INVESTIGATION OF FREQUENCY RESPONSE OF UNDERDAMPED SECOND-ORDER SYSTEMS BY MEANS OF VECTOR ALGEBRA

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NOMENCLATURE

arg = argument of a complex number
b = damping constant
f = forcing function (input)
k = spring constant
m = mass
s = Laplace parameter
t = time
x = displacement of mass, m
R = magnification ratio
α = normalized amplitude ratio
ζ = relative damping
σ = real part of s
ψ = phase angle
ω = frequency (imaginary part of s)
ω_d = system damped frequency $= \sqrt{(\omega_n^2 - \sigma^2)}$
ω_f = input frequency
ω_n = system natural (undamped) frequency $= \sqrt{(k/m)}$

INTRODUCTION

Every second-order system contains two energy-storing devices such as mass and spring in mechanical systems, or inductor and capacitor, in electrical systems. When no friction or electrical resistance is present, such systems, when disturbed, will exhibit a sustained oscillatory motion. However, in all physical systems some amount of friction is present which results in dissipation of energy and subsequent damping of the oscillatory motion, or even in preventing the oscillations from taking place altogether. We speak, in such cases, of underdamped or overdamped systems. Whether they belong to one or the other category, depends on the amount of the so-called relative damping defined by the ratio

$$\zeta = \frac{\sigma}{\omega_n}. \tag{1}$$

When $\zeta < 1$, we say the system is underdamped; when $\zeta > 1$, the system is overdamped; when $\zeta = 1$, the system is said to be critically damped. The subject of this investigation are the underdamped linear systems subjected to forced sinusoidal oscillations. Such systems exhibit 90° phase lag at the resonant frequency and magnification of the response amplitude which, with the exception of the case $\zeta = 0$, reaches its maximum at frequencies somewhat lower than the resonant frequency. One can determine the maximum value of this magnification, and the frequency at which it occurs, using well-known calculus methods. The resulting formulae can be found in a number of textbooks on Systems Dynamics [1–3]. Here, the vector algebra, leading to a quick graphical determination of these values, is used.

SYSTEM TRANSFER FUNCTION

The case of a second-order system subjected to some input (forcing function), $f(t)$, is represented mathematically by a second-order differential equation of the form:

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$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=f(t)\,. \tag{2}$$

Using the Laplace transform method we write the system transfer function as:

$$\frac{X(s)}{F(s)}=\frac{1}{ms^2+bs+k}=\frac{1/m}{(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)}\,. \tag{3}$$

The values of s, for which the transfer function becomes infinite, are known as its poles. They can be represented in the complex planes, $s=\sigma+j\omega$, by two points, $s_{1,2}=-\sigma\pm j\omega_d$, as shown in Figure 1.

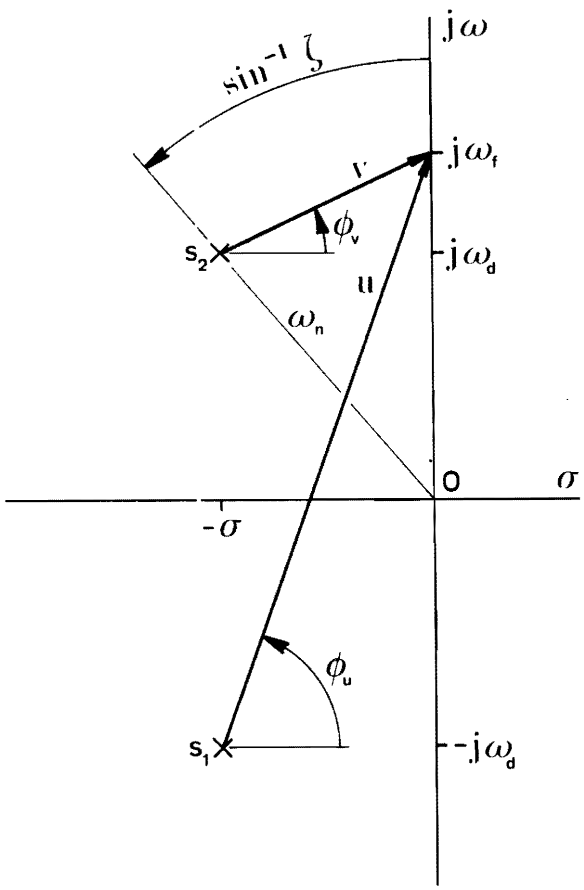

Figure 1. Poles of the Transfer Function and its Vectors **u** *and* **v**.

When the input is of the sinusoidal form, namely,

$$f(t)=f_m\cos\omega_f t \tag{4}$$

the parameter s in Equation 3 assumes the value of $j\omega_f$. Thus the transfer function can be written as

$$\frac{X(j\omega_f)}{F(j\omega_f)}=\frac{1/m}{(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)}\bigg|_{s=j\omega_f}. \tag{5}$$

The system response to such input is

$$x(t)=x_m\cos(\omega_f t+\psi) \tag{6}$$

where x_m is the amplitude of the response and ψ its phase angle measured with respect to the input. These quantities are readily obtained from Equation (5), namely,

$$\frac{X_m}{f_m/k}=\left|\frac{\omega_n^{\,2}}{(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)}\right|_{s=j\omega_f}=\alpha \tag{7}$$

and

$$\psi=\arg\left[\frac{\omega_n^{\,2}}{(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)}\right]_{s=j\omega_f}. \tag{8}$$

The quantity, α, in Equation (7) is known as the normalized amplitude ratio.

The system response to variable frequency of the input, ω_f, is called simply the frequency response.

VECTOR REPRESENTATION OF THE TRANSFER FUNCTION

The denominator of the transfer function, given by Equation (5), can be written as a product of two complex vectors, **u** and **v**, shown in Figure 1. Thus, Equations (7) and (8) can be written in the following form:

$$\alpha=\frac{\omega_n^{\,2}}{|\mathbf{u}\,\mathbf{v}|} \tag{9}$$

and

$$\psi=-\arg(\mathbf{u}\,\mathbf{v})=-(\phi_u+\phi_v)\,. \tag{10}$$

It is obvious that α attains its maximum value when the absolute value of the product $|\mathbf{u}\,\mathbf{v}|$, attains its minimum, while varying the frequency ω_f. Expressing **u** and **v** in terms of its real and imaginary components, namely,

$$\mathbf{u}=u_\sigma+j\,u_\omega \ \textit{and}\ \mathbf{v}=v_\sigma+j\,v_\omega\,, \tag{11}$$

we can show that

$$|\mathbf{u}\,\mathbf{v}|=\sqrt{\{(u_\sigma v_\sigma+u_\omega v_\omega)^2+(u_\sigma v_\omega-u_\omega v_\sigma)^2\}}$$
$$=\sqrt{\{|\mathbf{u}\cdot\mathbf{v}|^2+|\mathbf{u}\times\mathbf{v}|^2\}}\,. \tag{12}$$

Since the magnitude of the cross product, $\mathbf{u}\times\mathbf{v}$, is twice the area bounded by the triangle formed by these two vectors and the segments, s_1s_2 (see Figure 1), we can write

$$\mathbf{u}\times\mathbf{v}=2\sigma\omega_d=\text{const} \tag{13}$$

which means that its values is independent of ω_f.

Therefore, the term $|\mathbf{u}\ \mathbf{v}|$ attains its minimum when the dot product $\mathbf{u}\cdot\mathbf{v}$ is zero. Since

$$\mathbf{u}\cdot\mathbf{v} = u\ v \cos(\phi_u - \phi_v) \tag{14}$$

the dot product is zero when projection of $\mathbf{v}$ on $\mathbf{u}$ is zero, *i.e.*, when the two vectors are perpendicular, or, $\phi_u - \phi_v = 90°$.

From plane geometry, we know that a triangle inscribed in a semicircle, and whose base is equal to the diameter of that circle, is a right-angled triangle. Thus, the frequency at which the amplitude ratio, α, attains its maximum can be determined by constructing a semicircle of radius, $r = \omega_d$, as shown in Figure 2. Intersection of that semicircle with the $j\omega$-axis determines, $\omega_f = \omega_m$, for which the system response attains its maximum value.

From Figure 2 it is also clear that when $r = \omega_d < \sigma$, the semicircle will not intersect the $j\omega$-axis, *i.e.*, the dot product, $\mathbf{u}\cdot\mathbf{v}$, will not be zero and its minimum is attained when $\omega_f = 0$.

The frequency ω_f, for which the phase angle of the system response is $-90°$, can be determined by construction of another semicircle of radius $r = \omega_n$ and whose center lies at the origin O, of the s-plane, as shown in Figure 3.

Consider, in that figure, two triangles, $A\ s_1\ B$ and $A\ s_2\ B$. Both these triangles are identical and

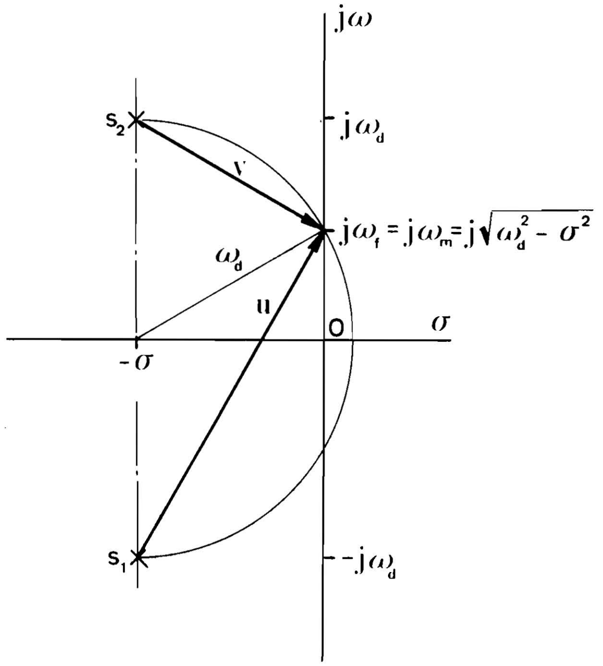

Figure 2. Graphical Determination of Input Frequency for α_{max}.

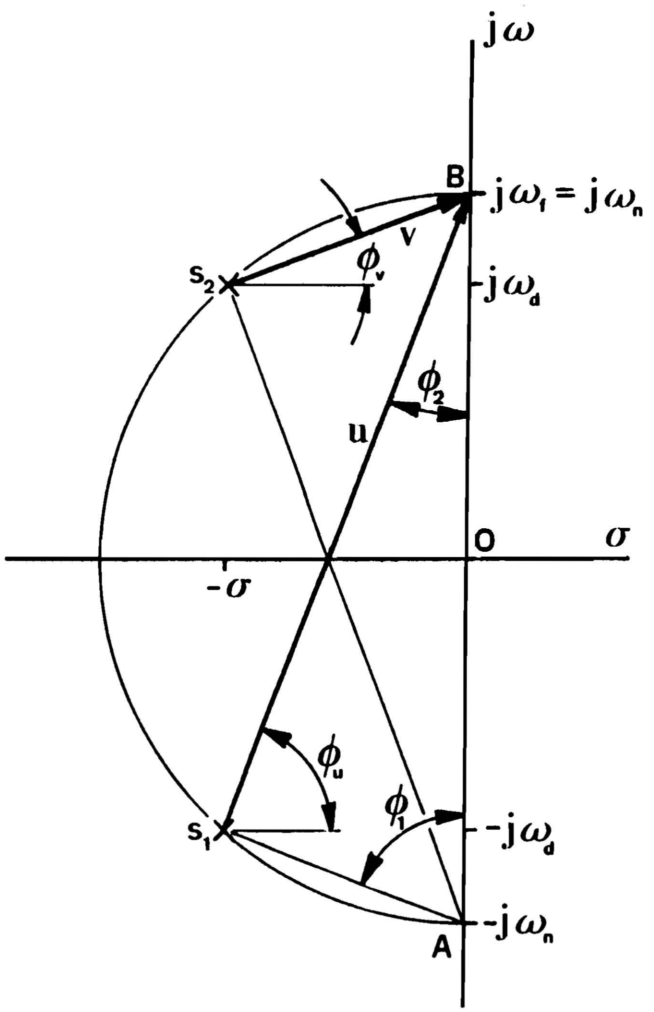

Figure 3. Graphical Determination of Input Frequency for $\psi = -90°$.

right-angled. Therefore, the angles ϕ_1 and ϕ_2 must satisfy the relationship

$$\phi_1 + \phi_2 = 90° . \tag{15}$$

But, $\phi_1 = \phi_u$ and $\phi_2 = \phi_v$, hence,

$$\phi_u + \phi_v = 90° . \tag{16}$$

This means that 90° phase lag takes place always at the resonant frequency, *i.e.*, $\omega_f = \omega_n$.

MAGNIFICATION RATIO

The signal (input) magnification ratio can be defined as the ratio of the amplitudes of the system maximum response and that for low frequencies ($\omega_f \to 0$). For $\omega_f = 0$, the amplitude of the system response can be written as

$$x_m(\omega_f = 0) = \frac{K}{|\mathbf{u}_o\ \mathbf{v}_o|} \tag{17}$$

where $\mathbf{u}_o$ and $\mathbf{v}_o$ are the vectors $\mathbf{u}$ and $\mathbf{v}$, respectively, associated with the input frequency, $\omega_f = 0$, and K is the system constant.

Since

$$\mathbf{u}_o = \mathbf{v}_o = \sigma^2 + \omega_d^2 = \omega_n^2 \ , \tag{18}$$

we can write

$$x_m(\omega_f = 0) = \frac{K}{\omega_n^2} \ . \tag{19}$$

At $\omega_f = \omega_m$, the dot product of vectors $\mathbf{u}$ and $\mathbf{v}$ is zero, while their cross product value is given by Equation (13); hence, we can write

$$x_m(\omega_f = \omega_m) = \frac{K}{2\ \omega_d\ \sigma} \ . \tag{20}$$

Therefore the ratio of the amplitudes at the two frequencies is

$$R_m = \frac{x_m(\omega_f = \omega_m)}{x_m(\omega_f = 0)} = \frac{\omega_n^2}{2\ \omega_d\ \sigma} \ . \tag{21}$$

Using the well-known relationships between σ, ω_d, ω_n, and ζ, we can write

$$R_m = \frac{1}{2\zeta\sqrt{(1-\zeta^2)}} \tag{22}$$

which is in full agreement with results found in the literature [1–3]. Equation (22) is valid only for the range, $0 \leqslant \zeta \leqslant \sqrt{2}/2$, *i.e.*, the range for which points of intersections of the $r = \omega_d$ circle and the j-axis exist (see Figure 2).

Of interest is also magnification of the input at the resonant frequency (Figure 3). The magnitude of the denominator of the transfer function as, given by Equation (12), is the resultant of the dot and cross products of vectors $\mathbf{u}$ and $\mathbf{v}$. It has been stated before (Equation 13), that the cross product value is independent of the input frequency and has a constant value of $2\sigma\omega_d$. While in the previous case the dot product was zero, here, it is not. Its value can be determined from the consideration of its components, namely,

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_\sigma v_\sigma + u_\omega v_\omega = \sigma^2 + (\omega_n + \omega_d)(\omega_n - \omega_d) \\ &= \sigma^2 + \omega_n^2 - \omega_d^2 = 2\sigma^2 \ .\end{aligned} \tag{23}$$

Substituting the derived values for the dot and cross products into Equation (12), we obtain

$$|\mathbf{u}\ \mathbf{v}| = 2\sigma\omega_n \ . \tag{24}$$

Therefore, the magnification ratio at the resonant frequency is

$$R_r = \frac{\omega_n^2}{2\sigma\omega_n} = \frac{1}{2\zeta} \ . \tag{25}$$

CONCLUSIONS

By means of the method presented here, the input frequencies for which the system response attains its maximum, or when its phase angle $\psi = -90°$, and the corresponding magnification ratios, can be easily and quickly determined. Although the results obtained are well known, in the opinion of the author this investigation illustrates well the usefulness of the vector method in analysis of the system frequency response and, thus, has tutorial merit.

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