g \in G$ and a unique vertex v of T such that $u = g(v)$.

(8) If $x \in E(X)$, then there exists $g \in G$ and $y \in E(Y)$ such that $x = g(y)$. If G acts on X without inversions, then y is unique.

(9) The set $G(Y) = \{g(y): g \in G,\ y \in Y\} = X$. Also $G(E(Y)) = \{g(y): g \in G,\ y \in E(Y)\} = E(X)$.

(10) The set $G(V(T)) = \{g(v): g \in G,\ v \in V(T)\} = V(X)$.

Definition 2.2. Let G, X, T, and Y be as above. For each $v \in V(X)$ let v^* be the unique vertex of T such that $G(v, v^*) \neq \varnothing$. It is clear that $v^* = v$, if $v \in V(T)$, and in general $(v^*)^* = v^*$. Also if $G(u, v) \neq \varnothing$, then $u^* = v^*$ for $u,\ v \in V(X)$.

Note that $G(v) \cap T = \{v^*\}$, for all $v \in V(X)$.

3. THE STRUCTURE OF GROUPS ACTING ON TREES

In this section and the rest of the paper G will be a group acting on the tree X in general, *i.e.*, action with inversions is possible, T be a tree of representatives, and Y be a fundamental domain such that Y contains T.

Given this we can now introduce the following notation needed throughout this paper.

(1) For each $v \in V(T)$, let $\langle \tilde{G}_v | rel\, G_v \rangle$ stand for any presentation of G_v *via* the map $\theta_v: F_v \to G_v$, where F_v is a free group of base $\tilde{G}_v$.

(2) For each edge y of $E(Y)$ we have the following:

(*a*) Define $[y]$ to be an element of $G(t(y), t(y)^*)$, that is, $[y](t(y)^*) = t(y)$, to be chosen as follows:
If $o(y) \in V(T)$ then: (*i*) $[y] = 1$ if $y \in E(T)$; (*ii*) $[y](y) = \bar{y}$ if $G(\bar{y}, y) \neq \varnothing$.
If $o(y) \notin V(T)$ then: $[y] = [\bar{y}]^{-1}$ if $G(\bar{y}, y) = \varnothing$, otherwise $[y] = [\bar{y}]$.
It is clear that $[y][\bar{y}] = 1$ if $G(\bar{y}, y) = \varnothing$, otherwise $[y][\bar{y}] = [y]^2$.

(*b*) Let $-y = [y]^{-1}(y)$ if $o(y) \in V(T)$, otherwise let $-y = y$. Now define $+y = [y](-y)$. It is clear that $t(-y) = t(y)^*$, $o(+y) = o(y)^*$ and $\overline{(+y)} = -(\bar{y})$.

(*c*) Let S_y be a word in $\tilde{G}_{o(y)^*}$ of value $[y][\bar{y}]$. It is clear that $S_{\bar{y}} = S_y$.

(*d*) Let E_y be a set of generators of G_{-y} and $\tilde{G}_y$ be a set of words in $\tilde{G}_{t(y)^*}$ mapping onto E_y, *i.e.* $\theta^{-1}_{t(y)^*}\ (G_{-y}) = \tilde{G}_y$.

(*e*) Define $\phi_y: G_{-y} \to G_{+y}$ by $\phi_y(g) = [y]g[y]^{-1}$, $g \in G_{-y}$ and define $\psi_y: \tilde{G}_y \to \tilde{G}_{\bar{y}}$ by taking the word which represents the element g of E_y to the word which represents the element $[y]g[y]^{-1}$.

(*f*) Let $yG_y y^{-1} = G_{\bar{y}}$ stand for the set of relations $ywy^{-1} = \psi_y(w),\ w \in \tilde{G}_y$.

(3) Let $P(Y)$ stand for the set of generating symbols

(*i*) $\tilde{G}_v$, for $v \in V(T)$

(*ii*) y, for $y \in E(Y)$

and $R(Y)$ stand for the set of relations

(*i*) $\mathrm{rel}\, G_v$, for $v \in V(T)$

(*ii*) $yG_y y^{-1} = G_{\bar{y}}$, for $y \in E(Y)$

(*iii*) $y = 1$, for $y \in E(T)$

(*iv*) $y\bar{y} = S_y$, for $y \in E(Y)$

(*v*) $y^2 = S_y$, for $y \in E(Y)$ such that $G(\bar{y}, y) \neq \varnothing$.

Note that if $G(\bar{y}, y) \neq \varnothing$, then $y \notin E(T)$.

Theorem 3.1. G is generated by the set $\{G_v, [y]: v \in V(T) \text{ and } y \in E(Y)\}$ and G has the presentation $\langle P(Y) | R(Y) \rangle$ *via* $\tilde{G}_v \to G_v$ and $y \to [y]$, for all $v \in V(T)$ and all $y \in E(Y)$.

Proof. See [1].

4. THE NORMAL FORM THEOREM OF GROUPS ACTING ON TREES

Definition 4.1. By a word of G we mean an expression w of the form $w = g_0.y_1.g_1. \dots .y_n.g_n$, $n \geq 0$, where $y_i \in E(Y)$, $i = 1,\dots,n$, such that:

(1) $g_0 \in G_{o(y_1)^*}$,

(2) $g_i \in G_{t(y_i)^*}$, for $i = 1,\dots,n$,

(3) $t(y_i)^* = o(y_{i+1})^*$, for $i = 1,\dots,n-1$.

We define $o(w) = o(y_1)^*$ and $t(w) = t(y_n)^*$.

We define n to be the length of w, and denote it by $|w|$. The inverse w^{-1} of w is defined to be the word:

$$w^{-1} = g_n^{-1}.\bar{y}_n.g_{n-1}^{-1}. \dots .g_1^{-1}.\bar{y}_1.g_0^{-1}.$$

It is clear that $|w^{-1}| = |w|$. Also $o(w^{-1}) = t(w)$ and $t(w^{-1}) = o(w)$ and $(w^{-1})^{-1} = w$.

w is called a reduced word of G if w contains no subword of the form

(1) $1.y_i.g_i.\bar{y}_i.1$, if $g_i \in G_{-y_i}$, for $i = 1,\dots,n$,

or

(2) $1.y_i.g_i.y_i.1$, if $g_i \in G_{-y_i}$ with $G(\bar{y}_i, y_i) \neq \emptyset$, for $i = 1,\dots,n$.

It is clear that if w is reduced, then w^{-1} is reduced.

If $o(w) = t(w)$, then w is called a closed word of G of type $o(w)$. If w is closed then w^{-1} is closed.

The value $[w]$ of w is the element $[w] = g_0[y_1]g_1 \dots [y_n]g_n$ of G.

It is clear that $[w^{-1}] = [w]^{-1}$.

If $w_1 = h_n.y_{n+1}.h_{n+1}. \dots .y_m.h_m$ is a word of G such that $t(w) = o(w_1)$, then $w.w_1$ is defined to be the word $w.w_1 = g_0.y_1.g_1. \dots .y_n.g_nh_n.y_{n+1}.h_{n+1}. \dots .y_m.h_m$.

It is clear that $[w.w_1] = [w][w_1]$ and $(w.w_1)^{-1} = w_1^{-1}.w^{-1}$.

Definition 4.2. The performance of the following operations is called a y-reduction on a word w of G, where y is an edge of Y occurs in w:

(1) replacing the form $y.g.\bar{y}$ by $[y]g[y]^{-1}$, if $g \in G_{-y}$,

or

(2) replacing the form $y.g.y$ by $[y]g[y]$, if $G(\bar{y}, y) \neq \emptyset$ and $g \in G_{-y}$.

It is clear that the y-reduction on a word w of G yields a reduced word w_1 of G such that $[w] = [w_1]$, $o(w) = o(w_1)$ and $t(w) = t(w_1)$.

Lemma 4.3. For any element g of G and vertices u and v of $V(T)$ there exists a reduced word w of G such that $g = [w]$, $o(w) = u$ and $t(w) = v$.

Proof. Let $g \in G$ and u, $v \in V(T)$.

By Theorem 3.1, g can be expressed as a product: $g_0[y_1]g_1 \dots [y_n]g_n$, where $g_i \in G_{u_i}$ for some vertices $u_0, u_1, \dots, u_n$ in T and edges $y_1, \dots, y_n$ in Y.

By taking the unique reduced paths in T between u and u_0, u_0 and $o(y_1)^*$, between $t(y_1)^*$ and $u_1, \dots,$ between $t(y_n)^*$ and u_n, and between u_n and v, and the identities of $G_{t(y_i)^*}$, we may choose this product so that $w = g_0.y_1.g_1. \dots .y_n.g_n$ is a word of G such that $g = [w]$, $o(w) = u$ and $t(w) = v$. Now applying a finite number of y-reductions on w yields a reduced word w^* of G such that $g = [w^*]$, $o(w^*) = u$ and $t(w^*) = v$. This completes the proof of the Lemma.

Definition 4.4. For $y \in E(Y)$ define A_y to be a right transversal for G_{-y} in $G_{t(y)^*}$ subject only to the condition that 1 is the representative for the coset G_{-y}.

Definition 4.5. A word $w = g_0.y_1.g_1. \dots .y_n.g_n$ of G is called normal if it is reduced and satisfies the following:

(1) $g_0 \in G_{o(y_1)^*}$.

(2) $g_i \in A_{y_i}$, for $i = 1,\dots,n$

(3) If $y_{i+1} = \bar{y}_i$ for some i, $1 \leq i \leq n-1$, then $g_i \neq 1$.

(4) If $y_{i+1} = y_i$ for some i, $1 \leq i \leq n-1$ and $G(\bar{y}_i, y_i) \neq \emptyset$, then $g_i \neq 1$.

Theorem 4.6. (The Normal Form Theorem). Every element of G is the value of a unique normal word of G of type v for an arbitrary vertex v of $V(T)$. Moreover, if w is a non-trivial closed reduced word of G, then $[w]$ (the value of w) is not the identity element of G.

Proof. See [5].

Lemma 4.7. If $y \in E(Y)$ and $y \notin E(T)$, then $[y] \notin G_v$, for all $v \in V(T)$.

Proof. We notice that if $y \in E(T)$ then $[y] = 1$ and consequently $[y] \in G_v$, for any $v \in V(T)$.

Now let $y \notin E(T)$. We need to show that $[y] \notin G_v$, for any $v \in V(T)$. Assume on the contrary that $[y] \in G_v$, for $v \in V(T)$. Then $[y]$ is the value of the word: $w = 1.y_1.1. \ldots .y_n.1.y.1.x_1.1. \ldots 1.x_m.1$, where $y_1, \ldots, y_n$ is the unique reduced path in T from v to $o(y)^*$ and $x_1, \ldots, x_m$ is the unique reduced path in T from $t(y)^*$ to v. Since the word $w.[y]^{-1}$ is closed of value the identity element of G, therefore by Theorem 4.6, w is not reduced. Therefore some y-reduction is applicable to $w.[y]^{-1}$. This occurs in the case $\bar{y} = y_n$ or $\bar{y} = x_1$. This means that $y \in E(T)$. This contradicts the assumption that $y \notin E(T)$. Hence $[y] \notin G_v$.

This completes the proof of the Lemma.

Lemma 4.8. If $w_1 = g_o.y_1.g_1. \ldots .y_n.g_n$ and $w_2 = h_o.x_1.h_1. \ldots .x_m.h_m$ are two reduced closed words of G of the same value and type, then $n = m$, $y_i = x_i$, (or $y_i = \bar{x}_i$ if $G(\bar{x}_i, x_i) \neq \emptyset$) for $i = 1, \ldots, n$, and there is a unique sequence $\pi_1, \pi_2, \ldots, \pi_{2n}$, where $\pi_{2i} \in G_{-y_i}$ and $\pi_{2i-1} \in G_{+y_i}$ for $i = 1, \ldots, n$ such that $g_o = h_o\pi_1$, $g_i = \pi_{2i}h_i\pi_{2i+1}$ for $i = 1, \ldots, n-1$, and $g_n = \pi_{2n}h_n$. Also if $g_ng_o \in G_{-y_n}$ then $h_nh_o \in G_{-y_n}$.

Proof. See [5].

5. THE CONJUGACY THEOREM OF GROUPS ACTING ON TREES

Definition 5.1. Let $w = g_o.y_1.g_1. \ldots .y_n.g_n$ be a closed word of G. For $i = 1, 2, \ldots, n$ we call the word $g_i.y_{i+1}.g_{i+1}. \ldots .y_n.g_ng_o.y_1.g_1. \ldots .y_i.1$ or the word $1.y_{i+1}.g_{i+1}. \ldots .y_n.g_ng_o.y_1.g_1. \ldots .y_i.g_i$ a cyclic permutation of w. If w is reduced and $|w| \leq 1$, or the word $1.y_n.g_ng_o.y_1.1$ is reduced then we call w a cyclically reduced word of G.

We observe that if w is cyclically reduced and $|w| > 1$, then w^n is cyclically reduced, where n is an integer, and $|w^n| = |n||w|$. Moreover, every cyclic permutation of w is cyclically reduced.

Also we note that if $w = g_o.y.g$ is such that $o(y)^* = t(y)^*$ and $gg_o \in G_{-y}$, then w^2 is cyclically reduced if $G(\bar{y}, y) = \emptyset$, while w^2 is not cyclically reduced if $G(\bar{y}, y) \neq \emptyset$.

Definition 5.2. If w_1 and w_2 are two words of G of the same value, *i.e.* $[w_1] = [w_2]$ then we write $w_1 \approx w_2$ and say that w_1 is equivalent to w_2.

Lemma 5.3. Let w be a cyclically reduced word of G, and w_o a closed reduced word of G such that w and w_o are the same type and $w \approx w_o$. Then w_o is cyclically reduced.

Proof. By Lemma 4.8, $|w| = |w_o|$. If $|w_o| \leq 1$ then by definition w_o is cyclically reduced. So let $|w_o| > 1$.

Let $w = g_o.y_1.g_1. \ldots .y_n.g_n$ and $w_o = h_o.x_1.h_1. \ldots .x_n.h_n$. We need to show that the word $1.x_n.h_nh_o.x_1.1$ is reduced. Since w is cyclically reduced, therefore $g_ng_o \notin G_{-y_n}$.

By Lemma 4.8, $h_nh_o \notin G_{-y_n}$. So $1.x_n.h_nh_o.x_1.1$ is reduced.

Therefore w_o is cyclically reduced. This completes the proof of the Lemma.

Definition 5.4. Two words w_1 and w_2 of G are conjugate denoted $w_1 \sim w_2$ if w_1 and w_2 define conjugate elements of G, *i.e.* if $[w_1]$ and $[w_2]$ are conjugate in G.

Lemma 5.5. Let w_1 and w_2 be two cyclically reduced words of G with $|w_2| \geq 1$. Then $w_1 \sim w_2$ if and only if any cyclic permutation of w_1 can be obtained by taking a suitable cyclic permutation of w_2, and then conjugating by an element of G_{-y}, where y is the last edge in the cyclic permutation of w_2.

Proof. Suppose first that any cyclic permutation w_1^* of w_1 can be obtained by taking a suitable cyclic permutation w_2^* of w_2 followed by conjugating by an element of G_{-y}, where y is the last edge in w_2^*, *i.e.*, $w_1^* \approx h.w_2^*.h^{-1}$, for $h \in G_{-y}$.

Since $w_1 \sim w_1^*$, and $w_2 \sim w_2^*$, it follows that $w_1 \sim w_2$.

Next suppose that $w_1 \sim w_2$.

Let w_1^* and w_2^* be any cyclic permutations of w_1 and w_2 respectively. Therefore $w_1^* \sim w_2^*$. Then by Lemma 4.3, $w_1 \approx w.w_2^*.w^{-1}$, where w is a reduced word of G such that $o(w) = o(w_1^*)$ and $t(w) = o(w_2^*)$. We use induction on $|w|$ to prove our result.

If $|w| = 0$, then the result follows from Lemma 4.8.

Suppose that $|w| \geq 1$.

Let $w = g_o.y_1.g_1. \ldots .y_n.g_n$, $n \geq 1$, and $w_2^* = h_o.x_1.h_1. \ldots .x_m.1$, $m \geq 1$. Since w_1^* is cyclically reduced by Lemma 5.3, some cancellation must be applicable to $w.w_2^*.w^{-1}$. This suggests the consideration of the following cases:

Case 1:

$x_1 = \bar{y}_n$, (or $x_1 = y_n$ if $G(\bar{y}_n, y_n) \neq \emptyset$), and $g_nh_o \in G_{-y_n}$. *Then* $w_1^* \approx w_o.w_2'.w_0^{-1}$, where $w_o = g_o.y_1.g_1. \ldots .y_{n-1}.\phi_{y_n}(g_ng_o)$, and $w_2' = h_1.x_1. \ldots .x_m.g_n^{-1}.x_1.\phi_{y_n}(g_ng_o)$. Since $x_1.\phi_{y_n}(g_nh_o) \approx g_nh_o.x_1$, we have $w_1^* \approx w_o.(h_1.x_2. \ldots .x_m.h_o.x_1.1)^{-1}.w_o^{-1}$. Since $h_1.x_2. \ldots .x_m.h_o.x_1.1$ is a cyclic permutation of w_2^*, the result follows by the induction hypothesis.

Case 2:

$x_m = y_n$, (or $x_m = \bar{y}_n$ if $G(\bar{y}_n, y_n) \neq \emptyset$), and $g_n \in G_{-y_n}$. This case is similar to Case 1.

This completes the proof of the Lemma.

Definition 5.6. An element of G is called cyclically reduced if it is the value of a cyclically reduced word of G. In view of Lemma 5.3, this concept is well defined.

Theorem 5.7. (The Conjugacy Theorem). Every element of G is conjugate to a cyclically reduced element of G. Moreover, suppose that g is a cyclically reduced element of G and v is the type of a cyclically reduced word of G of value g. Then

(*i*) if g is conjugate to an element h in G_{-y}, where $y \in E(Y)$ then g is in G_v and there are sequences of edges $y_1, y_2, \ldots, y_n$ of Y and of elements $h_1, h_2, \ldots, h_n$ of G satisfying:

(1) $o(y_1)^* = v$,

(2) $t(y_i)^* = o(y_{i+1})^*$, for $i = 1, \ldots, n-1$,

(3) $t(y_n)^* = t(y)^*$,

(4) $h_i \in G_{-y_i}$, for $i = 1, \ldots, n$,

(5) g and $\phi_{y_1}(h_1)$ are conjugate by an element of G_v,

(6) h_i and $\phi_{y_{i+1}}(h_{i+1})$ are conjugate by an element of $G_{o(y_{i+1})^*}$ for $i = 1, \ldots, n-1$,

(7) h_n and h are conjugate by an element of $G_{t(y_n)^*}$;

(*ii*) if g is conjugate to an element g' of G_u, for $u \in V(T)$, but not conjugate to any element of G_{-y} for any $y \in E(Y)$ such that $t(y)^* = u$, then $u = v$, $g \in G_v$, and, g and g' are conjugate in G_v;

(*iii*) let $w = g_o.y_1.g_1. \ldots .y_n.g_n$, $n \geq 1$ be a cyclically reduced word of G. Then g is conjugate to $[w]$ if and only if there is a cyclic permutation $w_i = g_i.y_{i+1}. \ldots .y_n.g_ng_o.y_1.g_1. \ldots .y_i.1$ of w and element h of G_{-y_i} such that g and $[w_i]$ are conjugate by the element h.

Proof. Let g be an element of G. We need to show that g is conjugate to a cyclically reduced element of G. Let g' be an element in the conjugacy class of G containing g such that g' is represented by a closed reduced word w of G of shortest length. We need to show that w is cyclically reduced.

Let $w = g_o.y_1.g_1. \ldots .y_n.g_n$.

If $n = 0$ then $w = g_o$ and w is cyclically reduced.

Let $n \geq 1$. If $g_ng_o \in G_{-y_n}$ and $y_1 = \bar{y}_n$ (or $y_1 = y_n$ if $G(\bar{y}_n, y_n) \neq \emptyset$) then $w_o = g_1.y_2.g_2. \ldots .y_{n-1}.g_{n-1}\phi_{y_n}(g_ng_o)$ is a closed reduced word of G and of value conjugating g. But w_o has length smaller than w. Contradiction. Thus w is a cyclically reduced word of G.

To prove (*i*), suppose that g is a cyclically reduced element of G such that g is conjugate to an element h in G_{-y} for $y \in E(Y)$.

Then by Lemma 4.3, $g = [w]h[w]^{-1}$, where $w = g_o.y_1.g_1. \ldots .y_n.g_n$ is a reduced word of G such that $o(w) = v$ and $t(w) = t(y)^*$.

If $n = 0$, then $v = t(y)^*$, $h \in G_v$ and the sequence g, h is the required type, since $g = g_ohg_o^{-1}$.

Let $n \geq 1$.

For each i, $1 \leq i \leq n$ define w_i to be the word

$$w_i = (g_i.y_{i+1}.g_{i+1}. \ldots .y_n.g_n).h.(g_i.y_{i+1}.g_{i+1}. \ldots .y_n.g_n)^{-1}.$$

Let $h_i = [w_i]$.

Suppose there is a largest integer q such that $h_q \notin G_{-y_q}$ but $h_{q+1} \in G_{-y_{q+1}}$. Then $h_j \in G_{-y_j}$, if $j > q$, for the existence of $j > q$ with $h_j \notin G_{-y_j}$ would contradict the maximality of q. If q exists, then the word $(g_o.y_1.g_1. \ldots).h_q.(g_o.y_1.g_1. \ldots)^{-1}$ is reduced of type v and value g, but not cyclically reduced. This contradicts Lemma 5.3. Hence q does not exist.

Therefore $h_j \in G_{-y_j}$ for $1 \leq j \leq n$. In any event, the edges $y_1, \ldots, y_n$ and the elements g, $h_1, h_2, \ldots, h_n$, h are of the required type and,

$h_i = g_i[y_{i+1}]h_{i+1}[y_{i+1}]^{-1}g_i^{-1} = g_i\phi_{y_{i+1}}(h_{i+1})g_i^{-1}$, for $1 \leq i \leq n-1$, and

$g = g_o[y_1]h_1[y_1]^{-1}g_o^{-1} = g_o\phi_{y_1}(h_1)g_o^{-1}$. Moreover, $h_n = g_nhg_n^{-1}$, and $g_n \in G_{t(y_n)^*}$.

To prove (*ii*), suppose that g is conjugate to an element f of G_u but not conjugate to any element of G_{-y}, for any $y \in E(Y)$ such that $t(y)^* = u$. Then by Lemma 4.3, $g = [w]f[w]^{-1}$, where $w = g_o.y_1.g_1. \ldots .y_n.g_n$ is a reduced word of G such that $o(w) = v$ and $t(w) = u$. Suppose that $n \geq 1$. Then g is the value of the word $w_o = w.f.w^{-1}$.

If $(1.y_n.g_n).f.(1.y_n.g_n)^{-1}$ is reduced, then w_o is reduced but not cyclically reduced. This contradicts Lemma 5.3. Hence $(1.y_n.g_n).f.(1.y_n.g_n)^{-1}$ is not reduced. Therefore $g_nfg_n^{-1} \in G_{-y_n}$, and g is conjugate to $g_nfg_n^{-1}$.

This contradicts the hypothesis.

Hence $n=0$, $u=v$ and both f and $g=g_0 f g_0^{-1}$ are in G_v, and are conjugate in G_v.

The proof of (*iii*) follows from Lemma 5.4.

This completes the proof of Theorem 5.7.

In view of Theorem 5.7 we have the following corollaries.

Corollary 5.8. Consider the sequences: $y_1,\ldots,y_n$ of edges of Y, and $h_1,\ldots,h_n$ of elements of G described in case (*i*) of Theorem 5.7. Then these sequences can be chosen so that no pair y_i, h_i is repeated.

Moreover, if Y and G_y are finite for all $y\in E(X)$ then there are only finitely many sequences of distinct edges and elements mentioned above.

Proof. If $y_j=y_s$, and $h_j=h_s$ for $j\le s$ then $y_1,\ldots,y_j$, $y_{s+1},\ldots,y_n$, and g, $h_1,\ldots,h_j$, $h_{s+1},\ldots,h_n$, h are shorter sequences of the required types, since $t(y_j)^*=t(y_s)^*=o(y_{s+1})^*$, and $[y_j]h_j[y_j]^{-1}=[y_s]h_s[y_s]^{-1}$, h_{s+1} are conjugate by an element of $G_{o(y_s)^*}$.

Hence any sequences of the required types with minimal numbers of terms has distinct pairs.

Now since Y is finite, so is T.

Therefore for any $v\in V(T)$, the set $\{y\in E(Y):t(y)^*=v\}$ is finite. Since for any $y\in E(Y)$ G_{-y} is finite, there are only finitely many distinct pairs and hence only finitely many sequences without repeats.

6. THE TORSION THEOREM OF GROUPS ACTING ON TREES

Theorem 6.1. If G acts without inversions on X, then every element of G of finite order is in G_v, for some $v\in V(X)$.

Proof. Let g be an element of G of finite order. By Theorem 5.7, g is conjugate to $[w]$, where $w=g_0.y_1.g_1.\ \ldots\ .y_n.g_n$ is a cyclically reduced word of G.

Thus $g=h[w]h^{-1}$, where $h\in G$.

If $n=0$, then it is clear that $g\in G_v$, where $v=h(o(y_1)^*)$.

But if $n\ge 1$, then:

$$w^r=g_0.y_1.g_1.\ \ldots\ .y_n.g_n g_0.y_1.g_1.\ \ldots\ .y_n.g_n.\ \ldots\ .g_0.y_1.\ \ldots\ .y_n.g_n$$

which is a closed reduced word of G, since w is cyclically reduced. By Theorem 4.6 $[w]^r\ne 1$, *i.e.*, $[w]$ has infinite order.

Hence in this case g cannot have finite order.

Remark: In Theorem 6.1 above we excluded the case when the action of G on X is with inversions, for otherwise we get an edge y of Y in which $G(\bar{y},y)\ne\varnothing$. In this case we have $[y]^2\in G_{t(y)^*}$ and it is possible that $[y]^2=1$, *i.e.* $[y]$ is of order two, but $[y]\notin G_{t(y)^*}$.

Corollary 6.2. If G acts without inversions on X and $y\in E(Y)$, $y\notin E(T)$, then $[y]$ has infinite order.

Proof. If $y\in E(T)$ then $[y]=1$. If $y\notin E(T)$ then by Lemma 4.7, $[y]\notin G_v$, for all $v\notin V(T)$. Therefore by Theorem 6.1, $[y]$ has infinite order.

Corollary 6.3. If G acts without inversions on X and H is a finite subgroup of G then H is contained in G_v for some $v\in V(X)$.

Proof. The proof follows easily by virtue of Theorem 6.1.

7. ON CONJUGACY CLASSES OF GROUPS ACTING ON TREES

Let P denote the following property of a group H: If g in H has infinite order, then $g,g^2,g^3,\ldots$ are in different conjugacy classes, or equivalently, if $g^m\sim g^n$ then $|m|=|n|$, where m and n are integers.

Many classes of groups have property P. For example infinite cyclic groups have property P. Also if a group H has property P, then so is every subgroup of H. Finite groups are the trivial example of groups of property P. For more details see [6].

Theorem 7.1. Let G act on X without inversions such that G_v has property P for all $v\in V(X)$, and G_x is cyclic for all $x\in E(X)$. Then G has property P.

Proof. For any elements f and g of G, we write $f\sim g$ to mean that f is conjugate to g. Suppose that g in G has infinite order and that $g^{m_1}\sim g^{m_2}$. We need to show that $|m_1|=|m_2|$.

By Theorem 5.7, $g\sim f$, where f is a cyclically reduced element of G. Therefore f^{m_1} and f^{m_2} are cyclically reduced elements of G and $f^{m_1}\sim f^{m_2}$. Let w be a cyclically reduced word of G of type v and value f. Then w^{m_1} and w^{m_2} are cyclically reduced words of G of values f^{m_1} and f^{m_2} respectively.

So $|w^{m_1}| = |m_1||w|$ and $|w^{m_2}| = |m_2||w|$. If $|w| \geq 1$, then by Theorem 5.7 (*iii*), $|m_1| = |m_2|$. If $|w| = 0$, then f, f^{m_1} and f^{m_2} are in G_v.

We have two cases:

Case 1:

$f^{m_1} \notin G_{-y}$ for all $y \in E(Y)$ such that $t(y)^* = v$.

Then by Theorem 5.7 (*ii*), $f^{m_1} \sim f^{m_2}$ in G_v. Since G_v has property P, we must have $|m_1| = |m_2|$.

Case 2:

$f^{m_2} \in G_{-y}$, where $y \in E(Y)$ such that $t(y)^* = v$.

Therefore, by Theorem 5.7 (i) there are two sequences of edges $y_1, \ldots, y_n$ of Y and elements $h_1, \ldots, h_n$ of G satisfying the conditions of Theorem 5.7 (i).

So
$f^{m_1} \sim \phi_{y_1}(h_1) \sim h_1 \sim \phi_{y_2}(h_2) \sim h_2 \sim \ldots \phi_{y_n}(h_n) \sim h_n \sim f^{m_2}$ where $f^{m_1} \sim \phi_{y_1}(h_1)$ by an element of G_v, $\phi_{y_i}(h_i) \sim h_i$ by the element $[y_i]$, for $1 \leq i \leq n$, $h_i \sim \phi_{y_{i+1}}(h_{i+1})$ by an element of $G_{o(y_{i+1})^*}$, for $1 \leq i \leq n-1$, and $h_n \sim f^{m_2}$ by an element of G_v.

By assumption, G_x is cyclic for all $x \in E(X)$. Therefore the G_{-y_i} are cyclic for $i = 1, \ldots, n$. Let h be a generator of the cyclic group G_{-y}, and suppose that $h_i \sim h^{\alpha_i}$ and $\phi_{y_i}(h_i) \sim h^{\beta_i}$, for $1 \leq i \leq n$, where α_i and β_i are integers. Therefore, $f^{m_1} \sim h^{\beta_1} \sim h^{\alpha_1} \sim h^{\beta_2} \sim h^{\alpha_2} \sim \ldots \sim h^{\beta_n} \sim h^{\alpha_n} \sim f^{m_2}$.

Since $\phi_{y_i}: G_{-y_i} \to G_{+y_i}$ given by $g \to [y_i]g[y_i]^{-1}$ is an isomorphism, therefore $|\beta_i| = |\alpha_i|$, for $1 \leq i \leq n$. Since $G_{o(y_i)^*}$ has property P, for $1 \leq i \leq n$, it follows that $|\alpha_i| = |\beta_{i+1}|$, for $1 \leq i \leq n-1$.

Therefore we have $f^{m_1} \sim h^{\alpha_n} \sim f^{m_2}$ or $f^{m_1} \sim h^{\alpha_n} \sim f^{-m_2}$ in G_v. Since G_v has property P, $|m_1| = |m_2|$.

This completes the proof of Theorem 7.1.

Corollary 7.2. Free groups have property P.

Proof. If G is a free group then there is a tree on which G acts such that the G-vertex stabilizers are trivial. By Theorem 7.1, G has property P.

Corollary 7.3. If $G \underset{i \in I}{\pi^*} (G_i; A_{jk} = A_{kj})$ is a tree product such that G_i has property P and A_{jk} is cyclic, then G has property P.

Proof. There is a tree on which G acts such that the G-vertex stabilizers are the conjugates of G_i and have property P, and the G-edge stabilizers are the conjugates of A_{jk} and are cyclic. Therefore by Theorem 7.1, G has property P.

Corollary 7.4. If $G = \langle H, t_i | \text{rel } H,\ t_i A_i t_i^{-1} = B_i \rangle$ is an *HNN* group such that H has property P and A_i is cyclic, then G has property P.

Proof. There is a tree on which G acts such that G is transitive on the set of vertices, and the G-vertex stabilizers are the conjugates of H and have property P, and the G-edge stabilizers are the conjugates of A_i and are infinite cyclic. Therefore by Theorem 7.1, G has property P.

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