ANTISYMMETRIC BONDED CONTACT PROBLEM FOR AN ELASTIC LAYER

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الخلاصـــة :

يهتم هذا البحث بدراسة مشكلة الانفعال strain المستوي لصفيحة مرنة مربوطة بطبقة صلبة على سطحها السفلي حيث تقع هذه الصفيحة تحت تأثير عزم ثني مبذول عبر كتلة صلبة مستطيلة مربوطة الى سطح الصفيحة العلوي . وأدى التحليل للصفيحة غير القابلة للانضغاط الى ايجاد نظام مكوّن من معادلتين تكامليتين مفردتين من النوع الأول للاجهاد العمودي على السطح البيني والاجهاد القصي على قاعدة الكتلة الصلبة . وتـَـمَّ حل هاتين المعادلتين التكامليتين رقميا . وتم حساب توزيعات الاجهاد ومعاملات شدة الاجهاد لأشكال هندسية متعددة .

ABSTRACT

This paper is concerned with the plane strain problem of an elastic layer bonded to a rigid substrate along its entire lower surface. The layer is under the action of a bending moment applied through a rigid rectangular block bonded to its upper surface. The analysis for the incompressible layer leads to a system of two singular integral equations of the first kind for the interface normal and shear stresses at the base of the rigid block. These integral equations are solved numerically, and the stress distributions and stress intensity factors are calculated for various geometries.

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1. INTRODUCTION

The contact problem for an elastic layer has attracted considerable attention in the past because of its possible application to a variety of structures of practical interest (*e.g.*, foundations, pavements in roads, and rolling mills). The contact between the layer and the substrate was assumed to be either perfect adhesion or frictionless which represent two extremes regarding the conditions in the contact zone between two bodies. Most of the previous works considered the frictionless contact problems with variations (see, for example, [1, 2] for extensive references). Some examples considering perfect adhesion may be found in [3–10].

In the analysis of contact of two (or more) bodies, usual assumptions are that the two contacting bodies are elastic and isotropic, and that they are in a plane or in an axisymmetric state. Further simplifying assumptions are that the curvature of one of the bodies is so small in the contact region that it may be represented by a half space or a layer, and that the other body is so much stiffer than the first one that it may be treated as a rigid body. It seems that the symmetric problem for a layer has been studied extensively where the load is primarily tension or compression. The antisymmetric problem in which the load is a bending moment, on the other hand, has not been treated properly yet.

This paper considers a linearly elastic, isotropic, and incompressible layer perfectly bonded to a rigid substrate along its entire lower surface (Figure 1). A rigid block with a flat surface is bonded perfectly to the upper surface of the layer through which a bending moment is transmitted to the layer.

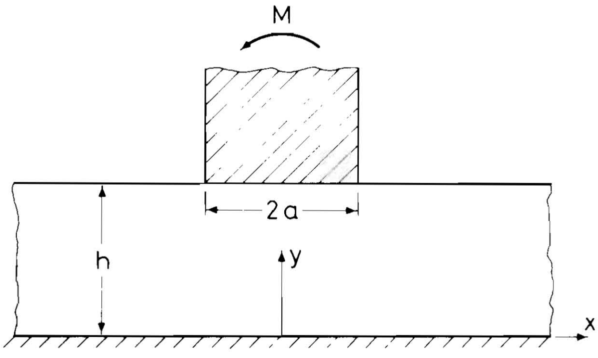

Figure 1. Antisymmetric Elastic Layer Problem.

2. FORMULATION OF THE PROBLEM

Consider the elastostatic plane strain problem for a layer shown in Figure 1. Material of the layer is linearly elastic, isotropic, and incompressible. The effect of gravitation is neglected. The layer of thickness h is perfectly bonded to a rigid substrate along its entire lower surface and to a rigid block of width $2a$ on its upper surface by means of which a bending moment M (per unit thickness in z-direction) is applied to the layer. Due to the antisymmetric nature of the problem, it is sufficient to consider the problem in $0 \leq x$ only. Under these circumstances, the governing equations of the plane elasticity must be solved subject to the following boundary conditions:

$$u(x,0) = 0, \quad v(x,0) = 0, \quad (0 \leq x < \infty), \tag{1a, b}$$

$$u(x,h) = 0, \quad v(x,h) = cx, \quad (0 \leq x < a),$$
$$\tau_{xy}(x,h) = 0, \quad \sigma_y(x,h) = 0, \quad (a < x < \infty), \tag{2a–d}$$

where u and v are the x and y components of the displacement vector and the constant c can be determined from the equilibrium condition

$$\int_{-a}^{a} \sigma_y(x,h)\, x\, \mathrm{d}x = M\,. \tag{3}$$

Solution of the governing equations for the antisymmetric layer may be obtained by using, for example, the classical Fourier transform method which yields the expressions:

$$u = \frac{2}{\pi}\int_0^\infty [(A+sy\,B)\mathrm{e}^{sy} + (C+sy\,D)\mathrm{e}^{-sy}]\cos(sx)\mathrm{d}s\,,$$

$$v = \frac{2}{\pi}\int_0^\infty [(A-B+sy\,B)\mathrm{e}^{sy} - (C+D+sy\,D)\mathrm{e}^{-sy}]\sin(sx)\mathrm{d}s\,, \tag{4a, b}$$

$$\sigma_x = \frac{4\mu}{\pi}\int_0^\infty s[-(A+B+sy\,B)\mathrm{e}^{sy} - (C-D+sy\,D)\mathrm{e}^{-sy}]\sin(sx)\mathrm{d}s\,,$$

$$\sigma_y = \frac{4\mu}{\pi}\int_0^\infty s[(A-B+sy\,B)\mathrm{e}^{sy} + (C+D+sy\,D)\mathrm{e}^{-sy}]\sin(sx)\mathrm{d}s\,,$$

$$\tau_{xy} = \frac{4\mu}{\pi}\int_0^\infty s[(A+sy\,B)\mathrm{e}^{sy} - (C+sy\,D)\mathrm{e}^{-sy}]\cos(sx)\mathrm{d}s\,, \tag{5a–c}$$

where μ is the shear modulus. The unknowns $A-D$ may be determined from the boundary conditions at $y=0$ and $y=h$. The mixed conditions (2) may be rewritten more appropriately in the form

$$\frac{d}{dx}\,u(x,h)=0\,,\qquad \frac{d}{dx}\,v(x,h)=c\,,\qquad (0\le x<a)\,,\quad (6a,b)$$

$$\sigma_y(x,h)=q_1(x)\,,\qquad \tau_{xy}(x,h)=q_2(x)\,,\qquad (0\le x<\infty)\,,\quad (7a,b)$$

such that

$$q_1(x)=q_2(x)=0\,,\qquad (a<x<\infty)\,,\quad (8a,b)$$

by introducing two new unknown functions q_1 and q_2, the normal and the shear stresses along the interface with the rigid block. Now, substituting Equations (4) and (5) into conditions (1) and (7), $A-D$ may be expressed in terms of q_1 and q_2 as

$$A=-C=(sh+sh\,e^{-2sh})Q_1+[1-sh+(1+sh)e^{-2sh}]Q_2\,,$$

$$B=-[1+(1-2sh)e^{-2sh}]Q_1+[1+(1+2sh)e^{-2sh}]Q_2\,,$$

$$D=(1+2sh+e^{-2sh})Q_1+(1-2sh+e^{-2sh})Q_2\,,\qquad (9a-d)$$

where

$$Q_1=\frac{e^{-sh}}{2\mu s\Delta}\int_0^a q_1(t)\sin(st)\,dt\,,$$

$$Q_2=\frac{e^{-sh}}{2\mu s\Delta}\int_0^a q_2(t)\cos(st)\,dt\,,$$

$$\Delta=1+(2+4s^2h^2)e^{-2sh}+e^{-4sh}\,,\qquad (10a-c)$$

so that Equations (8) are also satisfied. Hence the stresses and the displacements all are expressed in terms of q_1 and q_2. These expressions may be given in the form

$$\frac{\partial}{\partial x}\,v(x,y)=\frac{2}{\pi}\int_0^\infty\sum_{i=1}^{2} s\,K_{1i}(s,y)Q_i(s)\cos(sx)\,ds\,,$$

$$\frac{\partial}{\partial x}\,u(x,y)=-\frac{2}{\pi}\int_0^\infty\sum_{i=1}^{2} s\,K_{2i}(s,y)Q_i(s)\sin(sx)\,ds\,,\qquad (11a,b)$$

$$\sigma_x(x,y)=-\frac{4\mu}{\pi}\int_0^\infty\sum_{i=1}^{2} s\,K_{3i}(s,y)Q_i(s)\sin(sx)\,ds\,,$$

$$\sigma_y(x,y)=\frac{4\mu}{\pi}\int_0^\infty\sum_{i=1}^{2} s\,K_{4i}(s,y)Q_i(s)\sin(sx)\,ds\,,$$

$$\tau_{xy}(x,y)=\frac{4\mu}{\pi}\int_0^\infty\sum_{i=1}^{2} s\,K_{5i}(s,y)Q_i(s)\cos(sx)\,ds\,,\qquad (12a-c)$$

where

$$K_{i1}=(a_1+a_2a_8)a_9+(\delta_{i4}-\delta_{i1})(a_3+a_4a_8)/a_9\,,\qquad (i=1,4)\,,$$

$$K_{i2}=-(a_5+a_6a_8)a_9+(\delta_{i1}-\delta_{i4})(a_7+a_5a_8)/a_9\,,\qquad (i=1,4)\,,$$

$$K_{i1}=[(1+\delta_{i2})a_5+a_7a_8]a_9+(\delta_{i5}-\delta_{i2})(a_6+a_5a_8)/a_9\,,\qquad (i=2,5)\,,$$

$$K_{i2}=(a_4+a_3a_8)a_9+(\delta_{i5}-\delta_{i2})(a_2+a_1a_8)/a_9\,,\qquad (i=2,5)\,,$$

$$K_{31}=-[a_4+(2a_4-a_3)a_8]a_9-(2a_1-a_2+a_1a_8)/a_9\,,$$

$$K_{32}=[2a_4+a_5+(2a_3+a_5)a_8]a_9-[2a_2-a_7+(2a_1-a_5)a_8]/a_9\qquad (13)$$

in which δ_{ij} is the Kronecker delta and

$$a_1=1+sh-sy\,,\qquad a_2=a_1-2sh+2s^2hy\,,$$
$$a_3=a_2+2sh+2sy\,,\qquad a_4=1-sh+sy\,,$$
$$a_5=sh-sy\,,\qquad a_6=a_5-2s^2hy\,,$$
$$a_7=a_5+2s^2hy\,,\qquad a_8=e^{-2sh}\,,$$
$$a_9=e^{sy}\,.\qquad (14)$$

3. THE INTEGRAL EQUATIONS

The two unknown functions q_1 and q_2 will be determined by using the conditions (6). If the expressions (11) are evaluated at $y=h$, one may notice that the kernels contain unbounded terms as $s\to\infty$. After separating these terms and evaluating their integrals separately, one may obtain the expressions

$$\frac{d}{dx}\,u_n(x,h)=\frac{1}{2\mu\pi}\int_{-a}^{a}\sum_{m=1}^{2}\left[\frac{\delta_{nm}}{t-x}+L_{nm}(x,t)\right]q_m(t)\,dt\,,\qquad (n=1,2),\quad (15a,b)$$

where $u_1=v$, $u_2=u$ and

$$L_{nn}=-2\int_0^\infty\frac{e^{-2sh}}{\Delta}[1+2(\delta_{1n}-\delta_{2n})sh+2s^2h^2+e^{-2sh}]\sin(t-x)s\,ds\,,\qquad (n=1,2)\,,$$

$$L_{12}=-L_{21}=4\int_0^\infty\frac{e^{-2sh}}{\Delta}s^2h^2\cos(t-x)s\,ds\,.\qquad (16)$$

Substitution of (15) into (6) results in the following system of two singular integral equations

$$\sum_{m=1}^{2} \frac{1}{\pi} \int_{-a}^{a} \left[\frac{\delta_{nm}}{t-x} + L_{nm}(x,t) \right] q_m(t) dt = 2\mu c\, \delta_{1n},$$

$$(-a<x<a), \qquad (n=1,2), \qquad (17a,b)$$

which must be solved subject to the equilibrium conditions

$$\int_{-a}^{a} q_n(t) dt = 0, \qquad (n=1,2). \qquad (18a,b)$$

In order to simplify the numerical analysis, introduce the following dimensionless variables

$$r = x/a, \qquad w = t/a. \qquad (19a,b)$$

Then, Equations (17) and (19) may be rewritten as

$$\sum_{m=1}^{2} \frac{1}{\pi} \int_{-1}^{1} \left[\frac{\delta_{nm}}{w-r} + k_{nm}(r,w) \right] p_m(w) dw = 2\mu c \delta_{1n},$$

$$(-1<r<1),$$

$$(n=1,2),$$

$$\int_{-1}^{1} p_n(w)\, dw = 0, \qquad (n=1,2), \qquad (20)$$

where

$$k_{nm}(r,w) = a\, L_{nm}(ar, aw), \qquad (n,\, m=1,2),$$

$$p_n(w) = q_n(aw), \qquad (n=1,2). \qquad (21)$$

Due to the singular term $(w-r)^{-1}$, the solution will be sought in terms of sectionally holomorphic functions [11]. The unknown functions p_1 and p_2 have square root singularity at $r=\pm 1$ [5–10, 12, 13]. Hence, one may write

$$p_n(w) = c^* p_0 g_n(w) (1-w^2)^{-1/2}, \qquad (n=1,2), \qquad (22)$$

where g_1 and g_2 are Hölder-continuous functions in $[-1, 1]$ and

$$c^* = 2\mu\, c/p_0,$$

$$p_0 = 3M/2a^2. \qquad (23)$$

Now, one can make use of the Gauss integration formula [14] and replace Equations (20) by the following system of linear algebraic equations

$$\sum_{i=1}^{N} C_i \sum_{m=1}^{2} \left[\frac{\delta_{nm}}{w_i - r_j} + k_{nm}(r_j, w_i) \right] g_m(w_i) = \delta_{1n},$$

$$(j=1,\ldots,N-1;\, n=1,2),$$

$$\sum_{i=1}^{N} C_i g_n(w_i) = 0, \qquad (n=1,2), \qquad (24)$$

where

$$C_1 = C_N = \frac{1}{2(N-1)}, \quad C_i = \frac{1}{N-1}, \qquad (i=2,\ldots,N-1),$$

$$w_i = \cos\left[\frac{i-1}{N-1}\pi\right], \qquad (i=1,\ldots,N),$$

$$r_j = \cos\left[\frac{j-1}{2N-2}\pi\right], \qquad (j=1,\ldots,N-1). \qquad (25)$$

The constant c^* is determined from condition (3) as

$$c^* = \frac{2/3\,\pi}{\sum_{i=1}^{N} C_i w_i g_1(w_i)}. \qquad (26)$$

4. NUMERICAL RESULTS

Some of the calculated results are shown in Figures 2–9. Figures 2 and 3 show the variations of the normalized stresses q_1/p_0 and q_2/p_0 along the rigid block-layer interface for various a/h ratios. Note that for fixed values of p_0 and h, the applied bending moment M increases with increasing a/h ratio. As can be observed from these figures, stress distributions depend heavily on a/h ratio. As a/h increases, q_1 increases over the central portion of the interface whereas it decreases near the edges. On the other hand, q_2 generally increases with increasing a/h

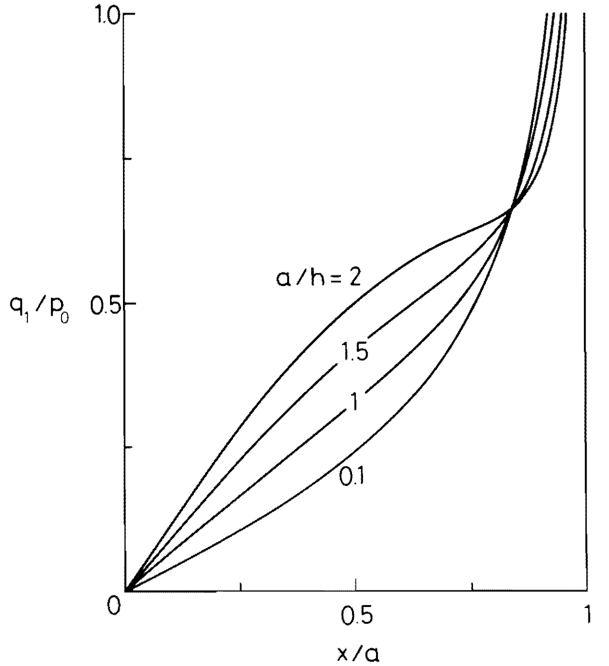

Figure 2. Normal Stress Between the Layer and the Rigid Block.

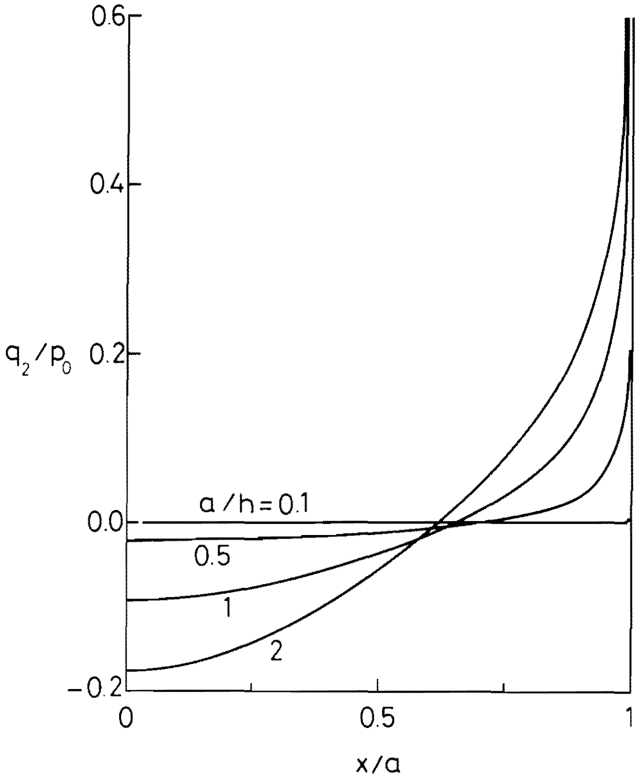

Figure 3. Shear Stress Between the Layer and the Rigid Block.

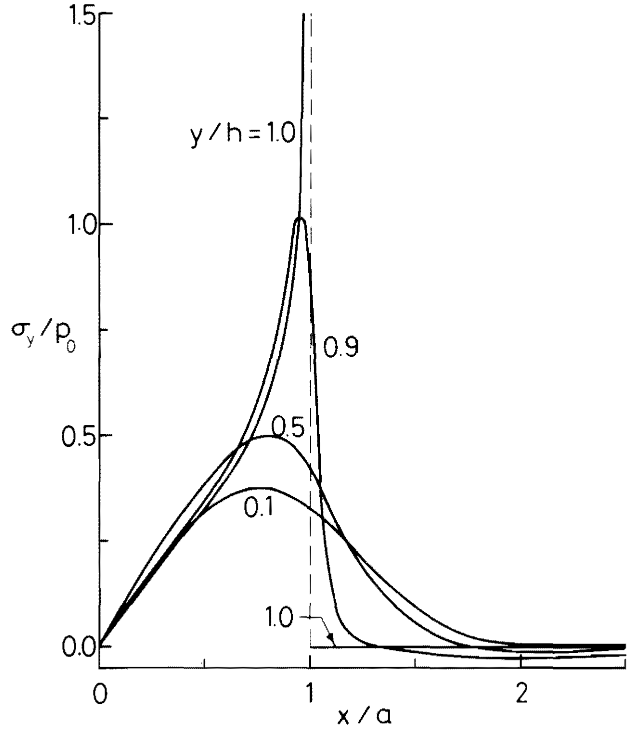

Figure 5. Normal Stress Distributions when a = h.

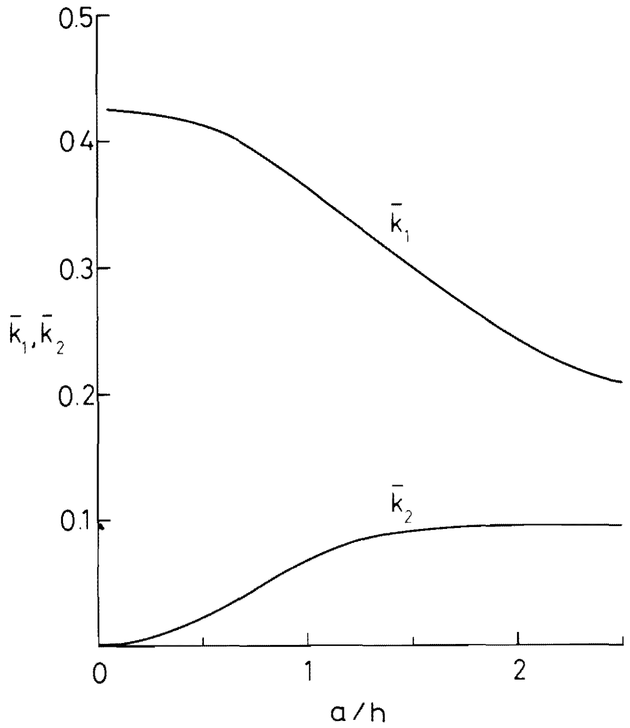

Figure 4. Stress Intensity Factors at the Corners of the Rigid Block.

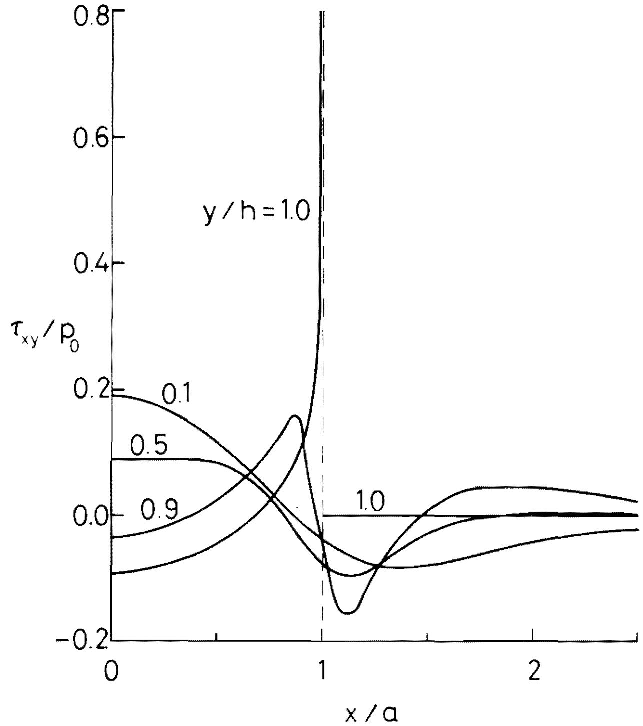

Figure 6. Shear Stress Distributions when a = h.

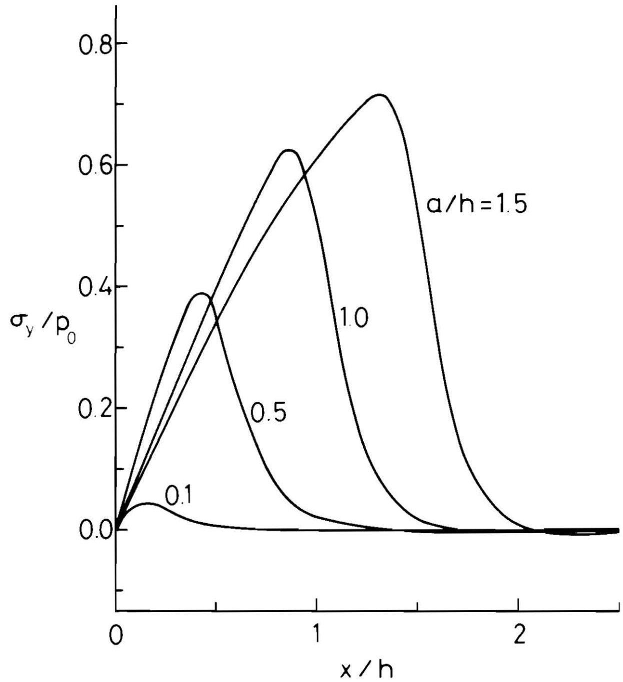

Figure 7. Normal Stress Distributions at y = 0.7 h.

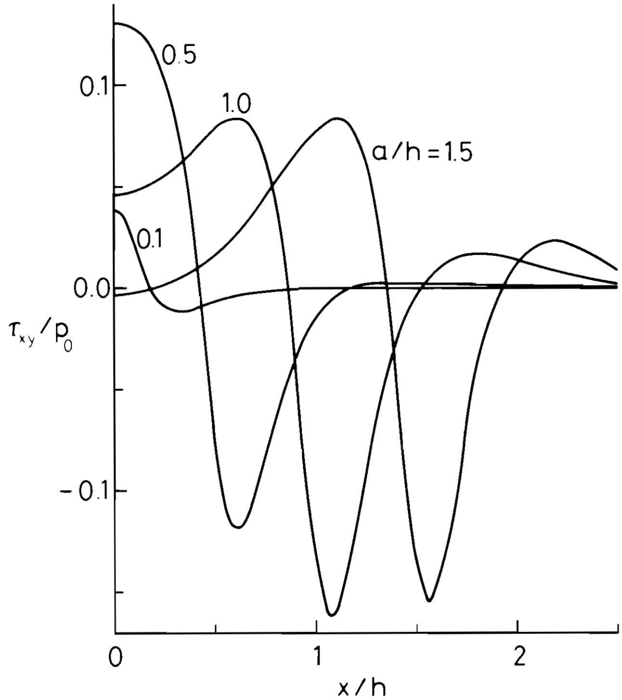

Figure 8. Shear Stress Distributions at y = 0.7 h.

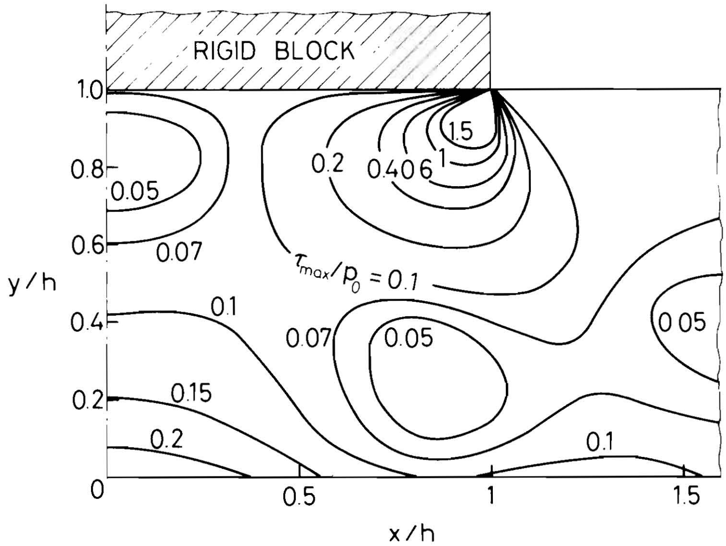

Figure 9. Maximum Shear Stress Contours when a = h.

ratio. One may be aware of the fact that the points near the edge on the upper surface of the layer (for $x>0$) will have a tendency to move horizontally toward the center $x = 0$. However, this tendency is restricted along the interface by the shear stresses resulting from the bonded contact with the rigid block. As the thickness of the rigid block increases, size of the restricted region becomes larger and consequently the required shear stresses for restriction of horizontal displacement will be larger. A similar behavior had been observed for axial loading in [7]. Both normal and shear stresses tend to infinity as the end point $x = a$ is reached. Stress state near the corners of the rigid block can be described in terms of the stress intensity factors defined by

$$k_1 = \lim_{x \to a} [2(a-x)]^{1/2}\, \sigma_y(x, h),$$

$$k_2 = \lim_{x \to a} [2(a-x)]^{1/2}\, \tau_{xy}(x, h). \qquad (27)$$

Figure 4 shows the variations of the normalized stress intensity factors

$$\bar{k}_i = k_i / p_0 \, a^{1/2}, \qquad (i = 1,2), \qquad (28)$$

with a/h. As a/h approaches zero, the constraint on the upper surface of the layer due to perfect bonding with the rigid block regarding the horizontal displacement simply disappears and the shear component of the stress intensity factor, $\bar{k}_2$, vanishes. As a/h ratio increases, $\bar{k}_1$, decreases and $\bar{k}_2$, increases.

As can be observed from Figures 2 and 3, both normal and shear stresses tend to infinity as the end point $x = a$ is reached. Therefore, predicting the performance of the bond is not straightforward using a strength of materials approach. In other words, stress can no longer be used as a design parameter when the corners of the rigid block are sharp. In this case, one has to employ the principles of fracture mechanics in predicting the critical pull-off state. More specifically, one has to consider, for example, the Griffith's energy balance theory (see, for example, [15]) and use the strain energy release rate as a parameter in predicting the failure. Note that the strain energy release rate can be expressed in terms of the stress intensity factors given in Figure 4.

Figures 5 and 6 show the normal and shear stress distributions, respectively, at various levels in the layer when $a = h$. Distributions are smoother at lower levels and they possess significant variations as the level under consideration gets closer to the upper surface. The most extensive disturbance seems to take place around $x = a$.

Figures 7 and 8 show the stress distributions at a level of $y = 0.7h$ for several a/h ratios. As a/h decreases, the peaks in these distributions which are around $x = a$ move toward the center. The normal stress σ_y / p_0 seems to decrease with decreasing a/h ratio. This is due to the fact that σ_y is normalized using p_0 and for fixed values of p_0 and h, the applied bending moment M decreases with decreasing a/h ratio. If one keeps M constant while decreasing a/h, p_0 will increase more rapidly. Therefore, for smaller a/h ratios σ_y will have higher peaks.

Finally, Figure 9 shows the contours of equal maximum shear stress τ_{max}/p_0 when $a = h$. One may note the accumulation of high shear stress contours around the corner $x = a$, $y = h$. Magnitude of the maximum shear stress becomes quite insignificant when $x > a$.

Results for the antisymmetric problem (bending) given in this paper can be superimposed with those for the symmetric problem (axial loading) given in [7] to obtain the results for the general asymmetric problem (combined loading).

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