A CONDITION IMPLYING SEPARABILITY OF A REFLEXIVE BANACH SPACE

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الخلاصة :

يبيّن هذا البحث أنه إذا كان X فضاء بناخ إنعكاسي ، وكان هناك مؤثر متباين ومتراص من X إلى فضاء بناخ آخر Y ، فإن X قابل للانفصال .

ABSTRACT

It is shown that if X is a Banach space then X is separable if there is a Banach space Y and a compact one-to-one operator from X and Y.

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INTRODUCTION

A reflexive Banach space is not necessarily separable. An example of a nonseparable reflexive Banach space is a general Hilbert space. In this note we give a condition under which a reflexive Banach space becomes separable. We show by example that reflexivity is necessary.

Throughout this note X and Y are Banach spaces. X^* is the dual space of X. $L(X,Y)$ is the set of all bounded linear operators from X into Y. The set of all compact one-to-one operators in $L(X,Y)$ is denoted by $K_0(X,Y)$. A family $A \subseteq X^*$ is called total if and only if $y \in X$, and $f(y) = 0$, for all f in A, together imply that $y = 0$.

THEOREM

Let X and Y be Banach spaces such that X is reflexive. Suppose that $K_0(X,Y)$ is not empty. Then X is separable.

Proof. Since T is compact, it is continuous. By (Reference [1], Theorem 15, p.422), T is weakly continuous. Let B be the closed unit ball of X and let $T|B$ denote the restriction of T to B. Since X is reflexive, B is weakly compact (Reference [1], Theorem 7, p. 425). Since T is continuous and one-to-one, $T|B$ is continuous and one-to-one which implies — by (Reference [1], Lemma 8, p. 18) — that $T|B$ is a weak homeomorphism. Hence $(T|B)^{-1}$ — the inverse of $(T|B)$ — is weakly continuous. Since T is compact, $T(B)$ is separable. Thus $T(B)$ contains a countable subset M. Since the weak closure of a set contains its norm closure, M is weakly dense in $T(B)$. Hence the set $J = (T|B)^{-1}(M)$ is weakly dense in B and countable (Reference [3], 3.10, p. 33). Let H be the set of all convex combinations with rational coefficients of elements of J, then H is countable and — by (Reference [1], Corollary 14, p. 422) — norm dense in X. This implies that X is separable.

Example. Let $X = l_\infty$ be the Banach space of all bounded sequences $x = \{x_n\}_{n=1}^{\infty}$. Then X is not reflexive and not separable. Let l_1 be the Banach space of all sequences $Y = \{y_n\}_{n=1}^{\infty}$ for which the norm

$$\|Y\| = \sum_{n=1}^{\infty} \|Y_n\|$$

is finite. Then $l_1^* = X$ and $l_1 \subseteq X^*$. Now consider the set $A = \{e_i \in l_1 : e_i$ has value one in the ith place and zero elsewhere$\}$. Then A is a countable subset of X^*. We show that it is total. Let $t = \{t_i\}_{n=1}^{\infty}$ be a nonzero element of X; then at least one of the entries of t is not zero, say t_j. Hence the element e_j in A satisfies $e_j(t) \neq 0$ since

$$e_j(t) = \sum_{i=1}^{\infty} \delta_{ij} t_i.$$

Thus A is total and countable in X^*, and so by (Reference [2], p. 810), $K_0(X,Y)$ is not empty for any Banach space Y.

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