ITERATIVE CALCULATION OF THE SECOND DERIVATIVE OF THE GRAVITY FIELD USING RICHARDSON'S FORMULA

E. M. Abdelrahman*

Geophysics Department
Eötvös Loránd University
Kun Béla tér 2, H-1083, Hungary

A. I. Bayoumi and H. M. El-Araby

Geophysics Department
Faculty of Science
Cairo University, Giza, Egypt

الخلاصـــة :

في هذا البحث تمَّ تعديل علاقة (ريشاردسون) لصورة جديدة حتى يمكن الحصول على نتائج أكثر دقة في حساب معاملات المشتقة الثانية للمجال التثاقلي. وقد توصل الباحثون إلى مجموعتين جديدتين من المعاملات يمكن استخدامها في حساب المشتقة الثانية للمجال التثاقلي للأرض. واسُتخدمت طريقة التحليل الترددي للحكم على دقة مجموعات المعاملات المشتقة باستخدام علاقة (ريشاردسون) الجديدة ومفكوك (تايلور). كذلك تمَّ حساب معامل المضاهاة بين تردُّدِ كل مجموعةِ معاملاتٍ وتردُّدِ المشتقة الثانية النظري لاستكمال متطلبات المقارنة من جميع الأوجه. كما تمَّ حساب معامل المضاهاة بين عديد من مجموعاتٍ من المعاملاتِ التي سبق أن اشتُقَّت بواسطة جمعٍ كبير من العلماء وتردد المشتقة الثانية النظرى.

ووُجد في جميع الحالات أن مجموعة المعاملات التي اشتُقَّت بالطريقة الجديدة تفوق جميعَ مجموعاتِ معاملاتِ المشتقةِ الثانية للمجال التثاقلي التي سبق أن اشتُقَّت بواسطة كثيرٍ من العلماء.

ABSTRACT

An improved version of Richardson's formula has been introduced in a new iterative form to increase the convergence of the Taylor series, which is the fundamental basis for calculation of the second derivative of the potential field. Two sets of filter coefficients have been developed. These coefficients have good responses in the frequency domain when compared with the theoretical amplitude response of the second derivative operator. A comparative study of the derived sets along with other coefficient sets is also presented.

* Address for correspondence:
Geophysics Department
Faculty of Science
Cairo University, Giza, Egypt

ITERATIVE CALCULATION OF THE SECOND DERIVATIVE OF THE GRAVITY FIELD USING RICHARDSON'S FORMULA

INTRODUCTION

Richardson's formula [1] was used for obtaining an improved approximation of the second vertical derivative of the gravity field from two known approximations of the field [2]. However, both articles [1, 2] used this formula in a non-iterative form which in turn makes Taylor series converge very slowly when using more gravity values, thus rendering the problem more difficult when calculating large numbers of data sets of second vertical derivative operators. In spite of this, no rigorous calculations are required in deriving second derivative coefficient sets when Richardson's formula is used [2]. This illustrates the desirability of the application of Richardson's formula compared to other approaches [3–12], which use tedious techniques in computing second derivative coefficient sets.

The purpose of this paper is to develop, for the calculation of the second derivative, weight coefficient sets which yield the best results. We have found suitable weight sets by making use of Richardson's formula, which is presented here in an iterative form to increase the convergence of Taylor series very rapidly. Two new coefficient sets have been developed by this technique. We also present a comparative study on the frequency responses of the derived sets, and those of the existing coefficient sets.

THE ITERATIVE FORM OF RICHARDSON'S FORMULA

Let us approximate the second vertical derivative of the gravity field g_{zz} by an $O(h^n)$ formula $g_{zz}(h)$, where $O(h^n)$ is the truncation error and is given as:

$$O(h^n) = g_{zz} - g_{zz}(h) = \tau(h), \tag{1}$$

where h is the grid spacing.

Suppose that we know in fact that the truncation error is of the form

$$\tau(h) = Ch^n + O(h^m), \text{ where } C \neq 0 \text{ and } m > n \tag{2}$$

so that

$$g_{zz} = g_{zz}(h) + Ch^n + O(h^m). \tag{3}$$

If we use the approximation formula with a particular h and also with a larger h (say, H) and we let q denote the ratio

$$q = H/h, \text{ so that } H = qh, \text{ where } q > 1 \tag{4}$$

then $\tau(H) = \tau(qh) = C(qh)^n + O((qh)^m)$ $= Cq^nh^n + O(h^m)$. [1]

So

$$g_{zz} = g_{zz}(H) + Cq^nh^n + O(h^m). \tag{5}$$

Subtracting (5) from $q^n \cdot (3)$ to eliminate the h^n term gives

$$q^ng_{zz} - g_{zz} = q^ng_{zz}(h) - g_{zz}(H) + O(\mathrm{h}^{\mathrm{m}}). \tag{6}$$

Finally, dividing by $q^n - 1$ gives Richardson's formula

$$g^1_{zz}(h) = \frac{q^ng_{zz}(h) - g_{zz}(H)}{q^n - 1} \tag{7}$$

which shows how to form a weighted sum of two $O(h^n)$ approximations to get a higher-order approximation.

Since $g_{zz}(h)$ is an $O(h^n)$ approximation of g_{zz} then $g^1_{zz}(h)$, is mth order, that is

$$g_{zz} - g^1_{zz}(h) = O(h^m). \tag{8}$$

In this case we can use (7) to get a still higher-order approximation [1, 2],

$$g^2_{zz}(h) = \frac{q^mg^1_{zz}(h) - g^1_{zz}(H)}{q^m - 1}, \tag{9}$$

that is

$$g_{zz} - g^2_{zz}(h) = O(h^o), \tag{10}$$

and so on.

Because q is held fixed in the above series (Equations (7) and (9)) and $0 > m > n$, the Taylor series will converge very slowly when using more gravity values, thus rendering the problem more difficult in calculations involving large amounts of data sets of second vertical operators. If, however, we put Richardson's formula in the following iterative form

$$g_{zz}^{R+1}(h) = \frac{q^n g_{zz}^R(h) - g_{zz}^R(H)}{q^n - 1}, \tag{11}$$

(where $g_{zz}(h)$ is an $O(h^n)$ approximation of the exact second vertical derivative g_{zz}, h, and H are two different step sizes ($H>h$), $q = H/h$, and $R = 0, 1, 2, 3, \ldots$, is the number of iterations), a more accurate weighted sum of two $O(h^n)$ approximations can be obtained, provided that both n and q are held fixed during the iteration process.

APPLICATIONS

Let the plane of observation be horizontal everywhere and the gravity anomaly $g(x,y,z)$ be continuous and infinitely differentiable at all the points of the free space $z = 0$. Let $g(x,y,z)$ satisfy Laplace's equation

$$g_{xx} + g_{yy} + g_{zz} = 0. \tag{12}$$

From (12), one can write

$$g_{zz} = -(g_{xx} + g_{yy}), \tag{13}$$

where g_{zz} is the exact second vertical derivative of the observed gravity anomaly data.

Let $P(x,y)$ be the point at which the second derivative is to be computed. Let us cover the whole area with a mesh of square grids of spacing h, with $P(x,y)$ as one of the grid nodes. Then using Laplace's equation and by making use of Taylor's formula, we get

$$g_{zz} = (4g(0) - 4\bar{g}(h))/h^2 + O(h^2), \tag{14}$$

where

$$g(0) = g(x,y)$$

$$\bar{g}(h) = (g(x+h,y) + g(x-h,y) + g(x,y+h) + g(x,y-h))/4.$$

Equation (14) can be written as

$$g_{zz} = g_{zz}^O + O(h^2),$$

or for practical use

$$g_{zz}^O(h) = (4g(0) - 4\bar{g}(h))/h^2 \tag{15}$$

where $g_{zz}^O(h)$ is an $O(h^2)$ approximation of the exact second derivative g_{zz} in Laplace's Equation (13). This means that $n = 2$.

Now, let us always use $q = H/h = 2$ to obtain optimum results, then Richardson's iterative formula will have the following form:

$$g_{zz}^{R+1}(h) = \frac{2^2 g_{zz}^R(h) - g_{zz}^R(H)}{2^2 - 1}. \tag{16}$$

For calculating the first improved approximation of the second vertical derivative function shown in (15) which is an $O(h^2)$ approximation of g_{zz}, we have only to know g_{zz}^O and $g_{zz}^O(H)$ from (15). Then (16) with $H = 2h$, gives

$$g_{zz}^1(h) = (15g(0) - 16\bar{g}(h) + \bar{g}(2h))/3h^2. \tag{17}$$

If we substitute (17) in (16) with $H = 2h$, also, then (16) gives the second iterative approximation

$$g_{zz}^2(h) = (225g(0) - 256\bar{g}(h) + 32\bar{g}(2h) - \bar{g}(4h))/36h^2. \tag{18}$$

Finally, the third improved approximation of the second derivative can be easily calculated by making use of equations (18) and (16) and is given by the following formula after simplification:

$$\begin{aligned} h^2(g_{zz}^3(h)) = &+7.81250000g(x,y) \\ &-9.48148148\bar{g}(h) \\ &+1.77777778\bar{g}(2h) \\ &-0.11111111\bar{g}(4h) \\ &+0.00231481\bar{g}(8h). \end{aligned} \tag{19}$$

The coefficient sets for calculating the second vertical derivative are presented in Table 1. Table 1 also shows the coefficient sets derived by Abdelrahman *et al.* method [2] using the non-iterative form of Richardson's formula, for the sake of comparison.

FREQUENCY ANALYSIS

The second derivative operation on gravity (or magnetic) data acts as a numerical filter. Therefore, frequency analysis makes it possible to judge the accuracy of a coefficient set by matching its amplitude response with the theoretical amplitude response. Let us denote the two-dimensional filter (amplitude) response function of the second derivative operation by $h_t(u,v)$. Following Swartz [3] and Agarwal and Lal

Table 1. Coefficients of the Gravity Value at the Center Point and the Ring Average Value of Gravity by Using Various Formulas. Correlation Factors Between the Calculated Amplitude Response of Each Set and the Theoretical Second Derivative Response are also Given.

Weight coefficients for radii (r)	Using both approaches		Using present iterative approach		Using non-iterative approach [2]	
	$g^{O}_{zz}(h)$ Eq. (15)	$g^{1}_{zz}(h)$ Eq. (17)	$g^{2}_{zz}(h)$ Eq. (18)	$g^{3}_{zz}(h)$ Eq. (19)	$g^{2}_{zz}(h)$ ($m = 4$)	$g^{3}_{zz}(h)$ ($o = 6$)
0	−4.000000	+5.000000	+6.250000	+7.812500	+5.250000	+5.312500
h	−4.000000	−5.333333	−7.111111	−9.481481	−5.688889	−5.779189
$2h$		+0.333333	+0.888888	+1.777778	+0.444444	+0.474074
$4h$			−0.027777	−0.111111	−0.005555	−0.007407
$8h$				+0.002314		+0.000022
Correlation factor	0.954537	0.968360	0.978044	0.983670	0.971134	0.971785

[12], $h_t(u,v)$ can be calculated from

$$h_t(u,v) = u^2 + v^2 \tag{20}$$

where $u/2\pi$ and $v/2\pi$ are the frequencies in cycles per unit of length in the x and y directions, respectively.

The transfer function in (20) differs strikingly from the usual ones encountered in linear system theory since it increases without limit as u and v increase. In our case, let the sampling rate be h data points per unit length; then the cutoff or Nyquist frequency will be given by

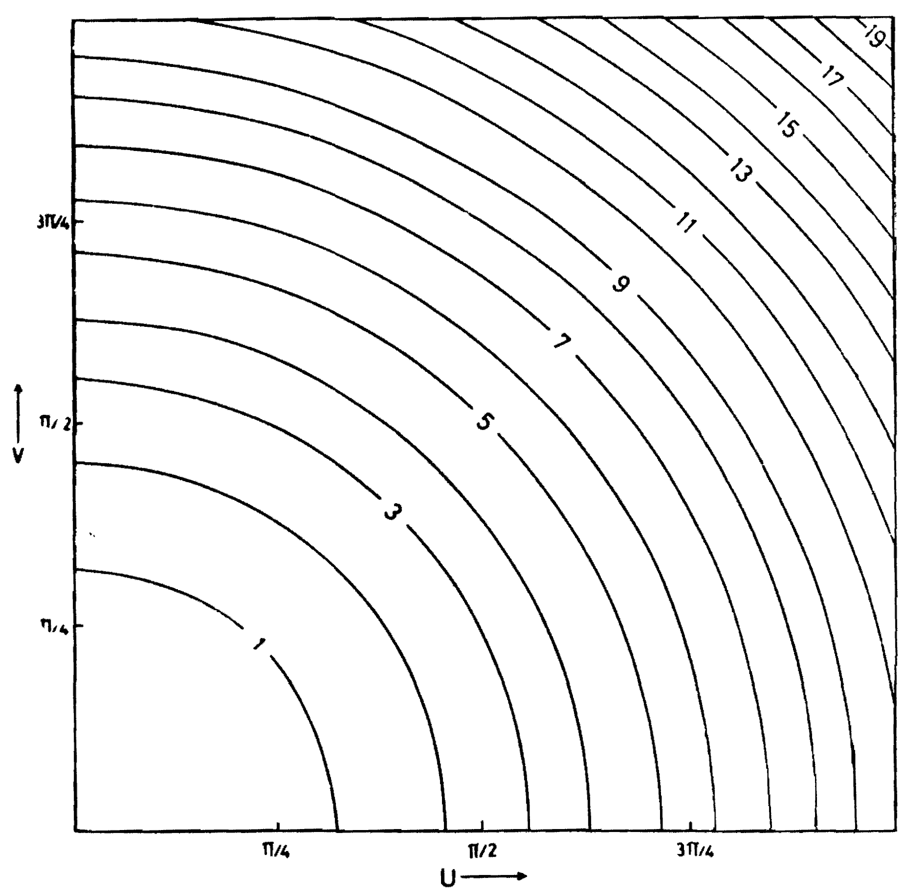

Figure 1. Theoretical Second Derivative Amplitude Response.

$$-\pi/h \leq u \leq \pi/h \text{ and } -\pi/h \leq v \leq \pi/h \quad (21)$$

Equation (20) represents a circle, thereby indicating the circular symmetric nature of the amplitude response of the second derivative (Figure 1).

All operators analyzed here can be presented as sets of radial weights (a_n) to be applied to average data values $\bar{g}(r_n)$ about a circle of radius r_n centered on the point at which filtered output is desired. The filtered output at a point (x,y) is then given by the sum of the products of a_n and $\bar{g}(r_n)$, a common factor of the weights is the reciprocal of the square of the distance of grid spacing. So the general formula for calculating the second derivatives is given by

$$\partial^2 g(x,y)/\partial z^2 = (1/h^2) \sum_{n=0}^{N} a_n \bar{g}(r_n) \quad (22)$$

where $N + 1$ is the number of averaging circles employed and the circle r_0 is the point (x,y) at which filtered output is desired.

Calculation of the frequency response function of Equation (22) requires first computing the Fourier transform of the average gravity value $\bar{g}(r)$ over circle of radius r. The operation may be transformed to cartesian coordinates by computing a weighted sum of several values in the surroundings of the reference point, observed on a regular grid. Then the average gravity can be expressed as follows

$$\bar{g}(r) = (1/m)\sum_k \sum_l g(x + K, y + L) \quad (23)$$

where m is the number of data points on the particular circle and the summation is performed over all k and L such that $k^2 + L^2 = r^2$. The filter response $h_r(u,v)$ of Equation (23) is given by Swartz [13] and Agarwal and Lal [12] as

$$h_r(u,v) = (1/m)\sum_k \sum_L e^{-i(uk + Lv)} \quad (24)$$

where $i = \sqrt{(-1)}$.

For calculating the two-dimensional amplitude responses of Equation (22), we have only to replace the average gravity value by the corresponding filter response given by Equation (24). All the derived coefficient sets are compared in this way and the corre-

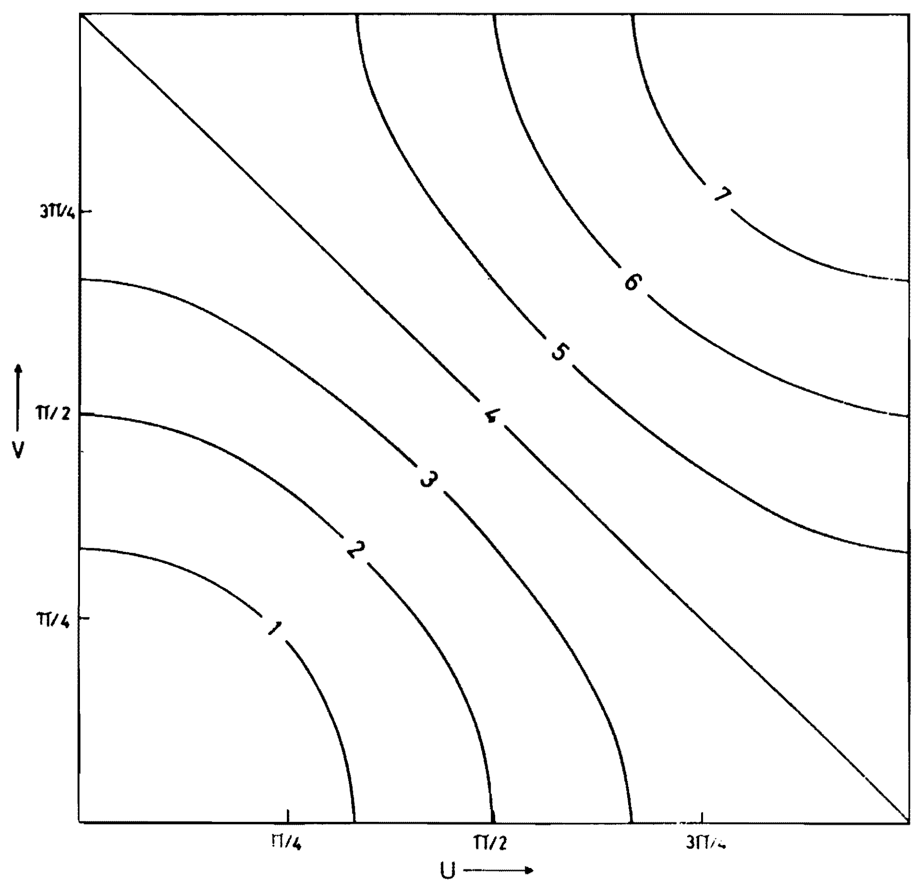

Figure 2. Amplitude Response of Equation (15).

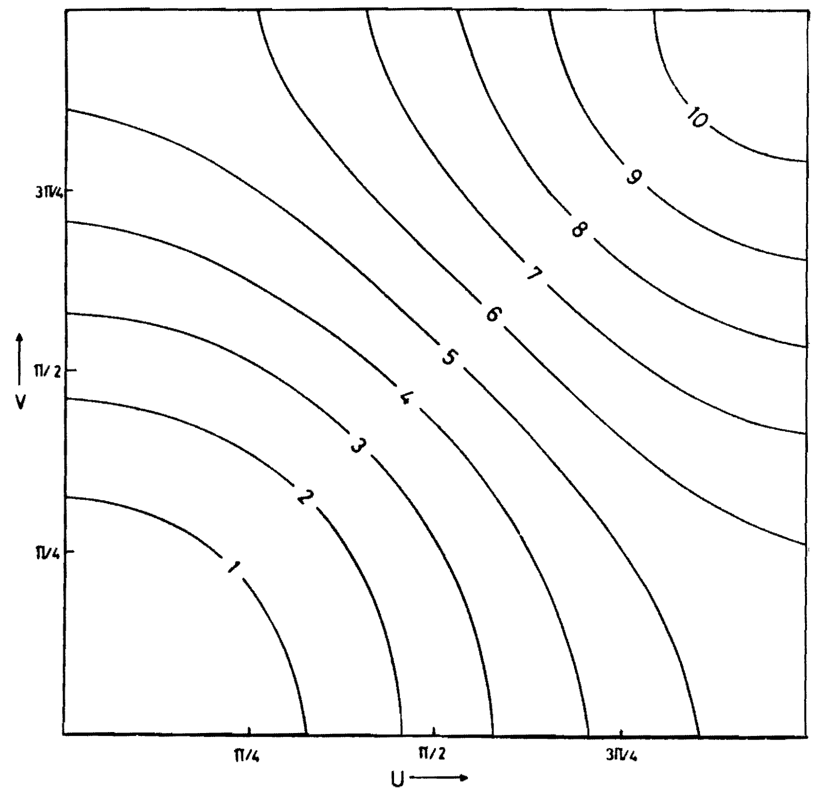

Figure 3. Amplitude Response of Equation (17).

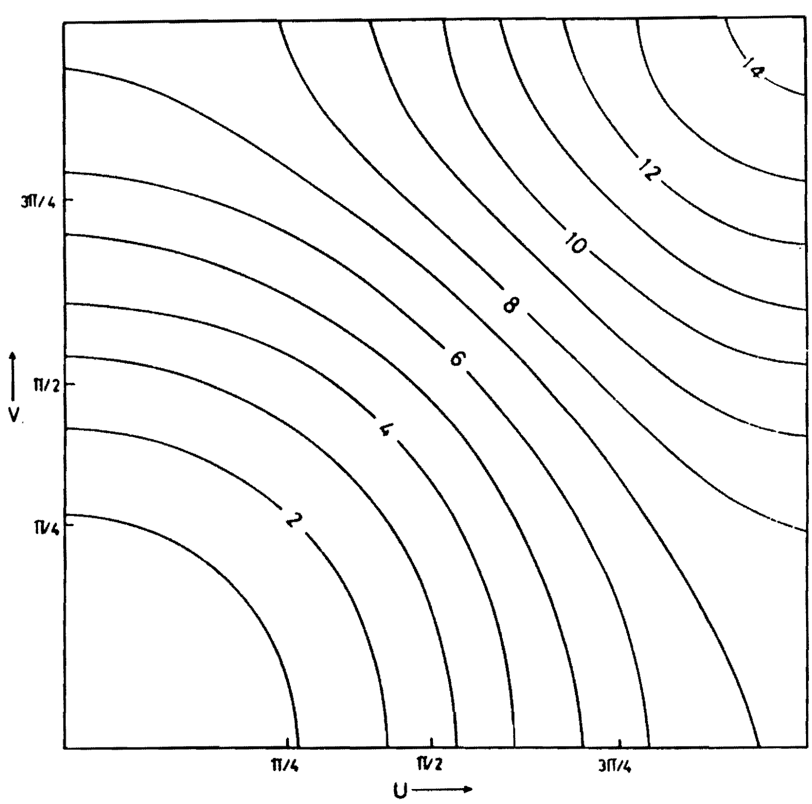

Figure 4. Amplitude Response of Equation (18).

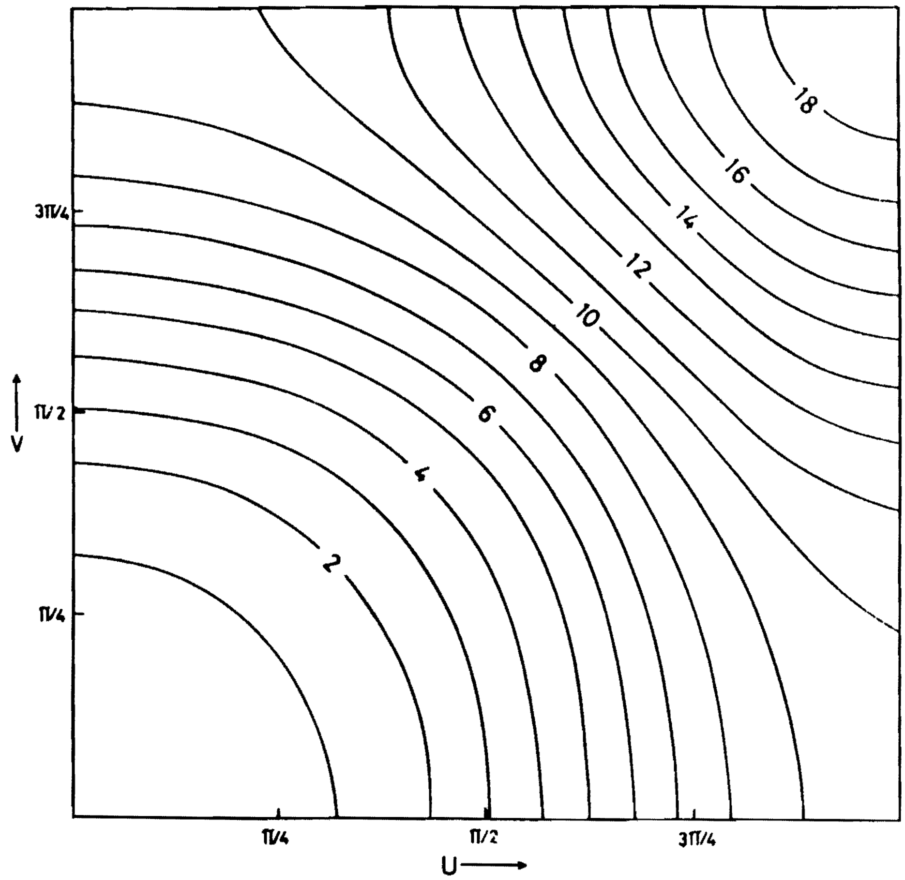

Figure 5. Amplitude Response of Equation (19).

sponding amplitude responses are shown in Figures 2 to 5. All responses (Figures 1–5) are computed at 169 points on a square grid with spacing $\pi/12$.

However, for the sake of complete comparison, the overall similarity between the calculated amplitude response of each set shown in Table 1 and the theoretical amplitude response of the second derivative operation computed at the 169 points on the same square grid with the same spacing as that of the calculated response has been determined. The simplest way to compare two maps is to compute the correlation factor between the mapped variables [14]. The equation used for this purpose is:

$$\text{Corr.} = S_1 - S_2/(S_3 \cdot S_4)^{\frac{1}{2}} \qquad (25)$$

where

$$S_1 = \sum_{u=1}^{M} \sum_{v=1}^{Q} h_t(u,v) \cdot h_r(u,v),$$

$$S_2 = \left[\sum_{u=1}^{M} \sum_{v=1}^{Q} h_t(u,v) \cdot \sum_{u=1}^{M} \sum_{v=1}^{Q} h_r(u,v) \right] \Big/ M \cdot Q$$

$$S_3 = \sum_{u=1}^{M} \sum_{v=1}^{Q} h_t^2(u,v) - \left(\sum_{u=1}^{M} \sum_{v=1}^{Q} h_t(u,v) \right)^2 \Big/ M \cdot Q$$

and

$$S_4 = \sum_{u=1}^{M} \sum_{v=1}^{Q} h_r^2(u,v) - \left(\sum_{u=1}^{M} \sum_{v=1}^{Q} h_r(u,v) \right)^2 \Big/ M \cdot Q.$$

The similarity between the calculated and the theoretical responses verified by the highest correlation, may generally be considered a criterion for determining the best coefficient set for calculating the second derivative of the gravity field. For each ring system, the numerical value of the correlation factor between the computed amplitude response and the theoretical response of the second derivative is also presented in Table 1. The advantage of this particular map comparison method is that it allows a more objective judgement on the accuracy of the coefficient sets.

We have also computed the correlation factor between each of the previously proposed set of weights and the theoretical response of the second derivative operation using the same number of data points with the same spacing mentioned above. Results in this particular case are listed in Table 2.

DISCUSSION OF THE RESULTS

It can be seen that the amplitude responses of the coefficient sets developed by the help of Taylor series and iterative Richardson's formula (Figures 4 and 5, Table 1) are close to the theoretical amplitude response of the second derivative operation. With the same data points and the same ring system (Table 1), when Richardson's formula in its iterative form is used, the Taylor series converges very rapidly compared to the same formula in its non-iterative form.

The figures in Tables 1 and 2 clearly indicate the superiority of the coefficient sets derived by use of Richardson's formula in its iterative or non-iterative form, to those previously proposed by many authors. The present coefficients show closer fit to the theoretical second derivative response than others, as indicated by their higher correlations.

It may be of interest here to indicate that the frequency analysis carried out by Abdelrahman *et al.* [2] using Mesko' method [15], which assumes that average gravity value over a circle is computed from an infinite number of points, established also that the Richardson's non-iterative approximation coefficient sets (Table 1) estimate the second derivative more accurately than the approximation sets of Rao *et al.* [11] (in total 60 sets of coefficients). This indicates that our iterative approximation coefficient sets estimate the second derivative more accurately than the approximation sets of Rao *et al.*

It is clear from Table 2 that the application of the filters proposed by other authors to data of high accuracy would reject valuable geological information because they have very poor negative correlations.

CONCLUSION

The results obtained here are of interest in the field of geophysics, although further detailed examination is required. Our aim is mainly to find a numerical approach for determining the second derivative coef-

Table 2. Numerical Values of the Correlation Factors Between the Amplitude Responses of the Previously Proposed Set of Weights and the Theoretical Amplitude Response of the Second Derivative.

Formula	Source	Correlation factor
	Center point and circles with radii 1, and $\sqrt{2}$	
1	Reference [4], Equation (10)	0.921243
2	Reference [4], Equation (13)	0.923949
3	Reference [12], Equation (25)	0.923949
	Center point and circles with radii 1, and $\sqrt{5}$	
4	Reference [5], Equation (14)	−0.316433
	Center point and circles with radii 1, $\sqrt{2}$, and 2	
5	Reference [4], Equation (15)	0.881871
	Center point and circles with radii 1, $\sqrt{2}$, and $\sqrt{5}$	
6	Reference [5], Equation (13)	0.299000
7	Reference [5], Equation (15)	−0.300000
8	Reference [6],	0.931354
9	Reference [7], Equation (16)	0.863104
10	Reference [9], equation (25)	0.846400
	Center point and circles with radii 1, $\sqrt{2}$, 2, and $\sqrt{5}$	
11	Reference [9], Equation (25)	0.107200
	Center point and circles with radii 1, $\sqrt{2}$, 2, $\sqrt{5}$, and $\sqrt{8}$	
12	Reference [12], Equation (26)	0.939292
	Center point and circles with other different radii	
13	Reference [3], Equation (27)	−0.148918
14	Reference [10]	0.854560
15	Reference [12], Equation (27)	0.956842
16	Reference [8]	0.948233

ficient set which uses the least number of rings for obtaining average gravity values and at the same time yield better results in the frequency domain without using extensive computations. The new iterative Richardson's formula has been used to increase the convergence of the Taylor series which is the main basis of the calculation of the second derivative. It makes the problem more easier from the points of view of calculating many sets of $-(g_{xx}+g_{yy})$ operators and the handling of large amounts of data.

ACKNOWLEDGMENT

The authors would like to express their thanks to the reviewers and the editors for their keen interest, valuable comments on the manuscript, and for their improvements to this work. The first author would like to thank Professor Academic A. Meskó, Head and Chairman of Geophysics Department, Eötvös Loránd University, for his keen interest and encouragement.

REFERENCES

[1] M. J. Maron, *Numerical Analysis. A Practical Approach.* New York: Macmillan, 1982, p. 273.

[2] E. M. Abdelrahman, A. I. Bayoumi and H. M. El-Araby, "A Comparative Study of the Frequency Responses of Some Existing Three-Dimensional Second Derivative Coefficient Sets with New Improvements", *Arab Gulf J. Scient. Res., Math. Phys. Sci.*, **A7** **(1)** (1989), p. 35.

[3] L. J. Peters, 1949, "The Direct Approach to Magnetic Interpretation and its Practical Application", *Geophysics*, **14** (1949), p. 290.

[4] R. G. Henderson and I. Zietz, 1949, "The Computation of Second Vertical Derivatives of Geomagnetic Fields": *Geophysics*, **14** (1949), p. 508.

[5] T. A. Elkins, 1951, "The Second Derivative Method of Gravity Interpretation". *Geophysics*, **16** (1951), p. 29.

[6] V. Baranov, "Calcul du gradient vertical du champ de gravite on du champ magnetique. Mesure a la surface du sol", *Geophys. Prospecting*, **1** (1953), p. 171-191.

[7] O. Rosenbach, 1953, "A Contribution to the Computation of the Second Derivative from Gravity Data", *Geophysics*, **18** (1953), p. 894.

[8] R. G. Henderson, "A Comprehensive System of Automatic Computation in Magnetic and Gravity Interpretation", *Geophysics*, **25** (1960), p. 569.

[9] M. K. Paul, 1961, "On Computation of Second Derivatives from Gravity Data", *Pure and Applied Geophys.*, **48** (1961), p. 7.

[10] E. K. Darby and X. X. Davies, 1967, "The Analysis and Design of Two-Dimensional Filters for Two-Dimensional Data", *Geophy. Prospecting*, **15** (1967), p. 383–406.

[11] B. S. R. Rao, R. I. V. Murthy, and S. J. Reddy, 1970, "A Simple Formula for the Second Derivative Method of Interpretation", *Pure and Appl. Geophys.*, **80(3)** (1970), p. 5.

[12] B. N. P. Agarwal and T. Lal, "Application of Rational Approximation in Calculation of the Second Derivative of Gravity Field", *Geophysics*, **36** (1971), p. 571.

[13] C. A. Swartz, "Some Geometrical Properties of Regional Maps", *Geophysics*, **19** (1954), p. 46.

[14] J. C. Davis, *Statistics and Data Analysis in Geology.* New York: Wiley, 1973.

[15] A. Meskó, 1965, "Some Notes Concerning the Frequency Analysis for Gravity Interpretation", *Geophys. Prospect.*, **13** (1965), p. 475.

Paper Received 29 July 1989; Revised 10 February 1990.