MINIMIZING THE STEERING ERROR OF AN ACKERMANN LINKAGE

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الخلاصـــة :

يتناول هذا البحث دراسة تركيب (تجميع) حركة الآليات رباعية القضبان، بالإضافة إلى دراسة إحدى طرق التفضيل المستخدمة في تقليل أخطاء التوجيه في آلية (اكرمان) للتوجيه.

ويصف هذا المقال أيضا طريقة التركيب السابق ذكرها والتي ستُستخدم من أجل إعداد صحيفة بيانات للاستفادة منها عند اللزوم. وفي النهاية استعملت إحدى طرق المفاضلة غير الخطية لتقليل مجموع مربعات أخطاء التوجيه لعدة زوايا مختلفة للآلية السابق ذكرها.

ويشتمل المقال أمثلة عددية لتوضيح مدى فائدة الطريقة المقترحة في تصميم آليات التوجيه.

ABSTRACT

In this paper the kinematic synthesis of a four bar linkage has been used together with an optimization method to minimize the steering error of an Ackermann steering linkage. A method which is based on the kinematic synthesis of a four bar linkage and which is used for the purpose of producing data sheets has been described. Finally, a nonlinear optimization method has been used to minimize the sum of the squares of the steering errors for various steering angles. Numerical examples are presented to illustrate the usefulness of the method proposed for the design of steering linkages.

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1. INTRODUCTION

One of the activities of an automotive engineer is the design of steering linkages that conform to the Ackermann principle. Donkin [1] provided some information about the design of a simple steering linkage in the form of a few data sheets. Donkin's work was of limited use because the ranges of the parameters he used were too restrictive. Guntur [2] presented a complete set of data sheets; these data sheets were useful in designing the steering linkages to conform to the Ackermann principle. Guntur [2] treated the problem of designing a steering linkage as a problem in the kinematic synthesis of a four bar linkage and used Freudenstein's equations [3].

Many solutions for the design of the steering linkages may be obtained using the data sheets in reference [2]. However, one has to determine which of these solutions is the most suitable for a particular application. The objective of the present paper is to present a least-square error minimization approach that enables a designer to choose a solution that minimizes the sum of the squares of the steering errors over a given range of the steering angles and for a given set of other design constraints. Thus the design of a steering linkage is posed not only as a problem of the kinematic synthesis of a four bar linkage but also as a problem of optimization.

2. THE BASIC DESIGN CONSIDERATIONS FOR A STEERING LINKAGE

There are two configurations of a steering linkage. In the configuration shown in Figure 1, the tie-rod is behind the front axle of the vehicle. Figure 2 shows the second configuration in which the tie-rod is in front of the front axle of the vehicle.

The maximum angle through which the axis of a front wheel is moved from its straight-ahead position

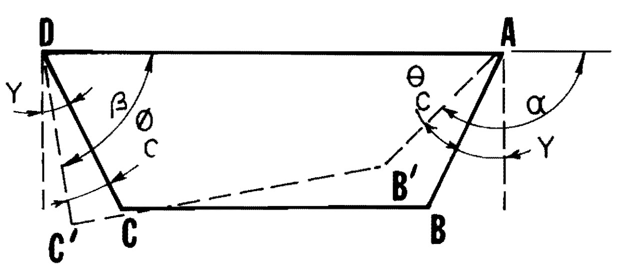

Figure 1. The Ackermann Steering Linkage with the Tie-Rod Behind Front Axle, AD (AD = L, BC = M, AB = CD = M).

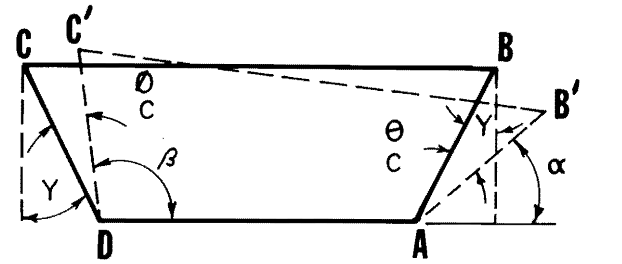

Figure 2. The Ackermann Steering Linkage with the Tie-Rod in Front of The Front Axle, AD (AD = L, BC = M, AB = CD = M).

in either of the two directions is called its steering angle at the full-lock. A designer may select the steering angle at the full-lock for the outer wheel to attain the desired minimum turning radius.

In Figure 3 are shown the positions of the two front wheels for the Ackermann steering. The condition that has to be satisfied by the steering angles of the inner and the outer wheels for the Ackermann steering may be expressed as follows [4]:

$$\cot\phi_c - \cot\theta_c = \frac{L}{W} \tag{1}$$

where,

ϕ_c, is the correct steering angle of the outer wheel,
θ_c, is the correct steering angle of the inner wheel,
L, is the track at the front, and
W, is the wheel base of the vehicle.

If a small turning radius is necessary, the steering angle of the outer wheel should conform to the Ackermann principle at full lock; otherwise, the vehicle may be unable to attain the desired turning radius. In other words the turning radius, r should be given by the following equation:

$$r = W\sqrt{(1+\cot^2\phi_c)}, \tag{2}$$

where r is the turning radius (or AG in Figure 3).

One may rewrite Equation (2) as follows

$$\phi_c = \cot^{-1}\left\{\frac{\sqrt{(r^2 - W^2)}}{W}\right\} \tag{2a}$$

Using Equations (1) and (2a) one may derive the following Equation for the correct steering angle of the inner wheel.

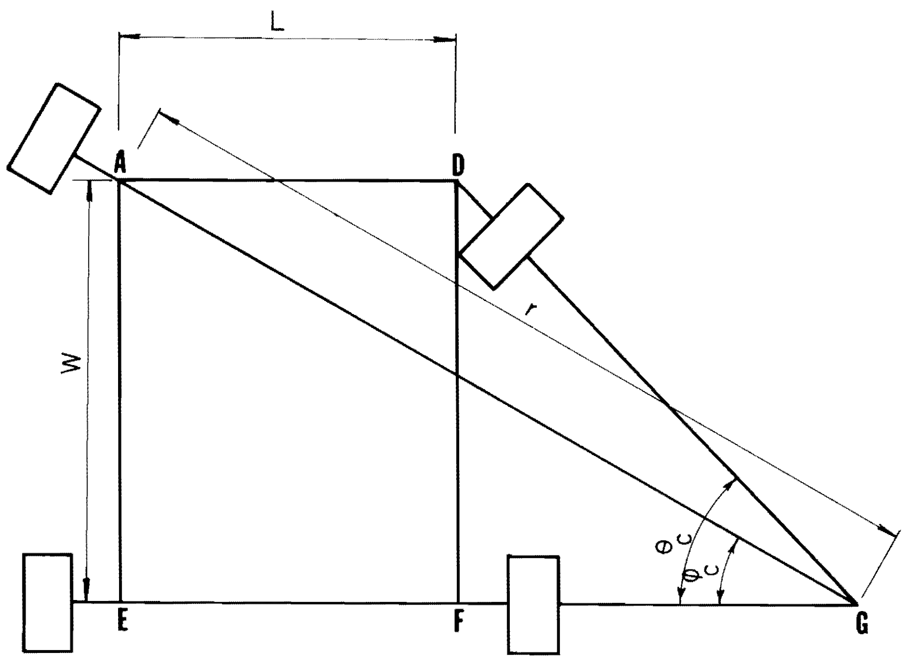

Figure 3. The Schematic Diagram of the Wheels of a Vehicle for the Ackermann Steering (AD = L, AE = W, AG = r).

$$\theta_c = \cot^{-1}\left\{\left(\frac{\sqrt{(r^2 - W^2)}}{W}\right) - \frac{L}{W}\right\} \tag{3}$$

If the minimum turning radius is not an important design consideration, the steering linkage may be designed to make the Ackermann steering angle equal to two-thirds of the angle at full lock of the inner wheel. Then the steering will be most accurate for the small steering angles that are most frequently used. In this case one uses the following equation to find the correct steering angle of the inner wheel.

$$\theta_c = \tfrac{2}{3}\theta_{max} \tag{4}$$

where,

θ_{max}, is the maximum steering angle at full lock of the inner wheel.

Using Equations (1) and (4), one may derive the following equation for the correcting steering angle of the outer wheel:

$$\phi_c = \cot^{-1}\left\{\cot\tfrac{2}{3}\theta_{max} + \frac{L}{W}\right\} \tag{5}$$

3. THE KINEMATIC SYNTHESIS OF A STEERING LINKAGE

There are two configurations of the steering linkage that have to be considered:

1. Tie-rod behind the front axle; and
2. Tie-rod in front of the front axle.

3.1. Tie-Rod Behind the Front Axle

This configuration of the linkage for the straight-ahead position is given by the full lines in Figure 1. For this case the following Equation may be derived.

$$M = L - 2m\sin\gamma, \tag{6}$$

where,

M is the length of the tie-rod,
m is the length of the steering arm, and
γ is the angle of the steering arm.

If the inner wheel is given a steering input equal to the correct steering angle of the inner wheel, the steering arms and the tie-rod are given by the dashed lines in Figure 1. The angles α and β in Figure 1 are given by the following equations.

$$\alpha = 90 + \gamma + \theta_c \tag{7}$$

$$\beta = 90 - \gamma + \phi_c. \tag{8}$$

In this case Freudenstein's equation (see Appendix 1) may be written as follows.

$$cos(\alpha - \beta) = \frac{L}{m}\{\cos\alpha - \cos\beta\} + A \tag{9}$$

where,

$$A = \frac{\{L^2 + 2m^2 - (L - 2m\sin\gamma)^2\}}{(2m^2)}$$

Equation (9) may be rewritten as follows.

$$\frac{m}{L} = \frac{\{2\sin\gamma - (\sin(\gamma + \theta_c) + \sin(\gamma - \phi_c))\}}{(\cos(\theta_c + 2\gamma - \phi_c) - \cos 2\gamma)} \quad (10)$$

3.2. Tie-Rod in Front of the Front Axle

When the tie-rod is forward of the front axle, the configuration of the linkage is shown by the full lines in Figure 2 for the straight-ahead position. For this case, the following equation may easily be derived:

$$M = L + 2\,m\sin\gamma. \quad (11)$$

If the inner wheel is given a steering input equal to the correct steering angle of the inner wheel, the steering arms and the tie-rod are given by the dashed lines in Figure 2. The angles α and β in Figure 2 are given by the following equations:

$$\alpha = 90 - \gamma - \theta_c. \quad (12)$$

$$\beta = 90 + \gamma - \phi_c. \quad (13)$$

Using the procedure presented in Appendix 1, one may write Freudenstein's equation as follows:

$$\cos(\alpha - \beta) = \frac{L}{m}\{\cos\alpha - \cos\beta\} + A \quad (14)$$

or

$$\frac{m}{L} = \frac{\{-2\sin\gamma + (\sin(\gamma + \theta_c) + \sin(\gamma - \phi_c))\}}{(\cos(\theta_c + 2\gamma - \phi_c) - \cos 2\gamma)}. \quad (15)$$

Equations (10) and (15) may be used to produce the data sheets given in reference [2].

4. DETERMINATION OF THE STEERING ERROR

The steering error occurs if the instant centers of rotation (in the plane of the road surface) of all the four wheels do not coincide with each other. In other words when the vehicle is moving along a curved path if G in Figure 3 is not the instant center of rotation of all the wheels there will be a steering error.

The steering linkage designed by using Equation (10) or (15) conforms to the Ackermann principle only for two steering angles. The steering error is zero when the steering angle of the inner wheel is either zero or equal to the correct steering angle. For other steering angles of the inner wheel the steering error is given by the following equation.

$$\varepsilon_i = \left(\cot\phi_i - \cot\theta_i - \frac{L}{W}\right) \quad (16)$$

where,

ε_i is the steering error for a given steering angle of the inner wheel, θ_i,
θ_i, is the steering angle of the inner wheel ($\theta_i = 1°, \ldots, \theta_n = \theta_{\max}$), and
ϕ_i, is the steering angle of the outer wheel for a given, θ_i.

It is customary to draw the steering error curve using the method described in reference [4]. The designer draws the steering error curve after designing the steering linkage to find out whether or not the steering linkage he designed gives a "reasonable" steering error for a given steering angle of the inner wheel.

It is necessary to establish a criterion to quantitatively analyze steering error produced by a given steering linkage. The following criterion based on sum of the squares of the steering errors is developed for this purpose. The total steering error is defined as the sum of the squares of the steering errors for various values of the steering angle of the inner wheel. For convenience, the steering angle of the inner wheel is varied in steps of 1 degree.

$$E^2 = \sum_{t=1}^{n} \varepsilon_t^2 \quad (17)$$

or

$$E^2 = \sum_{t=1}^{n}\left\{\cot\phi_t - \cot\theta_t - \frac{L}{W}\right\}^2 \quad (17a)$$

where,

E is the square root of the sum of the squares of the steering errors.

The steering angle for the outer wheel for a given steering angle of the inner wheel is determined by using the following method.

For any steering angle of the inner wheel the steering angle of the outer wheel may be obtained by using

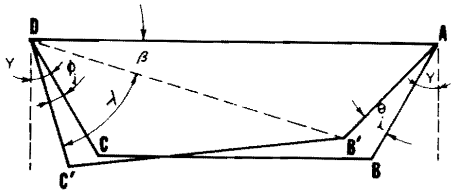

Figure 4. The Ackermann Steering Linkage when the Input Crank, AB *is Moved Through an Angle* θ_i (B'D = N).

the method described by Shigley [5]. For example, when the tie-rod is behind the front axle we may obtain the following equations (see Figure 4).

Considering the triangle DAB' and applying the cosine-law one may obtain the following equation.

$$N^2 = L^2 + m^2 - 2\,Lm \cos(90-\gamma-\theta_i) \quad (18)$$

where,

N is the side $B'D$ of the triangle DAB' in Figure 4.

Equation (18) may be rewritten as follows:

$$\frac{N}{L} = \sqrt{\left\{\frac{m^2}{L^2} + 1 - 2\,\frac{m}{L}\sin(\gamma + \theta_t)\right\}}. \quad (18a)$$

Considering the triangle ADB' in Figure 4 and using the cosine-law, one may obtain the following equation.

$$\beta = \cos^{-1}\left\{\frac{(N^2 + L^2 - m^2)}{(2NL)}\right\} \quad (19)$$

where,

β is the angle ADB' in Figure 4

Considering the triangle $B'DC'$ in Figure 4 and using the cosine-law one may obtain an expression for λ as follows.

$$\lambda = \cos^{-1}\left\{\frac{(N^2 + m^2 - M^2)}{(2Nm)}\right\} \quad (20)$$

where,

λ is the angle $B'DC'$ in Figure 4.

From Figure 4 we have,

$$\phi_i = \beta + \gamma + \lambda - 90. \quad (21)$$

Now the kinematic synthesis of the steering linkage may be used in conjunction with the optimization techniques [6,7] and a steering linkage may be designed to minimize the sum of the squares of the steering error.

5. DESCRIPTION OF THE OPTIMIZATION METHOD USED

The complex method of Box [8] is employed to minimize the sum of the squares of the steering errors.

The modified simplex or complex method is used to minimize a multivariable nonlinear function, subject to a given set of nonlinear inequality constraints. The general problem of finding the maximum of a function may be stated as follows.

Maximize,

$$F(Y_1, Y_2, Y_3, \ldots, Y_n)$$

Subject to,

$G_k \leq Y_k \leq H_k$

where,

$k = 1, 2, \ldots, M$ and $M \geq N$.

The independent variables are $Y_1, Y_2, \ldots, Y_N$ and the dependent variables are $Y_{|N+1|}, \ldots, Y_N$. The function to be maximized is dependent on the values of the independent variables. However, both the dependent and the independent variables are subjected to a given set of constraints. The dependent variables are linear or nonlinear functions of the independent variables.

The maximum value of the function is found by using an initial set of points scattered throughout the feasible region or the region in which the constraints are not violated. It may be noted here that the function has different values at different points in an N-dimensional space. Each point in this space corresponds to a certain value of the function and has N coordinates given by the values of the N independent variables.

An original "complex" of $K > M + 1$ points is generated consisting of a feasible starting point and $K - 1$ additional points are generated from random numbers and constraints for each of the independent variables using the following equations.

$$Y_{(i,j)} = G_j + r_{(i,j)}(H_i - G_j) \quad (22)$$

where,

$i = 1, 2, ..., N$,
$j = 1, 2, ..., K - 1$, and
$r_{(i,j)}$ are the random numbers between 0 and 1.

The constraints on both dependent and independent variables must be satisfied by the K points selected. When any constraint involving a depentent variable is not satisfied the corresponding point is moved one half of the distance to the centroid of the remaining points. The coordinates of the new point are given by the following relation.

$$Y_{(i,j)_{\text{new}}} = \frac{(Y_{(i,j)(\text{old})} + \bar{Y}_{(i,c)})}{2} \tag{23}$$

where,

i takes the values from $1, ..., N$, and
j indicates the point where the jth constraint is violated.

$Y_{(i,j)_{\text{old}}}$ are the coordinates of the point at which the jth constraint is not satisfied. $Y_{(i,c)}$ are the coordinates of the centroid of the remaining $K - 1$ points where constraints are satisfied.

The function $F(Y_1, Y_2, ..., Y_N)$ is calculated at each point. The point having the lowest value is replaced by a point which is α times as far from the centroid of the remaining points as the distance of the point being replaced. This new point is on the line joining the centroid of the remaining points and the point that is being replaced.

$$Y_{(i,j)_{\text{new}}} = \alpha(\bar{Y}_{(i,c)} - \bar{Y}_{(i,j)_{\text{old}}}) + \bar{Y}_{(i,c)} \tag{24}$$

The recommended value of α is 1.3 [6]. If a point repeatedly gives the lowest value of the function on consecutive trials, it is moved one half of the distance to the centroid of the remaining points.

The new point is checked against the constraints and adjusted if any constraint is violated. Convergence is assumed if the values of the function at all the points in the final "complex" do not differ from each other by a certain specified limit.

6. MINIMIZATION OF THE STEERING ERROR

The optimization technique outlined in the previous section is used to determine the maximum value of $-E^2$. In other words, the sum of the squares of the steering errors is minimized by using the complex method of the previous section. The constraints are as follows.

$$G_1 \leqslant \frac{L}{W} \leqslant H_1 \tag{25}$$

$$G_2 \leqslant \gamma \leqslant H_2 \tag{26}$$

$$G_3 \leqslant \frac{m}{L} \leqslant H_3 \tag{27}$$

where,

G_1, G_2, G_3 are the lower limits of the three constraints, and
H_1, H_2, H_3 are the higher limits of the three constraints.

The following two numerical examples are included to illustrate the use of the method. In these examples it is assumed that the tie-rod is behind the front axle.

The wheel base of the car in these examples is 3 m. In the first example the minimum turning radius is varied from 8 m to 12.5 m in steps of 1.5 m. The lower and upper limits of the inequality constraints are given by the following equations.

$$G_1 = 0.55$$
$$G_2 = 10°$$
$$G_3 = 0.1$$
$$H_1 = 0.75$$
$$H_2 = 50°$$
$$H_3 = 0.2$$

The results for this example are given in Table 1.

In the second example, the maximum steering angle of the inner wheel is varied from 25 to 45 degrees. The lower and upper limits in the inequality constraints are the same as those in the first example.

Table 1. Optimal Steering Linkage For Minimum Turning Radius.

Minimum Turning Radius	$\frac{m}{L}$	$\frac{L}{W}$	γ in degrees
8.0 m	0.116	0.584	23.1
9.5 m	0.131	0.588	24.4
11.0 m	0.108	0.631	27.0
12.5 m	0.110	0.634	27.8

Table 2. Optimal Steering Linkage For θ_{max}.

θ_{max} in degrees	$\frac{m}{L}$	$\frac{L}{W}$	γ in degrees
25	0.109	0.628	27.8
30	0.108	0.633	27.2
35	0.103	0.605	25.6
40	0.140	0.592	23.8
45	0.100	0.550	22.4

The results for the second example are given in Table 2.

For obtaining the results in Tables 1 and 2, a computer program given in reference [6] has been modified as follows. The subroutine for determining the objective function has been rewritten to yield the sum of the squares of the steering errors as the objective function. All the steps to determine the left hand side of Equation (17*a*) have been included in this subroutine. The subroutine to determine the constraints has been modified to take into account the inequality constraints given by the inequalities (25), (26), and (27).

The results in Table 1 indicate that as the minimum turning radius is decreased the values of γ and L/W for the optimal configuration of the steering linkage decrease.

The results in Table 2 indicate that as the maximum steering angle of the inner wheel increases both γ and L/W for the optimal configuration of the steering linkage decrease.

The results in Tables 1 and 2 indicate that the method in this paper is suitable to the problem of minimization of the steering error.

Although the results are presented only for two numerical examples in this paper the methodology described in this paper can easily be used to design the steering linkages of buses or trucks where the track width, L is fixed by road allowance but the wheelbase, W, may vary. In this case the inequality constraint (25) may be changed and suitable values for higher and lower limits may be used.

The first example given in this paper is particularly useful in designing steering linkages for subcompact vehicles. In this case one will try to design a steering linkage to obtain the minimum turning radius. The results in Table 1 also show the advantage of a subcompact vehicle using a short wheelbase.

The procedure presented in this paper can also be employed in case the tie-rod is forward of the front axle. The subroutine corresponding to the objective function has to be modified. The procedure that is required to write this subroutine is presented in the paper.

In cases where L/W is not a variable only two constraints will be used. The two constraints are given by the inequalities (26) and (27). This is a special case of the problem presented in this paper.

7. CONCLUSIONS

The method proposed in this paper which combines the kinematic synthesis and the minimization procedure for the steering error proves to be very useful in determining the optimal configuration of the steering linkage either for the minimum turning radius or for the maximum steering angle of the inner wheel.

The results indicate that in order to minimize the steering error both L/W and γ should be decreased if the minimum turning radius is to be decreased.

The results also indicate that in order to minimize the steering error both L/W and γ should be decreased if the maximum steering angle of the inner wheel is to be increased.

8. ACKNOWLEDGEMENTS

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APPENDIX I. The design of a four bar linkage for the coordinated motions of the crank.

The Derivation of Freudenstein's Equation

A relationship between the crank angles and the lengths of the various links in the four bar linkage can be obtained if the links are regarded as four vectors. Since the links form a closed quadrilateral, the sum of the components of the vectors along any line in the plane of the vectors should be equal to zero.

Thus equating the sum of the components of the vectors along the x-axis in Figure 5, we obtain the following relation.

$$b \cos \mu - c \cos \beta + d + a \cos \alpha = 0 \quad \text{(A.1)}$$

Equating the sum of the components of the vectors along the y-axis in Figure 5, we obtain another relation as follows.

$$b \sin \mu - c \sin \beta + a \sin \alpha = 0 \quad \text{(A.2)}$$

By squaring each term in Equation (A.1), one obtains the following equation.

$$b^2 \cos^2 \mu = (c \cos \beta - d - a \cos \alpha)^2. \quad \text{(A.3)}$$

By squaring each term in Equation (A.2), one obtains the following equation:

$$b^2 \sin^2 \mu = (c \sin \beta - a \sin \alpha)^2. \quad \text{(A.4)}$$

By adding Equations (A.3) and (A.4), we obtain the following equation:

$$b^2 = c^2 + d^2 + a^2 - 2dc \cos \beta - 2ac \cos \alpha \cos \beta - 2ca \sin \alpha \sin \beta + 2da \cos \alpha \quad \text{(A.5)}$$

By simplifying and rearranging the terms in Equation (A.5), we obtain the following relation.

$$R_1 \cos \alpha - R_2 \cos \beta + R_3 = \cos(\alpha - \beta) \quad \text{(A.6)}$$

where,

$$R_1 = \frac{d}{c}$$

$$R_2 = \frac{d}{a}$$

$$R_3 = \frac{(d^2 + a^2 - b^2 + c^2)}{(2ca)}.$$

Given any three coordinated motions of the cranks, R_1, R_2, and R_3 can be determined.

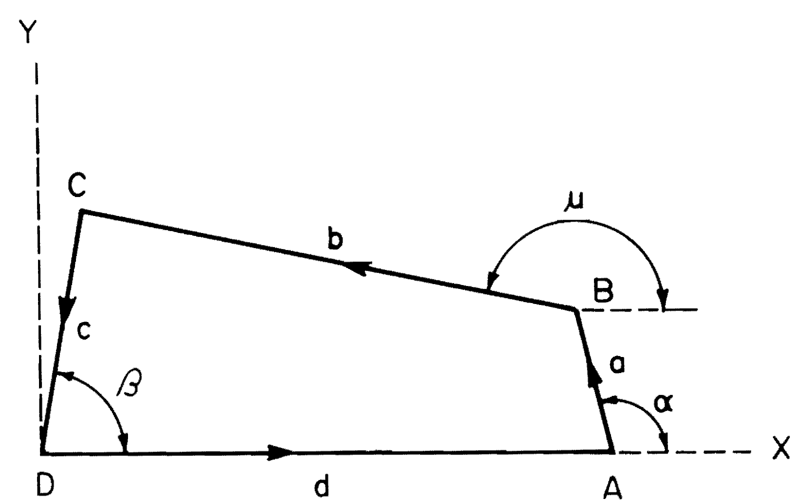

Figure 5. A Four-Bar Linkage in Which Links are Regarded as Vectors.