THE GENERALIZED MEHLER–FOCK TRANSFORMATION OF DISTRIBUTIONS

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الخلاصة :

عمّم تحويل ميلر وفوك المعمم

$$F(r) := \int_0^\infty f(x) P^{m,n}_{-(1/2)+ir}(\cosh x) \sinh x \, dx$$

ليشمل مجموعة الدوال المعممة حيث $P_k^{m,n}(z)$ هي دالة ليجندر المعممة ، وقد استنتجت نظرية عكسية بتفسير التقارب في حدود التوزيع الضعيف . طورت النظرية وطبقت على مسألة ديرشلت بشروط حدية توزيعية .

ABSTRACT

The generalized Mehler–Fock transformation

$$F(r) := \int_0^\infty f(x) P^{m,n}_{-(1/2)+ir}(\cosh x) \sinh x \, dx,$$

where $P_k^{m,n}(z)$ denotes the generalized Legendre function, is extended to a class of generalized functions. An inversion theorem is established by interpreting convergence in the weak distributional sense. The theory thus developed is applied to a Dirichlet problem with distributional boundary conditions.

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THE GENERALIZED MEHLER–FOCK TRANSFORMATION OF DISTRIBUTIONS

1. INTRODUCTION

The classical Mehler–Fock transformation has been successfully applied to deal with problems occurring in the mathematical theory of elasticity, particularly those concerned with analysis of stress in the vicinity of external cracks.

A generalization of the Mehler–Fock transformation has been given by Braaksma and Meulenbeld [1] in the following form:

$$f^*(r) := \int_0^\infty f(x) P^{m,n}_{-(1/2)+ir}(\cosh x) \sinh x \, dx, \tag{1.1}$$

where $P^{m,n}_{-1/2+ir}(\cosh x)$ is the generalized Legendre function defined by

$$P^{m,n}_k(z) = \frac{(z+1)^{n/2}}{\Gamma(1-m)(z-1)^{m/2}} F\left[k+\frac{n-m}{2}+1;\ -k+\frac{n-m}{2};\ 1-m;\ \frac{1-z}{2}\right] \tag{1.2}$$

for z not lying on the cross-cut along the real x-axis from 1 to $-\infty$ for complex values of the parameters k, m, and n. The corresponding inversion formula is

$$f(x) = \int_0^\infty \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x) f^*(r) dr, \tag{1.3}$$

where

$$\chi(r) = \Gamma\left(\frac{1-m+n}{2}+ir\right)\Gamma\left(\frac{1-m+n}{2}-ir\right)\Gamma\left(\frac{1-m-n}{2}+ir\right)\Gamma\left(\frac{1-m-n}{2}-ir\right)\times$$

$$[\Gamma(2ir)\Gamma(-2ir)\pi 2^{n-m+2}]^{-1}. \tag{1.4}$$

Note that Equation (1.1) reduces to the generalized Mehler–Fock transform when $m=n$, and to the Mehler–Fock transform when $m=n=0$ (see [2]).

The conditions of validity for Equations (1.1) and (1.3) are provided by the following theorem due to Braaksma and Meulenbeld (see [1], p. 245).

Theorem 1.1

Let m, n be complex numbers with $|\mathrm{Re}\ n| < 1 - \mathrm{Re}\ m$, and $f(t)$ a function such that for all $a > 1$

(i) $f(t)(t-1)^{-1/4}\log(t-1) \in L(1,a)$ if $\mathrm{Re}\ m = 0$;
(ii) $f(t)t^{-1/2} \in L(a,\infty)$.

Further, let this function be of bounded variation in a neighborhood of $t = x\ (x > 1)$.

Then $f(t)$ satisfies the relation

$$\int_0^\infty \chi(r) P^{m,n}_{-(1/2)+ir}(x) dr \int_1^\infty P^{m,n}_{-(1/2)+ir}(t) f(t) dt = \tfrac{1}{2}\{f(x-0)+f(x+0)\}. \tag{1.5}$$

In this paper, we extend this transformation to a class of generalized functions and prove the inversion theorem by interpreting convergence in the weak distributional sense. In the end, we develop an operational calculus that is applied to solve a certain boundary value problem. (The aforesaid transformation with $n=0$ was extended to generalized functions by Buggle [3] and the case $m=n=0$ was treated by Tiwari and Pandey [4].)

We make use of the following integral representation in our subsequent analysis.

$$P^{m,n}_{-(1/2)+ir}(\cosh t)=\frac{2^{(n-m+1)/2}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-m)}i^{m+1}\sinh^m t\int_0^t\frac{\cos r\phi}{(\cosh t-\cosh\phi)^{m+1/2}}F\left[\frac{n-m}{2},-\frac{n+m}{2};\tfrac{1}{2}-m;\frac{\cosh t-\cosh\phi}{1+\cosh t}\right]d\phi, \tag{1.6}$$

where $\operatorname{Re} m<\frac{1}{2}$. This can be obtained from the representation

$$F[a,b;c;z]=\frac{\Gamma(c)}{\Gamma(c-\mu)\Gamma(\mu)}\int_0^1 t^{\mu-1}(1-t)^{c-\mu-1}(1-tz)^{\lambda-a-b}F[\lambda-a,\lambda-b;\mu;tz]\,F\left[a+b-\lambda,\lambda-\mu;c-\mu;\frac{(1-t)z}{1-tz}\right]dt,$$

where

$$\operatorname{Re}c>\operatorname{Re}\mu>0,\quad z\neq 1,\quad |\arg(1-z)|<\pi,$$

on using

$$F[a,b;c;z]=(1-z)^{-a}F\left[a,c-b;c;\frac{z}{(z-1)}\right].$$

From Equation (1.6) we conclude that

$$|P^{m,n}_{-(1/2)+ir}(\cosh t)|\leqslant|P^{m,n}_{-1/2}(\cosh t)|,\quad \operatorname{Re} m<1/2. \tag{1.7}$$

From Equation (1.2) we have

$$P^{m,n}_{-(1/2)+ir}(\cosh t)=O(t^{-\operatorname{Re} m}),\quad t\to 0+. \tag{1.8}$$

Also, Equation (1.2), together with Equation (9) on p. 76 of [5], yields

$$P^{m,n}_{-(1/2)+ir}(\cosh t)=O(e^{-(1/2)t}),\quad t\to\infty. \tag{1.9}$$

Lastly, from Equation (1.2), and Equation (17) on p. 77 of [5], we obtain

$$P^{m,n}_{-(1/2)+ir}(\cosh t)=O(1),\quad r\to 0+ \tag{1.10}$$

$$=2^{(1/2)(n-m-1)}\pi^{-1/2}(\sinh t)^{-1/2}(ir)^{m-(1/2)}\times\{e^{irt}+ie^{-i(m\pi+rt)}+O(r^{-1})\},\quad r\to+\infty. \tag{1.11}$$

The function $\chi(r)$ defined by Equation (1.4) possesses the following asymptotic behavior.

$$\chi(r)=\begin{cases}O(r^2), & r\to 0+,\quad |\operatorname{Re} n|<1-\operatorname{Re} m & (1.12)\\ \dfrac{(ir)^{1-2m}}{\pi\, 2^{n-m+2}}[1+O(r^{-1})], & r\to\infty. & (1.13)\end{cases}$$

2. THE TEST FUNCTION SPACE $M^{\alpha}_{\beta}(I)$ AND ITS DUAL

Let I denote the open interval $(0,\infty)$. For a real number $\alpha\geqslant\operatorname{Re} m$ and real number $\beta\leqslant\frac{1}{2}$, let ζ be a continuous positive function on I such that

$$\zeta(t):=\zeta_{\alpha,\beta}(t):=\begin{cases}O(t^{\alpha}) & \text{as } t\to 0+\\ O(e^{\beta t}) & \text{as } t\to\infty.\end{cases}$$

Let $M^{\alpha}_{\beta}(I)$ be the collection of all infinitely differentiable complex valued functions ϕ defined on I such that for every non-negative integer k,

$$\gamma_k(\phi):=\sup_{0<t<\infty}|\zeta(t)\Delta_t^k\phi(t)|<\infty$$

where

$$\Delta_t^k = \left(D^2 + (\coth t)D + \frac{m^2}{2(1-\cosh t)} + \frac{n^2}{2(1+\cosh t)} \right)^k$$

and

$$D = \frac{d}{dt}.$$

One can readily see that $M_\beta^\alpha(I)$ is a linear space and γ_k, $k=0,1,2,\ldots$, is a seminorm while γ_0 is a norm. Therefore, the collection of seminorms $\{\gamma_k\}$, $k=0,1,2,\ldots$, is separating (see p. 8 of [6]). We equip $M_\beta^\alpha(I)$ with the topology generated by the seminorms $\{\gamma_k\}_{k=0}^\infty$. A sequence $\{\phi_\nu\}_{\nu=1}^\infty$ in $M_\beta^\alpha(I)$ converges to ϕ in $M_\beta^\alpha(I)$ if and only if for each k, $\gamma_k(\phi_\nu - \phi) \to 0$ as $\nu \to \infty$. A sequence $\{\phi_\nu\}_{\nu=1}^\infty$ is said to be a Cauchy sequence if for each k, $\gamma_k(\phi_\nu - \phi_\mu) \to 0$ as ν and μ both tend to infinity independently of each other. Following the technique of Pandey [7] it can be shown that $M_\beta^\alpha(I)$ is a sequentially complete locally convex topological vector space. D(I), the space of infinitely differentiable functions of compact support with the usual topology, is a linear subspace of $M_\beta^\alpha(I)$. The topology of D(I) is stronger than the topology induced on it by $M_\beta^\alpha(I)$. Hence the restriction of any $f \in M_\beta^{\alpha\prime}(I)$ to D(I) is in D$'(I)$, the dual space of D(I).

For $r \geq 0$, the generalized Legendre function $P_{-(1/2)+ir}^{m,n}(\cosh t)$ is an element of $M_\beta^\alpha(I)$, for $P_{-(1/2)+ir}^{m,n}(\cosh t)$ satisfies the differential equation

$$D^2 y + (\coth t)Dy + \left[\frac{m^2}{2(1-\cosh t)} + \frac{n^2}{2(1+\cosh t)} + (r^2 + \tfrac{1}{4}) \right] y = 0.$$

Therefore

$$\Delta_t P_{-(1/2)+ir}^{m,n}(\cosh t) = -(r^2 + \tfrac{1}{4}) P_{-(1/2)+ir}^{m,n}(\cosh t). \tag{2.1}$$

Using (2.1) we get

$$\gamma_k[P_{-(1/2)+ir}^{m,n}(\cosh t)] = \sup_{0<t<\infty} |\zeta(t) \Delta_t^k P_{-(1/2)+ir}^{m,n}(\cosh t)| \leq \sup_{0<t<\infty} |(r^2 + \tfrac{1}{4})^k \zeta(t) P_{-(1/2)+ir}^{m,n}(\cosh t)|.$$

Now, using Equations (1.2), (1.8), and (1.9) we have

$$\gamma_k[P_{-(1/2)+ir}^{m,n}(\cosh t)] < \infty. \tag{2.2}$$

3. THE DISTRIBUTIONAL GENERALIZED MEHLER–FOCK TRANSFORMATION

For $f \in M_\beta^{\alpha\prime}(I)$ where $\alpha \geq \mathrm{Re}\, m$ and $\beta \leq \frac{1}{2}$, define the distributional generalized Mehler–Fock transformation F of f by

$$F(r) := \langle f(t), P_{-(1/2)+ir}^{m,n}(\cosh t) \rangle, \tag{3.1}$$

where $r \geq 0$.

Lemma 3.1 For $\mathrm{Re}\, m < \frac{1}{2}$, $\alpha \geq \mathrm{Re}\, m$ and $\beta < \frac{1}{2}$, the functions $\zeta(t)(\partial/\partial r)^q P_{-(1/2)+ir}^{m,n}(\cosh t)$, $q=0,1,2,\ldots$ are uniformly bounded over $0<t<\infty$.

Proof. For $0 \leq \phi \leq t$, we have

$$\left| F\left(\frac{n-m}{2}, \frac{-(m+n)}{2}; \tfrac{1}{2} - m; \frac{\cosh t - \cosh \phi}{1 + \cosh t} \right) \right| \leq \sum_{k=0}^\infty \left| \frac{\left(\frac{n-m}{2}\right)_k \left[-\left(\frac{m+n}{2}\right) \right]_k}{k!\, (\frac{1}{2} - m)_k} \right| \left(\frac{\cosh t - 1}{\cosh t + 1} \right)^k$$

$$\leqslant \sum_{k=0}^{\infty} \left| \frac{\Gamma(\frac{1}{2}-m)}{\Gamma\left(\frac{n-m}{2}\right)\Gamma\left[-\left(\frac{m+n}{2}\right)\right]} k^{-3/2}[1+O(k^{-1})] \right| \leqslant \left| \frac{\Gamma(\frac{1}{2}-m)}{\Gamma\left(\frac{n-m}{2}\right)\Gamma\left[-\left(\frac{m+n}{2}\right)\right]} \right| \sum_{k=0}^{\infty} k^{-3/2}[1+O(k^{-1})] < \infty.$$

Therefore, from Equation (1.6) by differentiating with respect to r within the integral sign, we have

$$\left| \left(\frac{\partial}{\partial r}\right)^q P^{m,n}_{-(1/2)+ir}(\cosh t) \right| \leqslant C(\sinh t)^{m_r} \int_0^t \frac{\phi^q \mathrm{d}\phi}{(\cosh t - \cosh \phi)^{m_r+(1/2)}},$$

where C is a constant independent of t and r, and m_r denotes $\mathrm{Re}\, m$.

Now, using the integral representation given by Equation (8) on p. 156 of [5]:

$$P^{\mu}_{\nu}(\cosh t) = P^{\mu,0}_{\nu}(\cosh t) = (\tfrac{1}{2}\pi)^{-1/2} \frac{(\sinh t)^{\mu}}{\Gamma(\frac{1}{2}-\mu)} \int_0^t (\cosh t - \cosh \phi)^{-\mu-(1/2)} \cosh[(\nu+\tfrac{1}{2})\phi] \mathrm{d}\phi,$$

where $\mathrm{Re}\,\mu < \frac{1}{2}$, we have

$$\left| \zeta(t) \left(\frac{\partial}{\partial r}\right)^q P^{m,n}_{-(1/2)+ir}(\cosh t) \right| \leqslant C\zeta(t) t^q (\tfrac{1}{2}\pi)^{1/2}\, \Gamma(\tfrac{1}{2}-m_r) P^{m_r,0}_{-1/2}(\cosh t).$$

In view of the asymptotic estimates (1.8) and (1.9), the right-hand side is bounded uniformly for all $t \in (0, \infty)$ provided that

$$\alpha \geqslant \mathrm{Re}\, m, \quad \mathrm{Re}\, m < \tfrac{1}{2}, \quad \text{and} \quad \beta < \tfrac{1}{2}.$$

Theorem 3.2

For $f \in M^{\alpha'}_{\beta}(I)$, where $\alpha \geqslant \mathrm{Re}\, m$, $\mathrm{Re}\, m < \frac{1}{2}$, $\beta < \frac{1}{2}$, and $r \geqslant 0$, let $F(r)$ be defined by Equation (3.1). Then

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^q F(r) = \left\langle f(t), \left(\frac{\partial}{\partial r}\right)^q P^{m,n}_{-(1/2)+ir}(\cosh t) \right\rangle, \quad q = 0, 1, 2, \ldots \tag{3.2}$$

Proof. The proof is standard (see p. 30 of [8]).

Theorem 3.3

For $f \in M^{\alpha'}_{\beta}(I)$, $\alpha \geqslant \mathrm{Re}\, m$, $\mathrm{Re}\, m < \frac{1}{2}$, $\beta < \frac{1}{2}$, let $F(r)$ be defined by Equation (3.1). Then

$$F(r) = \begin{cases} O(1) & \text{as } r \to 0+ \\ O(r^{2q}) & \text{as } r \to \infty, \end{cases} \tag{3.3}$$

where q is a non-negative integer depending on f.

Proof. The proof is given by using the boundedness property of generalized functions (see p. 18 of [6]). Indeed

$$\begin{aligned} |F(r)| &\leqslant C \max_{0 \leqslant k \leqslant q} \gamma_k (P^{m,n}_{-(1/2)+ir}(\cosh t)) \\ &\leqslant C \max_k \sup_{0<t<\infty} |(r^2+\tfrac{1}{4})^k \zeta(t) P^{m,n}_{-(1/2)+ir}(\cosh t)| \\ &\leqslant C \max_k \sup_{0<t<\infty} |(r^2+\tfrac{1}{4})^k \zeta(t) P^{m_r,n}_{-1/2}(\cosh t)| \\ &\leqslant C' \max_k (r^2+\tfrac{1}{4})^k, \end{aligned}$$

from which the result follows.

Theorem 3.4

Let $f \in M_\beta^{\alpha'}(I)$, where $\alpha \geqslant \operatorname{Re} m$, $\operatorname{Re} m < \frac{1}{2}$, and $\beta < \frac{1}{2}$. Then for a fixed real number N,

$$\int_0^N \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh t) \langle f(x), P^{m,n}_{-(1/2)+ir}(\cosh x)\rangle \, dr$$

$$= \langle f(x), \int_0^N \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t)\, dr \rangle. \tag{3.4}$$

Proof. The expression (3.4) is meaningful since the integral on the right-hand side belongs to $M_\beta^\alpha(I)$. Indeed, for fixed $t > 0$, let

$$\theta(x, t) := \int_0^N \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t)\, dr.$$

Then

$$\Delta_x^k[\theta(x,t)] = \int_0^N \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh t) \Delta_x^k [P^{m,n}_{-(1/2)+ir}(\cosh x)]\, dr$$

$$= (-1)^k \int_0^N \chi(r)(r^2 + \tfrac{1}{4})^k P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t)\, dr.$$

Therefore

$$\sup_{0<x<\infty} |\zeta(x) \Delta_x^k[\theta(x,t)]| = \sup_{0<x<\infty} \left| \zeta(x) \int_0^N \chi(r)(r^2+\tfrac{1}{4})^k P^{m,n}_{-(1/2)+ir}(\cosh t) P^{m,n}_{-(1/2)+ir}(\cosh x)\, dr \right|$$

$$\leqslant C \int_0^N \left| P^{m,n}_{-1/2}(\cosh t) \Gamma\left(\frac{1-m+n}{2} + ir\right) \Gamma\left(\frac{1-m+n}{2} - ir\right) \Gamma\left(\frac{1-m-n}{2} + ir\right) \Gamma\left(\frac{1-m-n}{2} - ir\right) r \sinh 2\pi r (r^2 + \tfrac{1}{4})^k \right| dr$$

$< \infty$.

The fact that both sides of Equation (3.4) are equal can be proved by the Riemann-sums technique (see p. 186 of [6]).

4. INVERSION OF THE DISTRIBUTIONAL GENERALIZED MEHLER–FOCK TRANSFORMATION

For x, t, and y in $I = (0, \infty)$ and $N > 0$, define

(a) $G_N(t, x) := \int_0^N \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t) \sinh t\, dr;$

(b) $\psi_N(y, x) := \int_0^N \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x) R^{m,n}_{-(1/2)+ir}(y)\, dr,$

where

$$R^{m,n}_{-(1/2)+ir}(y) := \int_0^y \sinh t\, P^{m,n}_{-(1/2)+ir}(\cosh t)\, dt, \quad \operatorname{Re} m < 2.$$

Lemma 4.1 Let $|\operatorname{Re} n| < 1 - \operatorname{Re} m$. Then for fixed $y > 0$,

$$\lim_{N\to\infty} \psi_N(y, x) = \begin{cases} 1 & \text{for} \quad 0 < x < y \\ \frac{1}{2} & \text{for} \quad x = y \\ 0 & \text{for} \quad x > y. \end{cases}$$

Proof. In Theorem 1.1 we can set

$$f(\cosh t)=\begin{cases}1 & \text{for} \quad 0<t<y\\ 0 & \text{for} \quad t>y,\end{cases}$$

and get

$$\int_0^\infty P^{m,n}_{-(1/2)+ir}(\cosh t)\, f(\cosh t)\sinh t\,dt$$

$$=\int_0^y P^{m,n}_{-(1/2)+ir}(\cosh t)\sinh t\,dt$$

$$=R^{m,n}_{-(1/2)+ir}(y).$$

Now, using inversion Theorem 1.1 we get

$$\int_0^\infty \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh x)\, R^{m,n}_{-(1/2)+ir}(y)\,dr=\begin{cases}1 & \text{for} \quad 0<x<y\\ \frac{1}{2} & \text{for} \quad x=y\\ 0 & \text{for} \quad x>y.\end{cases}$$

The left-hand side of the equation is nothing but the limit of $\psi_N(y,x)$ as $N\to\infty$.

Lemma 4.2 Let a and b be any two real numbers satisfying $0<a<b$ and let $|\mathrm{Re}\, n|<1-\mathrm{Re}\, m$. Then

$$\lim_{N\to\infty}\int_a^b G_N(t,x)\,dt=\begin{cases}1 & \text{for} \quad x\in(a,b)\\ 0 & \text{for} \quad x\notin[a,b]\\ \frac{1}{2} & \text{for} \quad x=a,b.\end{cases}$$

Proof. We have

$$\int_a^b G_N(t,x)\,dt=\int_0^N \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh x)\,dr\int_a^b P^{m,n}_{-(1/2)+ir}(\cosh t)\sinh t\,dt$$

$$=\int_0^N \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh x)[R^{m,n}_{-(1/2)+ir}(b)-R^{m,n}_{-(1/2)+ir}(a)]\,dr$$

$$=\psi_N(b,x)-\psi_N(a,x).$$

Therefore, in view of Lemma 4.1, we obtain

$$\lim_{N\to\infty}\int_a^b G_N(t,x)\,dt=\begin{cases}1 & \text{for} \quad x\in(a,b)\\ 0 & \text{for} \quad x\notin[a,b]\\ \frac{1}{2} & \text{for} \quad x=a,b.\end{cases}$$

In the rest of this work, L will denote an arbitrarily large but fixed positive number.

Lemma 4.3 Let $b>a>0$ and let $|\mathrm{Re}\,n|<1-\mathrm{Re}\,m$. Then, for a fixed δ satisfying $0<\delta<\frac{1}{4}a$,

$$\text{(i)} \quad \int_a^{x-\delta} G_N(t,x)\,dt\to 0$$

uniformly for all $x\in(a,L]$ as $N\to\infty$;

$$\text{(ii)} \quad \int_{x+\delta}^b G_N(t,x)\,dt\to 0$$

uniformly for all $x\in[\delta,b-\delta]$ as $N\to\infty$.

Proof of (i). In view of the asymptotic behavior of Equations (1.10)–(1.13), we have

$$\left|\int_a^{x-\delta} dt \int_0^N \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t)\sinh t\, dr\right| \leqslant \left|\int_a^{x-\delta} dt\right| \left|\int_0^1 O(r^2) Q_r(t,x) dr\right|$$

$$+\int_a^{x-\delta} dt \left|\int_1^N C_{n,m} \sinh^{1/2} t \sinh^{-1/2} x \{e^{ir(t+x)}(1-e^{-2m\pi i})+2ie^{-im\pi}\cos r(x-t)+O(r^{-1})\} dr\right|,$$

where $C_{n,m}$ is a constant and $Q_r(t,x)$ is bounded for $0\leqslant r\leqslant 1$ and $(t,x)\in[a,L-\delta]\times[a,L]$.

Now estimating the right-hand side of the above inequality we observe that the left-hand side is an absolutely convergent double integral with respect to t and r. Hence, using Fubini's theorem and changing the order of integration, we have

$$\int_a^{x-\delta} G_N(t,x)dt = \int_0^N \chi(r)dr \int_a^{x-\delta} \sinh t\, P^{m,n}_{-(1/2)+ir}(\cosh t) P^{m,n}_{-(1/2)+ir}(\cosh x)dt$$

$$= \int_N^\infty \chi(r)dr \int_a^{x-\delta} \sinh t\, P^{m,n}_{-(1/2)+ir}(\cosh t) P^{m,n}_{-(1/2)+ir}(\cosh x)dt$$

$$- \int_N^\infty \chi(r)dr \int_a^{x-\delta} \sinh t\, P^{m,n}_{-(1/2)+ir}(\cosh t) P^{m,n}_{-(1/2)+ir}(\cosh x)dt$$

$$= T_1 - T_2 \text{ (say)}.$$

Now, since both r- and t-integrals are absolutely convergent, using Fubini's theorem, we can write

$$T_1 = \lim_{N\to\infty} \int_0^N \chi(r)dr \int_a^{x-\delta} (t-x)\sinh t\, P^{m,n}_{-(1/2)+ir}(\cosh t) P^{m,n}_{-(1/2)+ir}(\cosh x)dt$$

$$= \lim_{N\to\infty} \int_a^{x-\delta} G_N(t,x)dt = 0$$

by Lemma 4.2. Also, for sufficiently large N, substituting the asymptotic expressions for $P^{m,n}_{-(1/2)+ir}(\cosh t)$, $P^{m,n}_{-(1/2)+ir}(\cosh x)$ and $\chi(r)$, we see that

$$T_2 = \int_N^\infty dr \int_a^{x-\delta} (t-x) C_{n,m} \sinh^{1/2} t \sinh^{-1/2} x \{e^{ir(t+x)}(1-e^{-2m\pi i})+2ie^{-m\pi i}\cos r(x-t)+O(r^{-1})\}dt,$$

where $C_{n,m}$ is a finite constant. Now, integrating the inner integral by parts, it is not hard to see that

$$T_2 = O(N^{-1}), \quad \text{as} \quad N\to\infty$$

uniformly for all $x\in(a,L]$. Consequently, $T_1 - T_2 \to 0$ as $N\to\infty$ uniformly for all $x\in(a,L]$.

The proof of (ii) is similar to that of (i).

Lemma 4.4 Let $|\mathrm{Re}\, n| < 1 - \mathrm{Re}\, m$ and $\mathrm{Re}\, m < \frac{1}{2}$. Then for $0<a\leqslant t\leqslant b$, $0<c\leqslant x\leqslant d$ and $N>0$, the function $G_N(t,x)$ is bounded uniformly for all t, x, and N.

Proof. Let N be any positive number less than N_1 (say). Then

$$|G_N(t,x)| \leqslant \int_0^{N_1} |\chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t)\sinh t|dr$$

$$\leqslant \sup_{c\leqslant x\leqslant d} \left|P^{m,n}_{-1/2}(\cosh x)\right| \sup_{a\leqslant t\leqslant b} \left|P^{m,n}_{-1/2}(\cosh t)\sinh t\right| \sup_{0<rN_1} |\chi(r)| \int_0^{N_1} dr$$

$$< M_1,$$

M_1 being a positive constant independent of x, t, and N.

Next assume that N is a large number greater than or equal to N_1. Then

$$|G_N(t,x)| \leqslant \int_0^{N_1} |\chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t)\sinh t|\,dr + \int_{N_1}^{N} |\chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t)\sinh t|\,dr.$$

We have already proved that the first integral is bounded by M_1.

In view of the asymptotic behavior of

$$P^{m,n}_{-(1/2)+ir}(\cosh x),\ P^{m,n}_{-(1/2)+ir}(\cosh t),\ \text{and } \chi(r)$$

for large r and for fixed $x \geqslant c$, $t \geqslant a$, we have

$$\int_{N_1}^{N} \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t)\sinh t\,dr$$

$$=\frac{1}{8\pi^2}\left(\frac{\sinh t}{\sinh x}\right)^{1/2}\int_{N_1}^{N}[e^{irx}+ie^{-i(m\pi+rx)}][e^{irt}+ie^{-i(m\pi+rt)}][1+O(1/r)]dr$$

$$=\frac{1}{8\pi^2}\left(\frac{\sinh t}{\sinh x}\right)^{1/2}\int_{N_1}^{N}[e^{ir(x+t)}+ie^{-m\pi i}e^{ir(t-x)}+ie^{-m\pi i}e^{ir(x-t)}-e^{-2m\pi i}e^{-ir(x+t)}][1+O(1/r)]dr$$

$$=\frac{1}{8\pi^2}\left(\frac{\sinh t}{\sinh x}\right)^{1/2}\int_{N_1}^{N}[e^{ir(x+t)}+ie^{-m\pi i}e^{ir(t-x)}+ie^{-m\pi i}e^{ir(x-t)}-e^{-2m\pi i}e^{-ir(x+t)}]dr$$

$$+\frac{1}{8\pi^2}\left(\frac{\sinh t}{\sinh x}\right)^{1/2}\int_{N_1}^{N}[e^{ir(x+t)}+ie^{-m\pi i}e^{ir(t-x)}+ie^{-m\pi i}e^{ir(x-t)}-e^{-2m\pi i}e^{-ir(x+t)}]O(1/r)dr$$

$$=J_1+J_2 \text{ (say).}$$

J_1 can be expressed as a sum of four integrals, each of which is separately bounded. For instance,

$$\left|\int_{N_1}^{N} e^{ir(x+t)}dr\right| = \left|\frac{e^{iN(x+t)}}{i(x+t)}-\frac{e^{iN_1(x+t)}}{i(x+t)}\right| \leqslant \frac{2}{x+t}.$$

J_2 is also a sum of four integrals, each of which is separately bounded. For instance, the first term in J_2 is

$$\frac{1}{8\pi^2}\left(\frac{\sinh t}{\sinh x}\right)^{1/2} O\left[\int_{N_1}^{N}\frac{e^{ir(x+t)}}{r}dr\right]=\frac{1}{8\pi^2}\left(\frac{\sinh t}{\sinh x}\right)^{1/2} O\left[\frac{1}{r}\frac{e^{ir(x+t)}}{i(x+t)}\Bigg|_{N_1}^{N}+\int_{N_1}^{N}\frac{1}{r^2}\frac{e^{ir(x+t)}}{i(x+t)}dr\right].$$

The first term within the square bracket is

$$\left[\frac{e^{iN(x+t)}}{iN(x+t)}-\frac{e^{iN_1(x+t)}}{iN_1(x+t)}\right],$$

which is bounded. The modulus of the second term is less than

$$\frac{1}{|x+t|}\int_{N_1}^{N}\frac{1}{r^2}dr=\frac{1}{|x+t|}\left(\frac{1}{N_1}-\frac{1}{N}\right),$$

which is also bounded. The other terms can similarly be shown to be bounded. This completes the proof of the lemma.

Lemma 4.5 Let $\phi(t) \in D(I)$ with its support contained in $[a, b]$. Then for $0 < \delta < \frac{1}{4}a$,

(i) $$\int_a^{x-\delta} G_N(t,x)\,\phi(t)\mathrm{d}t \to 0$$

as $N \to \infty$ uniformly for all $x \in [a+\delta, L]$;

(ii) $$\int_{x+\delta}^{b} G_N(t,x)\,\phi(t)\mathrm{d}t \to 0$$

as $N \to \infty$ uniformly for all $x \in [\delta, b-\delta]$.

Proof of (i). Assume at first that $\phi(t)$ is an infinitely differentiable real valued function on $[a, x-\delta]$, $a+\delta \leqslant x \leqslant L$. Then ϕ is a function of bounded variation on $[a, x-\delta]$. Consequently, there exist monotonically increasing functions $p(x)$ and $q(x)$ on $[a, x-\delta]$, with $p(a)=q(a)=0$ such that

$$\phi(t) = \phi(a) + p(t) - q(t), \quad a \leqslant t \leqslant x-\delta$$

(see Theorem 6.27 on p. 120 of [9]). Hence

$$\int_a^{x-\delta} G_N(t,x)\,\phi(t)\mathrm{d}t$$

$$= \int_a^{x-\delta} p(t) G_N(t,x)\mathrm{d}t - \int_a^{x-\delta} q(t) G_N(t,x)\,\mathrm{d}x.$$

The result can now be proved by using the second mean value theorem of the integral calculus and Lemma 4.3(i).

The proof for a complex valued C^∞ function ϕ can be given by separating it into its real and imaginary parts.

The proof of (ii) is similar to that of (i).

Lemma 4.6 Let $\phi(t) \in D(I)$ with its support contained in the interval $[a, b]$, then

$$\int_a^b G_N(t,x)\,\phi(t)\,\mathrm{d}t \to \phi(x)$$

in $M_\beta^\alpha(I)$ as $N \to \infty$, provided that

$$\alpha \geqslant \operatorname{Re} m, \quad \beta < \tfrac{1}{2}, \quad \operatorname{Re} m < \min\left(\tfrac{1}{2}, 1 - |\operatorname{Re} n|\right).$$

Proof. It can be readily seen that

$$\Delta_x\, G_N(t,x) = \sinh t\, \Delta_t\, G_N^*(t,x)$$

where

$$G_N^*(t,x) = \int_0^N \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh x)\, P^{m,n}_{-(1/2)+ir}(\cosh t)\,\mathrm{d}r.$$

Now

$$\Delta_x \int_a^b G_N(t,x)\,\phi(t)\,\mathrm{d}t = \int_a^b \sinh t\, \Delta_t\, G_N^*(t,x)\,\phi(t)\,\mathrm{d}t$$

$$= \int_a^b G_N(t,x)\,\Delta_t\,[\phi(t)]\,\mathrm{d}t \text{ (by integration by parts).}$$

Therefore, operating Δ_x successively k times, we get

$$\Delta_x^k \int_a^b G_N(t,x)\,\phi(t)\,\mathrm{d}t = \int_a^b G_N(t,x)\,\phi_k(t)\,\mathrm{d}t,$$

where

$$\phi_k(t) = \Delta_t^k \phi(t).$$

Now, using Lemma 4.2 we can write

$$\lim_{N\to\infty} \zeta(x)\Delta_x^k \left[\int_a^b G_N(t,x)\,\phi(t)\,\mathrm{d}t - \phi(x) \right]$$

$$= \lim_{N\to\infty} \zeta(x) \int_a^b G_N(t,x)\,[\phi_k(t) - \phi_k(x)]\,\mathrm{d}t.$$

It is therefore reduced to proving that

$$\zeta(x) \int_a^b G_N(t,x)\,[\psi(t) - \psi(x)]\,\mathrm{d}t \to 0$$

uniformly for all x as $N\to\infty$ where $\psi(t)\in \mathrm{D}(I)$ with its support contained in $[a,b]$.

For a fixed $x \geqslant 2\delta$, where $0<\delta<\min(\frac{1}{2}, \frac{1}{4}a)$, we can write

$$\zeta(x) \int_a^b G_N(t,x)\,[\psi(t) - \psi(x)]\,\mathrm{d}t$$

$$= \zeta(x) \left(\int_a^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{b} \right) G_N(t,x)[\psi(t) - \psi(x)]\,\mathrm{d}t$$

$$= I_1 + I_2 + I_3 \text{ (say)}.$$

At first we consider I_2. For $x \geqslant b+\delta$ or $x \leqslant a-\delta$, I_2 is clearly zero. Therefore we consider I_2 for the case when $a-\delta<x<b+\delta$. We can write

$$|I_2| \leqslant \zeta(x) \int_{x-\delta}^{x+\delta} |G_N(t,x)|\,|\psi(t) - \psi(x)|\,\mathrm{d}t$$

$$\leqslant \delta\zeta(x) \sup_{a\leqslant\eta\leqslant b} |\psi'(\eta)| \int_{x-\delta}^{x+\delta} |G_N(t,x)|\,\mathrm{d}t$$

$$\leqslant \delta \mathrm{D}_1 \sup_{a\leqslant\eta\leqslant b} |\psi'(\eta)| \sup_{\substack{(3/4)a\leqslant x\leqslant b+(1/2)a \\ (1/2)a\leqslant t\leqslant b+(1/2)a}} |G_N(t,x)|.$$

Now using Lemma 4.4, we can find a constant $\mathrm{D}>0$ independent of δ such that

$$|I_2| < \mathrm{D}\delta.$$

For a given $\varepsilon>0$, we can choose $\delta = \min(\frac{1}{4}a, \frac{1}{2}, \varepsilon/\mathrm{D})$ and obtain

$$|I_2| < \frac{\varepsilon}{2}. \tag{4.1}$$

Next, consider

$$I_1 = \zeta(x) \int_a^{x-\delta} G_N(t,x)\,[\psi(t) - \psi(x)]\,\mathrm{d}t$$

$$= I_{1,1} - I_{1,2} \text{ (say)}.$$

Now, $I_{1,2}=0$ if $x \leqslant a$ and $x \geqslant b$. For $x\in(a,b)$,

$$|I_{1,2}| \leqslant \gamma_0(\psi) \left| \int_a^{x-\delta} G_N(t,x)\,\mathrm{d}t \right| \to 0 \quad \text{as} \quad N\to\infty$$

uniformly for all $x \in (a, b)$, in view of Lemma 4.3(i). Therefore,

$$\lim_{N\to\infty} I_{1,2} = 0$$

uniformly for all $x > \delta$.

Now, consider $I_{1,1}$. Since $I_{1,1} = 0$ if $x < a+\delta$, we have to consider $x \geqslant a+\delta$. In view of Lemma 4.5, $I_{1,1} \to 0$ uniformly for all $x \in [a+\delta, L]$ as $N \to \infty$. Hence, assume that $x > L$, where L is a large number greater than max $(2, b+\frac{1}{2})$. Since $\psi(t)$ is of compact support contained in $[a, b]$,

$$\begin{aligned} I_{1,1} &= \zeta(x) \int_a^{x-\delta} G_N(t, x)\psi(t)\,\mathrm{d}t \\ &= \zeta(x) \int_a^{b} G_N(t, x)\psi(t)\,\mathrm{d}t \\ &= \zeta(x) \int_a^{b} \psi(t) \sinh t \,\mathrm{d}t \int_0^N \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh x)\, P^{m,n}_{-(1/2)+ir}(\cosh t)\mathrm{d}r. \end{aligned}$$

Let N_1 be a large but fixed number such that $1 < N_1 < N$. Then

$$\begin{aligned} I_{1,1} &= \zeta(x) \int_a^{b} \psi(t) \sinh t \,\mathrm{d}t \times \\ &\quad \left(\int_0^{N_1} + \int_{N_1}^{N} \right) \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t)\,\mathrm{d}r \\ &= \zeta(x) \int_a^{b} \psi(t) \sinh t \,\mathrm{d}t \times \\ &\quad \int_0^{N_1} \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh x) P^{m,n}_{-(1/2)+ir}(\cosh t)\,\mathrm{d}r \\ &\quad + \zeta(x) \int_a^{b} \psi(t) \sinh t \,\mathrm{d}t \int_{N_1}^{N} \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x)\, P^{m,n}_{-(1/2)+ir}(\cosh t)\,\mathrm{d}r \\ &= J_{1,1} + J_{1,2} \text{ (say)}. \end{aligned}$$

Now

$$\begin{aligned} J_{1,1} &= \zeta(x) \int_a^{b} \psi(t) \sinh t \,\mathrm{d}t \int_0^{N_1} \chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x)\, P^{m,n}_{-(1/2)+ir}(\cosh t)\,\mathrm{d}r \\ &\leqslant \sup_{\substack{a \leqslant t \leqslant b \\ 0 \leqslant r \leqslant N_1}} |\psi(t) \sinh t\, \chi(r)\, P^{m,n}_{-1/2}(\cosh t)|\, (b-a) \mathrm{e}^{\beta x} \mathrm{O}(\mathrm{e}^{-1/2x}) \int_0^{N_1} \mathrm{d}r \\ &= \sup_{\substack{a \leqslant t \leqslant b \\ 0 \leqslant r \leqslant N_1}} |\psi(t) \sinh t\, \chi(r)\, P^{m,n}_{-1/2}(\cosh t)| (b-a) N_1 \exp\{-(\tfrac{1}{2}-\beta)x\} \\ &= \mathrm{O}[\exp\{-(\tfrac{1}{2}-\beta)x\}] \quad \text{as} \quad x \to \infty. \end{aligned}$$

Estimating as in the proof of Lemma 4.4, we can show that

$$J_{1,2}=O[\exp\{-(\tfrac{1}{2}-\beta)x\}] \quad \text{as} \quad x\to\infty.$$

Therefore, $I_{1,1}$ can be made less than $\varepsilon/2$ for all $N>0$ by choosing $x>L$. Thus

$$I_1\to 0 \quad \text{as} \quad N\to\infty. \tag{4.2}$$

Similarly, using Lemma 4.5(ii) it can be shown that

$$I_3\to 0 \quad \text{as} \quad N\to\infty \tag{4.3}$$

uniformly for all $x\geqslant 2\delta$.
Combining Equations (4.1), (4.2), and (4.3), we have

$$\lim_{N\to\infty} I=0 \tag{4.4}$$

where $I=I_1+I_2+I_3$, uniformly for all $x\geqslant 2\delta$.

For $0<x<2\delta$, write

$$\begin{aligned} I&=\zeta(x)\left(\int_a^{x+\delta}+\int_{x+\delta}^{b}\right) G_N(t,x)\,[\psi(t)-\psi(x)]\,\mathrm{d}t\\ &=J_1+J_2 \text{ (say).}\end{aligned}$$

Now

$$\begin{aligned} J_1&=\zeta(x)\int_a^{x+\delta} G_N(t,x)[\psi(t)-\psi(x)]\,\mathrm{d}t\\ &=0 \qquad \text{because} \qquad x+\delta<3\delta<a.\end{aligned}$$

Next consider J_2. Since $0<x<2\delta$, $\delta\leqslant\min(\frac{1}{4}a,\frac{1}{2})$ and $\psi(x)=0$ for $x<a$, therefore

$$\begin{aligned} J_2&=\zeta(x)\int_{x+\delta}^{b} G_N(t,x)\psi(t)\,\mathrm{d}t\\ &=\zeta(x)\int_a^b G_N(t,x)\psi(t)\,\mathrm{d}t\\ &=\zeta(x)\int_a^b \psi(t)\sinh t\,\mathrm{d}t\int_0^N \chi(r)\,P^{m,n}_{-(1/2)+ir}(\cosh x)\,P^{m,n}_{-(1/2)+ir}(\cosh t)\,\mathrm{d}r\\ &=\frac{\zeta(x)(2\cosh^2\frac{x}{2})^{n/2}}{(\Gamma(1-m))^2\,(2\sinh^2\frac{x}{2})^{m/2}}\int_a^b \psi(t)\sinh t\,\mathrm{d}t\,\frac{(2\cosh^2 t)^{n/2}}{(2\sinh^2\frac{t}{2})^{m/2}}\,\mathrm{d}t\,\times\\ &\int_0^N \chi(r)F\left(\mathrm{i}r+\frac{n-m+1}{2},-\mathrm{i}r+\frac{n-m+1}{2};1-m;\frac{1-\cosh x}{2}\right)F\left(\mathrm{i}r+\frac{n-m+1}{2},-\mathrm{i}r+\frac{n-m+1}{2};1-m;\frac{1-\cosh t}{2}\right)\mathrm{d}r\\ &=O(x^{\alpha-\mathrm{Re}\,m})\left\{\int_a^b \psi(t)\sinh t\,2^{(n-m)/2}\cosh^n\frac{t}{2}\sinh^{-m}\frac{t}{2}\,\mathrm{d}t\,\times\right.\\ &\left.\left[\int_0^{N_1} O(r^2)\,\mathrm{d}r+\int_{N_1}^{N} 2^{n-m-1}\pi^{-1}\{e^{ir(t+x)}+ie^{-m\pi i}e^{ir(x-t)}+ie^{-m\pi i}e^{ir(x-t)}-e^{-2m\pi i}e^{-ir(x+t)}\}\right]\mathrm{d}r\right\}\\ &=O(\delta^{\alpha-\mathrm{Re}\,m}), \qquad \alpha\geqslant\mathrm{Re}\,m,\end{aligned}$$

since the above integrals are bounded. Thus

$$\lim_{N\to\infty} I=0 \tag{4.5}$$

uniformly for all $x\in(0,2\delta)$.

Therefore, in view of Equations (4.4) and (4.5),

$$\lim_{N\to\infty} I=0 \quad \text{uniformly for all } x>0.$$

This completes the proof of the lemma.

Theorem 4.7 (Inversion)

Let Re $m<\min(\frac{1}{2}, 1-|\mathrm{Re}\, n|)$, $\alpha \geqslant \mathrm{Re}\, m$, and $\beta<\frac{1}{2}$. Assume that $f\in M_{\beta}^{\alpha\prime}(I)$ and $F(r)$ is the distributional generalized Mehler–Fock transformation of f defined by Equation (3.1). Then for each $\phi(t)\in \mathrm{D}(I)$,

$$\lim_{N\to\infty} \langle \int_0^N \chi(r)\, F(r)\, P^{m,n}_{-(1/2)+ir}(\cosh t) \sinh t\, dr, \phi(t)\rangle$$
$$=\langle f(t), \phi(t)\rangle.$$

Proof. Assume that the support of $\phi(t)$ is contained in the interval $[a,b]\subset(0,\infty)$. In view of Theorem 3.2, $F(r)$ is a continuous function of r. The integral

$$\int_0^N \chi(r)\, F(r)\, P^{m,n}_{-(1/2)+ir}(\cosh t) \sinh t\, dr$$

is therefore a continuous function of t, and as a consequence it generates a regular distribution. Hence we can write

$$\langle \int_0^N \chi(r)\, F(r)\, P^{m,n}_{-(1/2)+ir}(\cosh t) \sinh t\, dr, \phi(t)\rangle$$

$$=\int_a^b \phi(t) \sinh t\, dt \int_0^N \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh t)\langle f(x), P^{m,n}_{-(1/2)+ir}(\cosh x)\rangle dr$$

$$=\int_a^b \phi(t) \sinh t \langle f(x), \int_0^N \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh t)\, P^{m,n}_{-(1/2)+ir}(\cosh x)\, dr\rangle dt$$

(by Theorem 3.4)

$$=\langle f(x), \int_a^b \phi(t)\, G_N(t,x)\, dt\rangle$$

$$\to\langle f(x), \phi(x)\rangle$$

by Lemma 4.6. This completes the proof of the theorem.

Theorem 4.8 (Uniqueness)

Let f, $g\in M_{\beta}^{\alpha\prime}(I)$ and let $F(r)$, $G(r)$ be their generalized Mehler–Fock transforms respectively. If $F(r)=G(r)$ for all $r>0$, then $f=g$ in the sense of equality in $\mathrm{D}'(I)$.

The proof is trivial.

5. AN OPERATIONAL CALCULUS

In this section, we shall apply the preceding theory in solving certain differential operator equations. Define the operator

$$\Delta_t^*: M_\beta^{\alpha'}(I) \to M_\beta^{\alpha'}(I)$$

by the relation

$$\langle \Delta_t^* f(t), \phi(t) \rangle := \langle f(t), \Delta_t \phi(t) \rangle$$

for all $f \in M_\beta^{\alpha'}(I)$ and $\phi(t) \in M_\beta^\alpha(I)$ for $\alpha \geqslant \mathrm{Re}\, m$ and $\beta < \frac{1}{2}$. It can be readily seen that

$$\langle (\Delta_t^*)^k f(t), \phi(t) \rangle = \langle f(t), \Delta_t^k \phi(t) \rangle$$

for each $k = 1, 2, 3 \ldots$ In case f is a regular distribution generated by an element of D(I), then

$$\Delta_t^* = \mathrm{D}^2 - (\coth t)\mathrm{D} + \mathrm{cosech}^2\, t + \frac{m^2}{2(1-\cosh t)} + \frac{n^2}{2(1+\cosh t)}.$$

It can be proved that

$$M_{\bar{r}}(\Delta_x^*)^k f(x) = (-1)^k (\tfrac{1}{4} + r^2)^k M_{\bar{r}}[f(x)], \tag{5.1}$$

where $M_{\bar{r}}[f(x)]$ denotes the generalized Mehler–Fock transform of $f(x)$. Now we consider the operator equation

$$P(\Delta_x^*) u = g \tag{5.2}$$

where $g \in M_\beta^{\alpha'}(I)$ and P is any polynomial having no zeros on $-\infty < x \leqslant 0$.

We wish to find a generalized function $u \in M_\beta^{\alpha'}(I)$ satisfying the operator equation (5.2). Taking the generalized Mehler–Fock transform of both sides of Equation (5.2) and using Equation (5.1) we get

$$P[-(\tfrac{1}{4} + r^2)]\, U(r) = G(r)$$

where U and G are generalized Mehler–Fock transforms of $u(x)$ and $g(x)$ respectively. So that if $P[-(\frac{1}{4} + r^2)] \neq 0$, we can apply the inversion formula for the distributional Mehler–Fock transform and for each $\phi \in D(I)$, we get

$$\langle u, \phi \rangle = \lim_{N \to \infty} \left\langle \int_0^N \frac{G(r)}{P[-(\frac{1}{4} + r^2)]} \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh x) \sinh x\, \mathrm{d}r, \phi(x) \right\rangle. \tag{5.3}$$

By Theorem 3.3 we know that

$$|G(r)| \leqslant C r^{2q} \quad \text{as} \quad r \to \infty$$

for some non-negative integer q depending upon g. Now, let $Q(x)$ be a polynomial of degree greater than or equal to $q - \mathrm{Re}\, m + 2$ having no zeros on the negative real axis. Then, the convergence of the right-hand side of Equation (5.3) can be established as follows:

$$\left\langle \int_0^N \frac{G(r)}{P[-(\frac{1}{4} + r^2)]} \chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh x) \sinh x\, \mathrm{d}r, \phi(x) \right\rangle$$

$$= \left\langle Q(\Delta_x^*) \int_0^N \frac{G(r)\, \chi(r)}{P[-(\frac{1}{4} + r^2)] Q[-(\frac{1}{4} + r^2)]} P^{m,n}_{-(1/2)+ir}(\cosh x) \sinh x\, \mathrm{d}r, \phi(x) \right\rangle$$

$$= \left\langle \int_0^N \frac{G(r)\, \chi(r)}{P[-(\frac{1}{4} + r^2)] Q[-(\frac{1}{4} + r^2)]} P^{m,n}_{-(1/2)+ir}(\cosh x) \sinh x\, \mathrm{d}r, Q(\Delta_x)\, \phi(x) \right\rangle$$

(by integration by parts).

Let us suppose that the support of $\phi(x)$ is contained in $[A, B]$. Then, we can find a constant L such that for N_1, $N_2 > L$ we have

$$|J| = \left| \left\langle \int_{N_1}^{N_2} \frac{G(r)}{P[-(\frac{1}{4}+r^2)]}\chi(r)\, P^{m,n}_{-(1/2)+ir}(\cosh x)\sinh x\, dr, \phi(x) \right\rangle \right|$$

$$\leqslant C \int_{N_1}^{N_2} \left| \frac{r^{2q}\chi(r)}{P[-(\frac{1}{4}+r^2)]Q[-(\frac{1}{4}+r^2)]} \right| dr \int_A^B \left| P^{m,n}_{-(1/2)+ir}(\cosh x)\sinh x\, Q(\Delta_x)\,\phi(x) \right| dx.$$

Since for $x \in [A, B]$,

$$\left| P^{m,n}_{-(1/2)+ir}(\cosh x)\sinh x \right| \leqslant C_1 \quad \text{for all} \quad r \geqslant 0,$$

using the estimate (1.7) we can find a positive constant M such that

$$|J| \leqslant CM \int_{N_1}^{N_2} \frac{r^{2q-2\mathrm{Re}\,m+1}}{P[-(\frac{1}{4}+r^2)]Q[-(\frac{1}{4}+r^2)]} dr \to 0 \quad \text{as} \quad N_1, N_2 \to \infty.$$

Therefore

$$\lim_{N\to\infty} \left\langle \int_0^N \frac{G(r)\chi(r)}{P[-(\frac{1}{4}+r^2)]} P^{m,n}_{-(1/2)+ir}(\cosh x)\sinh x\, dr, \phi(x) \right\rangle$$

exists and by completeness of $D'(I)$ there exists $f \in D'(I)$ such that

$$\lim_{N\to\infty} \left\langle \int_0^N \frac{G(r)\chi(r)}{P[-(\frac{1}{4}+r^2)]} P^{m,n}_{-(1/2)+ir}(\cosh x)\sinh x\, dr, \phi(x) \right\rangle = \langle f, \phi \rangle. \tag{5.4}$$

Now for all $\phi \in D(I)$, we have

$$\lim_{N\to\infty} \left\langle P(\Delta_x^*) \int_0^N \frac{G(r)\chi(r)}{P[-(\frac{1}{4}+r^2)]} P^{m,n}_{-(1/2)+ir}(\cosh x)\sinh x\, dr, \phi(x) \right\rangle = \langle P(\Delta_x^*)f, \phi \rangle,$$

or

$$\lim_{N\to\infty} \left\langle \int_0^N G(r)\chi(r) P^{m,n}_{-(1/2)+ir}(\cosh x)\sinh x\, dr, \phi(x) \right\rangle = \langle P(\Delta_x^*)f, \phi \rangle.$$

Hence by our inversion Theorem 4.7, it follows that

$$\langle g, \phi \rangle = \langle P(\Delta_x^*)f, \phi \rangle.$$

This proves that f determined by Equation (5.4), which belongs to $D'(I)$ and is the restriction of $u \in M_\beta^{\alpha\prime}(I)$ to $D(I)$, satisfies the operator equation (5.2).

6. A DIRICHLET PROBLEM WITH A DISTRIBUTIONAL BOUNDARY CONDITION

In this section we discuss a boundary value problem associated with the Legendre function $P_{-(1/2)+ir}(z) \equiv P^{0,0}_{-(1/2)+ir}(z)$. Let us determine a function v which satisfies the equation

$$\frac{\partial^2 v}{\partial \tau^2} - \coth\tau \frac{\partial v}{\partial \tau} + \frac{\partial^2 v}{\partial \theta^2} + (\operatorname{cosech}^2\tau + \tfrac{1}{4})v = 0 \tag{6.1}$$

under the following boundary conditions.

(i) As $\theta \to 0+$, $v(\tau, \theta)$ converges in $D'(I)$, $I = (0, \infty)$ to some generalized function $g(\tau) \in (M_\beta^\alpha)'(I)$.
(ii) As $\theta \to \pi-$, $\partial v/\partial\theta$ converges to zero uniformly on every compact subset of $0 < \tau < \infty$.

Let $\bar{v}(r,\theta)$ be the distributional generalized Mehler–Fock transform of order zero of $v(\tau,\theta)$. Then by Equation (6.1), we get

$$\frac{\partial^2 \bar{v}}{\partial \theta^2} - r^2 \bar{v} = 0,$$

so that

$$\bar{v} = A(r)\cosh r\theta + B(r)\sinh r\theta.$$

In view of the boundary conditions (i) and (ii) we get

$$\begin{aligned}\bar{v} &= \langle g(\tau), P_{-(1/2)+ir}(\cosh\tau)\rangle(\cosh r\theta - \tanh r\pi \sinh r\theta)\\ &= G(r)(\cosh r\theta - \tanh r\pi \sinh r\theta),\end{aligned}$$

where

$$G(r) = \langle g(\tau), P_{-(1/2)+ir}(\cosh t)\rangle.$$

Now, applying the inversion theorem for the generalized Mehler–Fock transform, we have

$$\begin{aligned}&\langle v(\tau,\theta), \phi(\tau)\rangle\\ &= \lim_{N\to\infty} \left\langle \int_0^N \chi(r)G(r)(\cosh r\,\theta - \tanh r\pi \sinh r\theta)\, P_{-(1/2)+ir}(\cosh\tau)\sinh\tau\, dr, \phi(\tau)\right\rangle\end{aligned}$$

for each $\phi \in D(I)$. Consequently, in the conventional sense,

$$v(\tau,\theta) = \lim_{N\to\infty} \int_0^N \chi(r)G(r)\frac{\cosh r(\pi-\theta)}{\cosh r\pi} P_{-(1/2)+ir}(\cosh\tau)\sinh\tau\, dr. \tag{6.2}$$

Therefore

$$\begin{aligned}\frac{\partial v}{\partial \theta} &= \lim_{N\to\infty}\int_0^N \chi(r)G(r)\left(\frac{-r\sinh r(\pi-\theta)}{\cosh r\pi}\right)P_{-(1/2)+ir}(\cosh\tau)\sinh\tau\, dr \qquad (6.3)\\ &= \int_0^1 \chi(r)G(r)\left(\frac{-r\sinh r(\pi-\theta)}{\cosh r\pi}\right)P_{-(1/2)+ir}(\cosh\tau)\sinh\tau\, dr\\ &+ \int_1^N \chi(r)G(r)\left(\frac{-r\sinh r(\pi-\theta)}{\cosh r\pi}\right)P_{-(1/2)+ir}(\cosh\tau)\sinh\tau\, dr\\ &= I_1 + I_2 \text{ (say)}.\end{aligned}$$

Now, I_1 is easily shown to satisfy

$$|I_1| \leqslant C_1 \sinh(\pi-\theta),$$

and using Theorem 3.3,

$$|I_2| \leqslant C_2 \int_1^\infty e^{-r\theta} r^{2q-\operatorname{Re}m+3/2}\, dr.$$

Therefore the integral in Equation (6.3) converges absolutely and uniformly for all θ satisfying $0<\theta_0<\theta<\pi$.

Hence we can take the limit $\theta\to\pi-$ within the integral sign and verify the boundary condition (ii).

To verify the boundary condition (i), assume as in Section 5 that $Q(x)$ is a polynomial of degree greater than or equal to $q-\operatorname{Re}m+2$, having no zeros on the negative real axis. Then for each $\phi\in D(I)$ with support contained in $[a,b]$

we have

$$\langle v(\tau,\theta),\phi(\tau)\rangle$$

$$= \lim_{N\to\infty} \left\langle Q(\Delta_\tau^*) \int_0^N \frac{\chi(r)G(r)}{Q[-(\frac{1}{4}+r^2)]} \frac{\cosh r(\pi-\theta)}{\cosh r\pi} P_{-(1/2)+ir}(\cosh\tau)\sinh\tau\, dr, \phi(\tau)\right\rangle$$

$$= \lim_{N\to\infty} \int_0^N \frac{\chi(r)G(r)\cosh r(\pi-\theta)}{Q[-(\frac{1}{4}+r^2)]\cosh r\pi} dr \int_a^b P_{-(1/2)+ir}(\cosh\tau)\sinh\tau\, Q(\Delta_\tau)\phi(\tau)d\tau \qquad (6.4)$$

(by integration by parts).

Now in view of the asymptotic behavior of $\chi(r)$ and $G(r)$, the right-hand side converges uniformly with respect to θ, $0\leqslant\theta\leqslant\pi$. Therefore, letting $\theta\to 0+$ and interchanging the limiting operation with respect to N and θ in the right-hand side of Equation (6.4) we get

$$\lim_{\theta\to 0+} \langle v(\tau,\theta),\phi(\tau)\rangle$$

$$= \lim_{N\to\infty} \int_a^b Q(\Delta_\tau)\phi(\tau)\,d\tau \int_0^N \frac{\chi(r)G(r)}{Q[-(\frac{1}{4}+r^2)]} P_{-(1/2)+ir}(\cosh\tau)\sinh\tau\, dr$$

$$= \lim_{N\to\infty} \int_a^b \phi(\tau)\,d\tau \int_0^N \chi(r)\,G(r)\,P_{-(1/2)+ir}(\cosh\tau)\sinh\tau\, dr$$

(by integration by parts)

$$= \langle g, \phi\rangle \qquad \text{(by Theorem 4.7).}$$

Lastly, in view of the asymptotic orders of $\chi(r)$ and $G(r)$ and the fact that $0\leqslant\theta\leqslant\pi$, it can be readily justified that

$$\left(\frac{\partial^2}{\partial\tau^2} - \coth\tau\frac{\partial}{\partial\tau} + \frac{\partial^2}{\partial\theta^2} + \tfrac{1}{4} + \operatorname{cosech}^2\tau\right) v(\tau,\theta)$$

$$= \lim_{N\to\infty} \int_0^N \chi(r)\,G(r)\frac{\cosh r(\pi-\theta)}{\cosh r\pi}\left(\frac{\partial^2}{\partial\tau^2} - \coth\tau\frac{\partial}{\partial\tau} + \frac{\partial^2}{\partial\theta^2} + \tfrac{1}{4} + \operatorname{cosech}^2\tau\right) P_{-(1/2)+ir}(\cosh\tau)\sinh\tau\, dr.$$

Therefore, $v(\tau,\theta)$ as defined by Equation (6.2) satisfies the differential equation (6.1). The solution is unique in view of Theorem 4.8.

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