ATTENUATION OF GRAVITY WAVES IN THE PRESENCE OF A MAGNETIC FIELD IN A TURBULENT CONDUCTING FLUID

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الخلاصــة :

يدرس هذا البحث تأثير المجال المغناطيسي ومؤثرات أخرى على تخفيف موجات الجاذبية الداخلية والمولدة فى وسطٍ مائعٍ موصلٍ مضطرب ذي طبقات ضعيفة التكهرب ، بواسطة نظرية نموذج الانتشار الطولي . وباختيار عددٍ معيّن من بارامترات (متغيّر وسيط) توصلنا إلى النتائج التالية :

١) تتضاءل الموجات القصيرة بقوة مهما كان عمق المائع ومهما كانت قوة المجال المغناطيسي .
٢) تتضاءل الموجات الطويله بسرعة فى الموائع قليلة العمق .
٣) الزيادة في العمق لها تأثير ملحوظ على الموجات الطويلة فقط .
٤) يكون اعتماد التضاؤل النسبي فى الطول للموجات الطويله على طبقه المائع بعكس اعتماد التضاؤل النسبي في الطول للموجات القصيرة .
٥) الحد الأدنى فى الدورة والحد الأقصى فى التضائل النسبي فى الطول يحدثان فى موجات وسطية الطول ومكان حدوث هذه الحدود يعتمد على المجال المغناطيسي وعمق طبقة المائع ولزوجة التيار .

ABSTRACT

The attenuation of internal gravity waves by magnetic fields and by other diffusing effects, propagating in a weakly stratified, electrically conducting, turbulent flow, is studied using a linear theory based on the gradient diffusion model. It is shown that the damping length and period increase are, in general, dependent on magnetic field, Rayleigh number, thickness of the layer, and the eddy viscosity parameter. Subject quantitatively to the choice made for these parameters, the following conclusions are drawn: (*i*) the short waves are damped strongly for all depths of the fluid and for any strength of the magnetic field; (*ii*) long waves decay faster in shallow depths; (*iii*) increase in depth has no significant effect on attenuation of short waves but causes marked changes in long waves; (*iv*) the dependence of relative damping length on the fluid layer is in the opposite directions for long and short waves; (*v*) the period increase and relative damping length present a minimum and a maximum respectively at the intermediate wavelength, the positions of these extrema depending on the magnetic field, depth of the fluid layer, and eddy viscosity.

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NOTATION

a	Dimensionless wave number
g	Acceleration due to gravity
H	Depth of the fluid layer
H_0	Imposed magnetic field
k	Wave number
K	Thermal conductivity
K_{fr}	Eddy viscosity
K_{hr}	Eddy thermal diffusivity
K_{mr}	Eddy magnetic viscosity
K_{er}, K_{mer}	Magnetic diffusion coefficients
L	Wavelength of gravity waves
n	Mode number
P_0	Hydrostatic pressure
$q_i=(u,v,w)$	Mean velocity components of the fluid
$R=\dfrac{g\alpha\beta H^4}{K_f^2}$	Rayleigh number
t	Time
T	Temperature
x_e	Damping length
α	Volumetric expansion coefficient
$\beta=-\dfrac{dT_0}{dZ}$	Temperature gradient
$\eta_0=$	π^2+a^2
η_1	Dimensionless eddy heat diffusivity
η_2	Dimensionless magnetic diffusivity
η_3	Dimensionless eddy viscosity
μ	Permeability of fluid medium
ν	Kinematic viscosity of the fluid
ν_m	Magnetic viscosity
ρ	Density of the fluid
σ	Dimensionless growth rate
τ	Time constant characterizing damping rate
ω	Growth rate.

1. INTRODUCTION

Recently there has been greatly increased interest in the theory and modeling of transfer of energy and momentum by internal gravity waves in geophysics [1,2], particularly from the mantle into the outer core of the earth [3]. The fact that the outer core and the lower mantle might be thermally stably stratified was recognized by Braginskii [4]. Even if the lower mantle and outer core were stably stratified, this does not of course mean that all radial motions are impossible. Internal gravity waves modified by the Lorentz force can propagate in a stably stratified medium due to perturbation forces either in the interior fluid or on its boundary [3]. There is some evidence [3] that the core–mantle boundary is not smooth, but is bumpy on a tangential scale of the order of hundreds of kilometers and on a radial scale of one to two kilometers. If this is the case, then it is easy to visualize that internal gravity waves are generated by the disturbances at the core–mantle interface. Here it is known that the kinematic viscosity $\nu \sim 10^{-7}\mathrm{m^2s^{-1}}$, characteristic velocity is $\sim 4\times10^{-4}\mathrm{m\,s^{-1}}$, characteristic length is $\sim 10^6$m and magnetic viscosity $\nu_m \sim 2\times10^3\mathrm{m^2s^{-1}}$ so that the Reynolds number is $\sim 10^9$ and the magnetic Reynolds number is ~ 0.1. At such a high Reynolds number, the flow is turbulent and the induced magnetic field may be weak. A considerable amount of work has been done on internal gravity waves in the presence of a magnetic field (see references [1], [2] and references therein) under the assumption of laminar flow. However, not much attention has been given to the propagation of internal gravity waves in a turbulent conducting fluid in the presence of a magnetic field. In this paper, we study this under the assumption of low magnetic Reynolds number, in the hope that the results may be useful in a geophysical problem cited above. The results of this study may also be useful in industrial problems involving turbulent flow modified by the Lorentz force [5]. In this paper attention is focused on the understanding of the attenuation of internal gravity waves in the presence of a magnetic field, in addition to other diffusion effects. The inclusion of momentum, heat, and magnetic field advection terms in the basic equations, with a built-in statistical description, leads to the appearance of Reynolds, thermal, and magnetic stresses respectively, which results in an indeterminate system of governing equations and therefore requires a suitable closure model.

In the absence of magnetic fields, LeBlond [6] has investigated the damping of internal gravity waves in continuously stratified media using a differential closure model, based on a gradient diffusion model combined with the Prandtl mixing length hypothesis. In the study of internal gravity waves in MHD turbulence, much more insight is needed to overcome the indeterminacies arising from Reynolds, thermal, and magnetic stresses. We note that the assumption of small magnetic Reynolds number makes the problem somewhat simpler and in that case the indeterminacies can be resolved by obtaining additional information concerning the relationship between the mean and the fluctuating quantities. One of the most commonly exploited theories to obtain this relationship is that resulting from the gradient diffusion model, based on the Prandtl mixing length hypothesis (see Kollmann [7]). This model leads to the appearance of eddy viscosity, the eddy magnetic and thermal diffusivities which are assumed to be constant in the present paper. The validity of this mixing length hypothesis will depend on the individual problem considered i.e. on the physical phenomenon involved. Since the aim of the present paper is to study the attenuation of internal gravity waves involving a large body of conducting fluid, we assume, as in the case of an ordinary fluid [6], that the assumption of constant eddy viscosity and diffusion co-efficients may give reasonable results. Another assumption is about the validity of the linear model. In ordinary viscous flow, LeBlond [4] has pointed out that linearizing the equations of motion using eddy viscosity and diffusion co-efficients is justifiable because the stable stratification reduces the intensity of turbulence considerably. In the present problem, however, in addition to stable stratification, the Lorentz force also reduces the intensity of turbulence.

The results obtained in this paper, using the first order closure model with small magnetic Reynolds number, may give a physical insight when considering a more general MHD turbulence model using the second order closure model. The plan of the work in this paper is as follows. The wave equation, using the average processes and gradient diffusion model, is derived in Section 2. The wave solution and the corresponding attenuation is discussed in Section 3. The general conclusions are derived in Section 4. It is shown that the effect of magnetic fields, in addition to other diffusive effects, is to attenuate the internal gravity waves propagating in a turbulent flow.

2. MATHEMATICAL FORMULATION

We consider a weakly electrically conducting, Boussinesq fluid of thickness H and of infinite lateral extent. Let (X,Y,Z) be the right-handed co-ordinate system with Z-axis vertically upwards such that $Z=0$ is at the lower surface of the layer and gravity acts vertically downwards. The basic equations for this model under the Boussinesq approximation are (see Chandrasekhar [8]):

$$\frac{\partial q_i}{\partial t} + q_j\frac{\partial q_i}{\partial X_j} = -\frac{1}{\rho_0}\frac{\partial P}{\partial X_i} - \frac{\rho}{\rho_0} g\delta_{i3} + \frac{\mu}{\rho_0} H_j \frac{\partial H_i}{\partial X_j} + \nu\frac{\partial^2 q_i}{\partial X_j \partial X_j}, \tag{1}$$

$$\frac{\partial q_i}{\partial X_i} = 0, \tag{2}$$

$$\frac{\partial T}{\partial t} + q_j\frac{\partial T}{\partial X_j} = K\frac{\partial^2 T}{\partial X_j \partial X_j}, \tag{3}$$

$$\frac{\partial H_i}{\partial t} + q_j\frac{\partial H_i}{\partial X_j} - H_j\frac{\partial q_i}{\partial X_j} = \nu_m\frac{\partial^2 H_i}{\partial X_j \partial X_j}, \tag{4}$$

$$\frac{\partial H_i}{\partial X_i} = 0, \tag{5}$$

$$\rho = \rho_0[1-\alpha(T-T_0)], \tag{6}$$

where q_i are the velocity components, $P = p + \frac{1}{2}\mu H_i^2$ is the total pressure, ρ is the density, T is the temperature, ρ_0 is the density at the reference temperature $T = T_0$, g is the free-fall acceleration, H_i are the components of the magnetic field, ν is the kinematic viscosity, μ is the magnetic permeability, and ν_m is the magnetic diffusivity. The fluid is, on a time average, at rest and is permeated by a uniform magnetic field H_0 in the vertical Z-direction. The fluid is weakly stratified in density which is maintained by an unspecified but adequate source of heat. The basic state is described by quiescent state with hydromagnetic balance, that is

$$\mathbf{q}_0 = 0, \;\; P_0 = -\rho_0 g Z + \text{const.}$$

Superimposed on this is a horizontally homogeneous field of turbulence characterized by velocity and temperature fluctuations. Because of homogeneity, time averages are independent of horizontal co-ordinates.

We assume that the flat-crested (i.e. plane) internal waves propagate in turbulent field in the positive X-direction with angular frequency ω and wave number k. The amplitudes of these waves are assumed to be small enough so that we may neglect squares of mean fluctuating velocities as well as density disturbances due to the waves as compared to the main density field. At present, there are well-known examples of two-dimensional, or nearly two-dimensional, flows that arise in nature (Roberts and Stix [9]), where there will be a decisive physical effect that singles out the fluid direction as being special. In the present problem, the turbulent field will be separated from the waves by averaging in the y-direction. Since the turbulent variables are random in phase and homogeneous in horizontal space, the y-average will leave only the wave field. Denoting the averaged wave field by overscored variables and the total turbulent perturbations (including the extra perturbations caused by turbulent transport of fluid particles endowed with wave velocities) by primed variables, the total field is given by:

$$\begin{aligned} \mathbf{q} &= \bar{\mathbf{q}} + \mathbf{q}' \\ \rho &= \rho_0 + \bar{\rho} + \rho' \,, \\ T &= T_0 + \bar{T} + T' \,, \\ P &= P_0 + \bar{P} + P' \,, \\ \mathbf{H} &= \mathbf{H}_0 + \bar{\mathbf{h}} + \mathbf{h}' \,. \end{aligned} \tag{7}$$

By definition all the primed quantities have zero y-average.

Substituting (7) into (1) to (6) and taking the y-average of these equations, and using the Boussinesq approximation and linearizing w.r.t. overscored variables, we obtain equations (8−13).

$$\frac{\partial \bar{q}_i}{\partial t} + \frac{\partial \bar{p}}{\partial X_i} + \frac{\bar{\rho}}{\rho_0} g\delta_{i3} - \frac{\mu H_0}{\rho_0}\frac{\partial \bar{h}_i}{\partial X_3} = \frac{\partial}{\partial X_j}\left(\nu \frac{\partial \bar{q}_i}{\partial X_j} - \overline{q_j' q_i'}\right) - \frac{\partial}{\partial X_j}\left(\frac{\mu}{\rho_0}\overline{h_j' h_i'}\right), \tag{8}$$

$$\frac{\partial \bar{q}_i}{\partial X_i} = 0 \,, \tag{9}$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{w}\frac{\mathrm{d}T_0}{\mathrm{d}X_3} = \frac{\partial}{\partial X_j}\left(K\frac{\partial \bar{T}}{\partial X_j} - \overline{q_j' T'}\right), \tag{10}$$

$$\frac{\partial \bar{h}_i}{\partial t} - H_0\frac{\partial \bar{q}_i}{\partial X_3} = \frac{\partial}{\partial X_j}\left(\nu_m \frac{\partial \bar{h}_i}{\partial X_j} - \overline{q_j' h_i'} + \overline{h_j' q_i'}\right), \tag{11}$$

$$\frac{\partial \bar{h}_i}{\partial X_i} = 0 \,, \tag{12}$$

$$\bar{\rho} = -\alpha \rho_0 \bar{T} \,, \tag{13}$$

$$\rho' = -\alpha\rho_0 T'. \tag{14}$$

The Reynolds stress $\overline{q_j' q_i'}$, the magnetic stress $\overline{h_j' h_i'}$, the heat advection term $\overline{q_j' T'}$, the transport of magnetic field $\overline{q_j' h_i'}$, and the stretching of field lines by velocity field $\overline{h_j' q_i'}$ appearing in the above equations need further modeling. We introduce the concept of eddy viscosity and diffusion coefficients in the following form:

$$\begin{aligned} \frac{\partial}{\partial X_j}(-\overline{q_j' q_i'}) &= \frac{\partial}{\partial X_r}\left(K_f' \frac{\partial \bar{q}_i}{\partial X_r}\right), \\ \frac{\partial}{\partial X_j}(-\overline{h_j' h_i'}) &= \frac{\partial}{\partial X_r}\left(K_m' \frac{\partial \bar{h}_i}{\partial X_r}\right), \\ \frac{\partial}{\partial X_j}(-\overline{q_j' T'}) &= \frac{\partial}{\partial X_r}\left(K_h' \frac{\partial \bar{T}}{\partial X_r}\right), \\ \frac{\partial}{\partial X_j}(-\overline{q_j' h_i'}) &= \frac{\partial}{\partial X_r}\left(K_e' \frac{\partial \bar{h}_i}{\partial X_r}\right), \\ \frac{\partial}{\partial X_j}(-\overline{h_j' q_i'}) &= \frac{\partial}{\partial X_r}\left(K_{me}' \frac{\partial \bar{q}_i}{\partial X_r}\right), \end{aligned} \tag{15}$$

where K_f' is the viscosity co-efficient, K_m' is the eddy magnetic viscosity coefficient and K_h', K_e', and K_{me}' are the thermal and magnetic diffusion coefficients. The eddy coefficients for momentum, heat, and magnetic field, will, in general, be different from each other and are assumed to be spatially uniform. This assumption of uniform eddy coefficients does not lead to serious error in dealing with large body of fluid as in the viscous case of LeBlond [4]. This assumption has to be modified when we deal with small body of fluids as in many engineering applications where we have to use two point closure methods rather than the single point closure method used in the present paper.

Substituting (15) into Equations (8), (10) and (11), adding an extra r to the eddy coefficients to take their anisotropy into account and eliminating all variables except $\bar{q}_3(=\bar{w})$ we get the following wave equation:

$$\left[\left(\frac{\partial}{\partial t}-K_{\mathrm{er}}\frac{\partial^2}{\partial X_{\mathrm{r}}^2}\right)\left(\frac{\partial}{\partial t}-K_{\mathrm{fr}}\frac{\partial^2}{\partial X_{\mathrm{r}}^2}\right)-\frac{\mu}{\rho_0}\left(H_0\frac{\partial}{\partial X_3}-K_{\mathrm{mr}}\frac{\partial^2}{\partial X_{\mathrm{r}}^2}\right)\right.$$

$$\left.\times\left(H_0\frac{\partial}{\partial X_3}-K_{\mathrm{mer}}\frac{\partial^2}{\partial X_{\mathrm{r}}^2}\right)\right]\left(\frac{\partial}{\partial t}-K_{\mathrm{hr}}\frac{\partial^2}{\partial X_{\mathrm{r}}^2}\right)\nabla^2\bar{w}$$

$$+\,g\alpha\frac{\partial T_0}{\partial Z}\left(\frac{\partial}{\partial t}-K_{\mathrm{er}}\frac{\partial^2}{\partial X_{\mathrm{r}}^2}\right)\nabla_{\mathrm{h}}^2\bar{w}=0, \tag{16}$$

where $K_{\mathrm{fr}}=\nu+K'_{\mathrm{fr}}$, $K_{\mathrm{hr}}=K+K'_{\mathrm{hr}}$, $K_{\mathrm{mr}}=K'_{\mathrm{mr}}$, $K_{\mathrm{er}}=\nu_{\mathrm{m}}+K'_{\mathrm{er}}$, $K_{\mathrm{mer}}=K'_{\mathrm{mer}}$. We assume the waves of the form

$$\bar{w}=W(Z)\exp(\omega t-\mathrm{i}kx).$$

Then Equation (16) can be written as

$$\left[\left(\omega+K_{\mathrm{e1}}k^2-K_{\mathrm{e3}}\frac{\mathrm{d}^2}{\mathrm{d}Z^2}\right)\left(\omega+K_{\mathrm{f1}}k^2-K_{\mathrm{f3}}\frac{\mathrm{d}^2}{\mathrm{d}Z^2}\right)\right.$$

$$-\frac{\mu}{\rho_0}\left(H_0\frac{\mathrm{d}}{\mathrm{d}Z}+K_{\mathrm{m1}}k^2-K_{\mathrm{m3}}\frac{\mathrm{d}^2}{\mathrm{d}Z^2}\right)$$

$$\left.\times\left(H_0\frac{\mathrm{d}}{\mathrm{d}Z}+K_{\mathrm{me1}}k^2-K_{\mathrm{me3}}\frac{\mathrm{d}^2}{\mathrm{d}Z^2}\right)\right]$$

$$\times\left(\omega+K_{\mathrm{h1}}k^2K_{\mathrm{h3}}-\frac{\mathrm{d}^2}{\mathrm{d}Z^2}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}Z^2}-k^2\right)W$$

$$-\,k^2g\alpha\frac{\mathrm{d}T_0}{\mathrm{d}Z}\left(\omega+K_{\mathrm{e1}}k^2-K_{\mathrm{e3}}\frac{\mathrm{d}^2}{\mathrm{d}Z^2}\right)W=0\,. \tag{17}$$

To study the propagation behavior of internal Alfvén gravity waves we have to obtain the solution of Equation (17). For this we need proper boundary conditions at $Z=0$ and H. On both bounding surfaces the vertical velocity and the magnetic field perturbations vanish:

$$\text{i.e. } \bar{w}(\mathbf{W},t)=\bar{\mathbf{h}}(\mathbf{W},t)=0 \text{ at } z=0,\ H. \tag{18}$$

Since the boundaries are assumed to be stress free, we have

$$\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}Z^2}=0 \text{ at } Z=0,\ H. \tag{19}$$

The boundary conditions (18) and (19) using (6) to (17) can be expressed in terms of the vertical velocity alone. Thus

$$W=\frac{\mathrm{d}^2\bar{W}}{\mathrm{d}Z^2}=0 \text{ at } Z=0,\ H. \tag{20}$$

$$\left[\frac{\mu}{\rho_0}\frac{K_{\mathrm{m3}}K_{\mathrm{me3}}}{K_{\mathrm{e3}}}-K_{\mathrm{f3}}\right]D^4W$$

$$-\frac{\mu H_0}{\rho_0K_{\mathrm{e3}}}\left(K_{\mathrm{m3}}+K_{\mathrm{me3}}\right)D^3W$$

$$+\frac{\mu H_0}{\rho_0K_{\mathrm{e3}}}\left[k^2\left(K_{\mathrm{m1}}+K_{\mathrm{m3}}+K_{\mathrm{me1}}-K_{\mathrm{m3}}\frac{K_{\mathrm{e1}}}{K_{\mathrm{e3}}}\right)\right.$$

$$\left.-\,\omega\frac{K_{\mathrm{m3}}}{K_{\mathrm{e3}}}\right]DW=0, \text{ at } Z=0,\ H, \tag{21}$$

and

$$\left(K_{\mathrm{e3}}K_{\mathrm{f3}}-\frac{\mu}{\rho_0}K_{\mathrm{m3}}K_{\mathrm{me3}}\right)D^6W$$

$$+\frac{\mu H_0}{\rho_0}(K_{\mathrm{m3}}+K_{\mathrm{me3}})D^5W-[K_{\mathrm{f3}}(\omega+K_{\mathrm{e1}}k^2)$$

$$+K_{\mathrm{e3}}(\omega+K_{\mathrm{f1}}k^2)+k^2K_{\mathrm{e3}}K_{\mathrm{f3}}-\frac{\mu}{\rho_0}K_{\mathrm{m3}}K_{\mathrm{me3}}k^2$$

$$+\frac{\mu}{\rho_0}(H_0^2-K_{\mathrm{m3}}K_{\mathrm{me1}}k^2-K_{\mathrm{m1}}K_{\mathrm{me3}}k^2)]\,D^4W$$

$$-\left[\frac{\mu H_0}{\rho_0}(K_{\mathrm{m3}}+K_{\mathrm{me3}})k^2+\frac{\mu H_0}{\rho_0}k^2(K_{\mathrm{me1}}\right.$$

$$\left.+K_{\mathrm{m1}})\right]D^3W+\frac{\mu}{\rho_0}H_0k^4(K_{\mathrm{me1}}+K_{\mathrm{m1}})DW$$

$$+g\frac{K_{\mathrm{e3}}}{K_{\mathrm{h3}}}k^2\alpha\frac{\mathrm{d}T_0}{\mathrm{d}Z}W=0, \text{ at } Z=0,\ H. \tag{22}$$

The conditions (21) and (22) are obtained from the momentum and magnetic induction equations using the boundary conditions on magnetic field. Since the boundary conditions (21) and (22) are complicated it is mathematically difficult to discuss the propagation of waves analytically. However, when the induced magnetic field is small compared with the other terms, which is true for weakly conducting fluids, the governing wave equation and the corresponding boundary conditions reduce to simpler form and is analytically tractable. In the remaining part of this paper we consider this situation. Under this approximation the induced magnetic field $\bar{\mathbf{h}}+\mathbf{h}'$ can be neglected compared with the applied magnetic field H_0. Then $\overline{h_i'h_j'}$ term in Equation (8) and $\frac{\partial \bar{h}_i}{\partial t}$, $\overline{q_j'h_i'}$ and $h_j'q_i'$ terms in Equation (11) are negligibly small compared to the other terms (see Nihoul [7]). Now the governing wave equation takes the form

$$\left[\left(K_{e1}k^2 - K_{e3}\frac{\partial^2}{\partial Z^2}\right)\left(\omega + K_{f1}k^2 - k_{f3}\frac{\partial^2}{\partial Z^2}\right)\right.$$

$$\left. - \frac{\mu H_0^2}{\rho_0}\frac{\partial^2}{\partial Z^2}\right]\left(\omega + K_{h1}k^2 - K_{h3}\frac{\partial^2}{\partial Z^2}\right)\left(\frac{\partial^2}{\partial Z^2} - k^2\right)W$$

$$- k^2 g\alpha\frac{\partial T_0}{\partial Z}\left(K_{e1}k^2 - K_{e3}\frac{\partial^2}{\partial Z^2}\right)W = 0, \quad (23)$$

and the boundary conditions are

$$\frac{\partial^{2N}W}{\partial z^{2N}} = 0 \text{ for } N = 0,1,2,3 \text{ at } z = 0, H \quad (24)$$

with $K_{ej} = \nu_m$, $j = 1,2,3$.

3. WAVE SOLUTION

Equations (23) and (24), made dimensionless using

$$z' = Z/H,\ a = kH,\ K_f = ⅓\sum_{j=1}^{3} K_{fj},\ \sigma = \frac{\omega H^2}{K_f},$$

take the form

$$[(P_{21}a^2 - P_{23}D^2)(\sigma + P_{31}a^2 - P_{33}D^2) - A^2D^2](\sigma + P_{11}a^2 - P_{13}D^2)(D^2 - a^2)W$$

$$- a^2R(P_{21}a^2 - P_{23}D^2)W = 0, \quad (25)$$

where

$$P_{ij} = K_f^{-1}(K_{fj}\delta_{i3} + K_{ej}\delta_{i2} + K_{hj}\delta_{i1} + K_f\delta_{i0}),$$

$$A^2 = \left(\frac{\mu H_0^2}{\rho_0}\right)\bigg/\left(\frac{K_f}{H}\right)^2,$$

$$R = \left[\frac{\alpha g\left(\frac{dT_0}{dZ}\right)H^4}{K_f^2}\right]$$ is the modified Rayleigh number,

and $D = \frac{d}{dz'}$.

The corresponding boundary conditions are

$$D^{2N}W = 0 \text{ for } N = 0,\ 1,2,3 \text{ at } z' = 0,\ 1 \quad (26)$$

Since Equation (25) has constant coefficients, the solution of it satisfying the boundary conditions (26) is of the form

$$W = \text{const. } \sin(n\pi z') \text{ with } n = 1,\ 2,\ \ldots\ldots \quad (27)$$

Substituting (27) in (25), we get the dispersion relation of the form

$$\sigma^2 + \left(\eta_1 + \eta_3 + \frac{A^2n^2\pi^2}{\eta_2}\right)\sigma + \left(\eta_3\eta_1 + \frac{a^2R}{\eta_0} + \eta_1\frac{A^2n^2\pi^2}{\eta_2}\right) = 0, \quad (28)$$

where

$\eta_f = p_{f3}n^2\pi^2 + p_{f1}\,a^2$, $f = 0,1,2,3$.

Equation (28) has complex roots

$$\sigma = -½\left(\eta_1 + \eta_3 + \frac{A^2n^2\pi^2}{\eta_2}\right) \pm i\left[\frac{a^2R}{\eta_0} - ¼\left(\eta_1 - \eta_3 - \frac{A^2n^2\pi^2}{\eta_2}\right)^2\right]^{½} \quad (29)$$

In the absence of a magnetic field i.e. $A \to 0$, Equation (29) reduces to that obtained [6] for ordinary viscous flow. The real part of (29) is always negative and hence the waves are always damped. It is clear that the effect of a magnetic field is to damp the waves. The system has oscillatory solution only when σ is complex. The imaginary part of σ contains the terms arising from R and fractional processes. From (29) we observe that the frequency of the damped waves is smaller than that of the undamped waves ($\eta_1 = \eta_3 = 0$). The frequency of the waves is also reduced in the presence of a magnetic field. If

$$\frac{\left(\eta_1 - \eta_3 - \frac{A^2n^2\pi^2}{\eta_2}\right)^2\eta_0}{4a^2R} \geqslant 1, \quad (30)$$

σ is always real and only critically damped motions exist. We can characterize the damping rate by a time-constant τ, in which the wave amplitude decreases by a factor 1/e where

$$\tau = \frac{2H^2}{K_f\left(\eta_1 + \eta_3 + \dfrac{A^2 n^2 \pi^2}{\eta_2}\right)} \quad (31)$$

Comparing this with the one given by LeBlond [6] for viscous flow, we observe that the damping rate is much larger in the presence of a magnetic field than that in the case of ordinary viscous flows. This is similar to that obtained by Rudraiah [11] in the case of waves in turbulent flow through porous media.

Since the velocity of the wave is also affected by mixing, friction, and the magnetic field; it is more meaningful to compare the distance x_e travelled by the wave during the time τ with the wavelength $L(=2\pi/k)$:

$$\frac{x_e}{L} = \left[\frac{a}{\left(\eta_1 + \eta_3 + \dfrac{A^2 n^2 \pi^2}{\eta_2}\right)}\right]\left(\frac{R}{\eta_0}\right)^{1/2} \times \left[1 - \frac{\left(\eta_3 - \eta_1 + \dfrac{A^2 n^2 \pi^2}{\eta_2}\right)^2 \eta_0}{4a^2 R}\right]^{1/2} \quad (32)$$

Physically this represents, for small dissipation rates, the ratio of the average energy content of the wave to the amount of energy lost per cycle through the turbulent interaction.

To understand the influence of dissipative processes on the frequency of internal waves, we have drawn in Figure 1 the values of

$$\frac{\left(\eta_3 - \eta_1 + \dfrac{A^2 n^2 \pi^2}{\eta_2}\right)^2 \eta_0}{4a^2 R}$$

against the wave number k for different values of K_{f1} and H with $\frac{\Delta\rho_0}{\rho_0} = 10^{-3}$, $K_{f3} = 10^2 \mathrm{cm^2 s^{-1}}$ and $\frac{\eta_1}{\eta_3} = 0.1$. From the Figure we observe that as K_{f1} increases the value of

$$\left[\frac{\left(\eta_3 - \eta_1 + \dfrac{A^2 n^2 \pi^2}{\eta_2}\right)^2 \eta_0}{4a^2 R}\right]$$

increases and hence frequency decreases. In other words, as the eddy viscosity increases the frequency

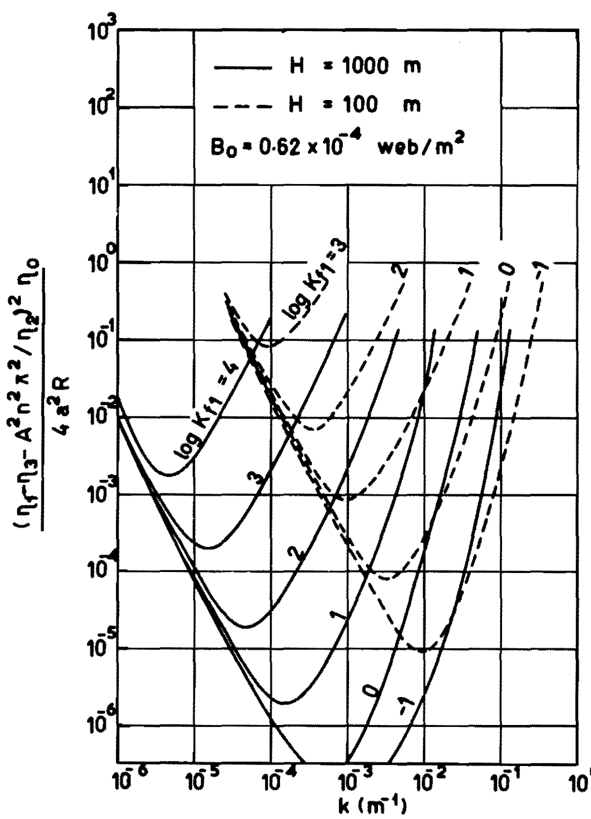

Figure 1. The Influence of Eddy Viscosity on the Frequency of Internal Waves.

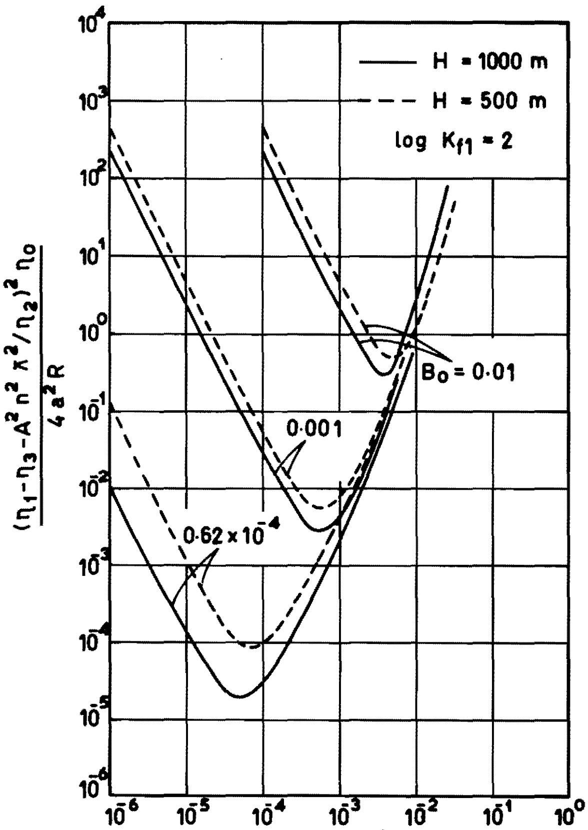

Figure 2. The Influence of Magnetic Field on the Frequency.

of the wave decreases. Also if the depth of the fluid decreases, the frequency of the wave decreases. From Figure 1, we also observe that the effect of eddy viscosity is very small on the frequency of long waves. Figure 2 depicts the effect of magnetic field on the frequency. As the magnetic field increases we observe that the frequency of the internal waves decreases. Figures 3 and 4 show the effect of eddy viscosity and magnetic field respectively on the rate of dissipation of internal waves. These figures reveal that as the eddy viscosity or magnetic field increases the waves are dissipated faster. One important conclusion is that the variation of depth and the magnetic field has no significant effect on short waves and the dissipation time decreases rapidly for large wavenumbers. This means that the short waves are damped strongly for all magnetic field and depths of the fluid. From Figures 5 and 6 we observe that the damping length decreases as the eddy viscosity and magnetic field increases. Also the dependence of

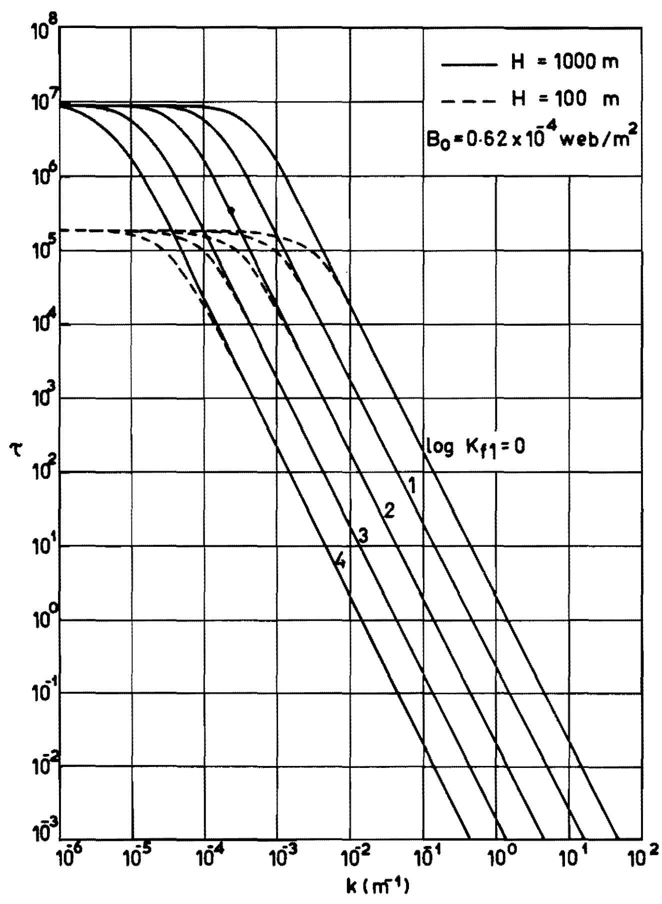

Figure 3. The Influence of Magnetic Field on the Rate of Dissipation of Internal Waves.

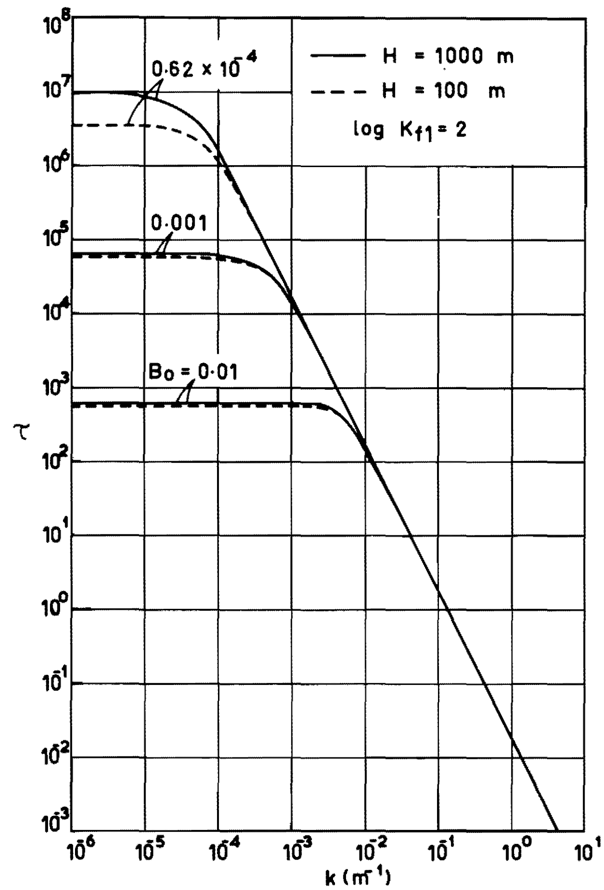

Figure 4. The Influence of Eddy Viscosity on the Rate of Dissipation of Internal Waves.

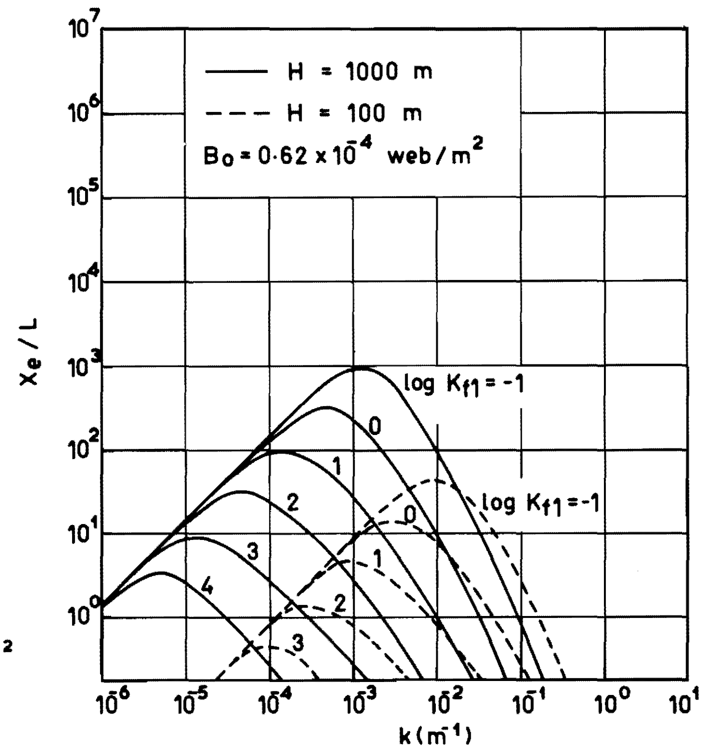

Figure 5. The Influence of Eddy Viscosity on Damping Length.

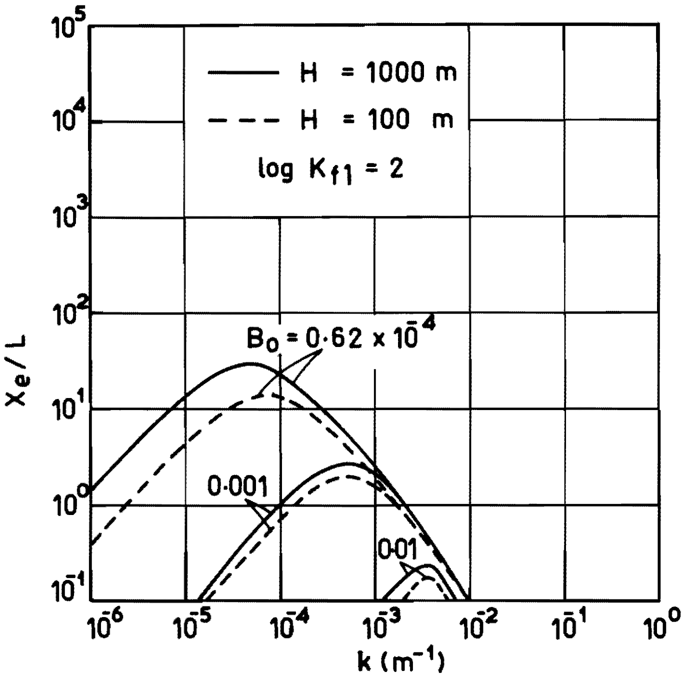

Figure 6. The Influence of Eddy Viscosity on Damping Length.

relative damping length on the fluid layer is in opposite direction for long and short waves. Further, as the depth decreases, the damping length decreases. From Figures 1 and 5 we observe that the period increase and relative damping length present a minimum and maximum respectively at the intermediate wavelength, the positions of these extrema depend on magnetic field, the depth of the fluid layer and eddy viscosity.

4. CONCLUSIONS

We have studied the propagation behavior of internal gravity waves, through turbulent mixing, in a stratified, weakly conducting fluid in the presence of a magnetic field. To study the behavior of waves, the differential model based on eddy diffusivity concept is used. We find that the short waves are damped strongly for all depths of the fluid and for all strengths of the magnetic field. Also the long waves decay faster in shallow depths and the effect of increase in depth has no significant effect on the attenuation of short waves but changes markedly in long waves. However, the waves are more damped due to the presence of a magnetic diffusive term ($A^2n^2\pi^2/\eta_2$) (this may be indicated by the shift of maximum frequency upwards and the steepness of the curves).

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