RATIONAL AND POLYNOMIAL APPROXIMATIONS FROM CHEBYSHEV AND LEGENDRE SERIES FOR LINEAR DIFFERENTIAL EQUATIONS

E. H. Doha*

Department of Mathematics,
Faculty of Science,
Cairo University,
Giza, Egypt

الخلاصة :

يتناول هذا البحث وصف طريقة عددية للحصول آنياً على التقريب بكثيرة حدود والتقريب بخارج قسمة كثيرتي حدود لأي دالة تحقق معادلة تفاضلية خطية مع شروطها الحدية الخاصة بها . يعتمد جوهر الطريقة على فك الدالة ومشتقاتها التي تظهر في المعادلة التفاضلية التي تحققها وذلك إما في صورة مفكوك كثيرة حدود تشيبيشيف أو كثيرة حدود لاجندر ، ثم تعين معاملات المفكوك بالتعويض في المعادلة التفاضلية ومساوات المعاملات .

أعطي في هذا البحث بعض الأمثلة العددية التي توضح كيفية تطبيق الطريقة لدوال تحقق معادلات تفاضلية من الرتبة الأولى والثانية .

تعتبر الطريقة بشكلها الحالي تعميماً لطريقة كلينشو (١٩٥٧) في المستوى المركب .

ABSTRACT

In this paper we describe a method for obtaining simultaneously rational and polynomial approximations for functions defined by linear differential equations with associated boundary conditions. The essence of the method is that an expansion in either Chebyshev or Legendre polynomials is assumed for the function and its derivatives occurring in the differential equation; the coefficients of expansion are then determined by substituting in the differential equation and equating the coefficients. Some numerical examples are given of the application to some first and second order differential equations.

The method in its present form may be considered as an extension of Clenshaw's method (1957) into the complex domain.

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*Present address: Department of Mathematics and Statistics, Faculty of Science, United Arab Emirates University, P.O. Box 15551, Al-Ain, U.A.E.

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1. INTRODUCTION

Our principal aim in this paper is to describe a method that enables us to obtain simultaneously an accurate polynomial and rational approximations for functions defined by linear differential equations. The latter will be considered as an extension of the method to the complex domain.

Suppose that we have a linear differential equation of order m of the form

$$\sum_{i=0}^{m} p_i(x)\frac{\mathrm{d}^i y}{\mathrm{d}x^i}=q(x) \tag{1}$$

where $q(x)$ and $p_i(x)$, $i=0, 1, 2, \ldots, m$, are functions in x. The complete system uniquely determining the function $y(x)$ needs m initial or boundary conditions together with Equation (1). Two methods for the numerical solution of Equation (1) are due to Lanczos [1] and Clenshaw [2]. Lanczos [1] appends a perturbation term proportional to a Chebyshev polynomial to the right-hand side of Equation (1) and solves the perturbed equation exactly; the result is a polynomial approximation to $y(x)$. Lanczos's method may also be extended to yield rational approximations to $y(x)$ valid for arbitrary complex values of x (see Fox and Parker [3] and Lanczos [4]).

On the other hand, Clenshaw [2] assumes that $y(x)$ and its derivatives may be expanded in Chebyshev series, and derives by means of recurrence relations the coefficients for the expansion of $y(x)$. The main aim of the following is to give a straightforward extension of Clenshaw's method into the complex domain, which consequently yields a rational approximation instead of a polynomial one.

There is no need to discuss the error analysis of this method, because the coefficients of the expansion are in general readily evaluated, and approximations of any specified accuracy are provided by mere truncation. An analysis similar to that given in this work could be made for expansions in terms of ultraspherical polynomials $C_n^{(\lambda)}(x)$. We hope to describe this analysis in a forthcoming paper.

The present method is described in Sections 2 and 3, and is illustrated by numerical examples in Section 4. In Section 5, we deal with the application of the method when the solution of the differential equation is expanded in terms of Legendre polynomials.

2. CLENSHAW'S METHOD OF SOLUTION

Suppose that the range of integration is the closed interval $[-1, 1]$, and that the required solution of the differential equation (1) is $y(x)$. Then if $y(x)$ is continuous in $[-1, 1]$, we can write

$$y(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty} a_n T_n(x) \tag{2}$$

where the coefficients a_n are to be determined. The s^{th} derivative of y, $y^{(s)}(x)$, is expanded formally as

$$y^{(s)}(x)=\tfrac{1}{2}a_0^{(s)}+\sum_{n=1}^{\infty} a_n^{(s)} T_n(x) \tag{3}$$

for $s=1, 2, \ldots, m$. $T_n(x)$ is the Chebyshev polynomial defined over the interval $[-1, 1]$ by

$$T_n(x)=\cos n\theta, \quad \theta=\cos^{-1} x. \tag{4}$$

The method of determining the coefficients a_n depends upon two simple relations. From

$$2\int T_n(x)\mathrm{d}x=\frac{1}{n+1}T_{n+1}(x)-\frac{1}{n-1}T_{n-1}(x) \tag{5}$$

it can be found easily that the equation relating the coefficients of $y^{(s)}$ to those of $y^{(s+1)}$, is

$$2na_n^{(s)}=a_{n-1}^{(s+1)}-a_{n+1}^{(s+1)} \tag{6}$$

and from

$$2xT_n(x)=T_{n+1}(x)+T_{n-1}(x), \tag{7}$$

if $C_n(y)$ denotes the coefficient of $T_n(x)$ in the expansion for $y(x)$ when $n>0$, and twice this coefficient when $n=0$, then

$$C_n(xy)=\tfrac{1}{2}(a_{n-1}+a_{n+1}) \tag{8}$$

and thus

$$C_n(x^2y)=\tfrac{1}{4}(a_{n-2}+2a_n+a_{n+2}). \tag{9}$$

Generalizing, we see that

$$C_n(x^k y)=2^{-k}\sum_{j=0}^{k}\binom{k}{j}a_{n-k+2j} \tag{10}$$

where $a_{-m}=a_m$ for all m. In this equation we may replace $y(x)$ by $y^{(s)}(x)$ provided a_n is changed by $a_n^{(s)}$. From Equation (10), the quantities $C_n(xy)$, $C_n(x^2y), \ldots, C_n(x^m y)$ can easily be found, and so in Equation (1) $C_n(p_i(x)(d^i y/dx^i))$ can rapidly be written down if $p_i(x)$ is a polynomial in x. In cases where $p_i(x)$, $i=0, 1, \ldots, m$, are not polynomials in x it is sometimes best to replace them by suitable polynomial approximations. Substituting Equations (2) and (3) into Equation (1) we obtain, by means of Equations (6) and (10), relations for the coefficients $a_n^{(s)}$ for $s=0, 1, 2, \ldots, m$ and all n. These relations and those obtained from the boundary or initial conditions are equivalent to an infinite set of linear equations in the unknowns $a_n^{(s)}$. The numerical solution of these equations can be performed by the two methods described in detail by Clenshaw [2]. These are the method of recurrence and the iterative method. The starting point of the method of recurrence is to assume that $a_n^{(s)}=0$ for $s=0, 1, 2, \ldots, m$ and $n>N$, where N is some arbitrary positive integer, and to assign arbitrary values to $a_N^{(s)}$. The values of $a_n^{(s)}$ for $n=N-1, N-2, \ldots, 0$, may then be obtained from the recurrence relations. Thus, apart from a small residual in the equations near $n=N$, all the equations are satisfied, except possibly a few in the neighborhood of $n=0$ which will not have been used if the recurrence process yields a_0 without making use of them. The solution of the problem requires the satisfaction of these remaining equations, and of the boundary conditions. This may be achieved by repeating the recurrence process with different arbitrary values of $a_N^{(s)}$, and taking the appropriate linear combination. For instance, if the solution we seek is the only solution of the differential equation with a convergent Chebyshev expansion, then a single trial solution obtained by recurrence will yield this required result when multiplied by a constant factor. This factor is usually given by the satisfaction of a boundary condition. The method is in general quick, the main disadvantage being that N may be chosen either too small or too large. In the former case the required accuracy for the coefficients may not be obtained, in which case the computation must be repeated with a larger N. If N is chosen too large, more computation than necessary will have been done. In general a solution by recurrence is direct and rapid although care must be taken that figures are not lost from the most significant end when linear combinations of solutions are taken. If this does occur, the solution may be improved using the iterative method.

The iterative method starts with some initial guess for the a_n that satisfies the boundary conditions. From these values, Equation (6) can be used to compute $a_n^{(1)}$, $a_n^{(2)}$, etc. When all $a_n^{(s)}$, $s=1, 2, \ldots, m$ have been found, these values can be used to compute a new a_n from the recurrence relation, again satisfying the boundary conditions. This procedure is continued until the desired accuracy is reached.

3. EXTENSION OF THE METHOD TO THE COMPLEX DOMAIN

To extend Clenshaw's method to the complex domain, we consider instead the function $y(zx)$, where x is the independent variable, $-1 \leqslant x \leqslant 1$, and z is regarded as a parameter that may take any real or complex value. The function $y(zx)$ satisfies the differential equation

$$\sum_{i=0}^{m} p_i(zx) \frac{d^i y(zx)}{z^i dx^i} = q(zx). \tag{11}$$

Clenshaw's method is applied to get a polynomial approximation in x, say $R_N(z, x)$, for $y(zx)$ in the form

$$R_N(z, x) = \frac{1}{\alpha(z)} \left[\tfrac{1}{2} a_0(z) + \sum_{n=1}^{N} a_n(z) T_n(x) \right]. \tag{12}$$

The coefficients $a_n(z)$ are now rational functions in z since the recurrence relations for the coefficients $a_n^{(s)}(z)$ involve such functions. The multiplying factor $1/\alpha(z)$ is another rational function in z resulting from the satisfaction of an initial condition. Finally, putting $x=1$ in Equation (12) yields a rational function $R_N(z, 1)=Y_N(z)$, say, which approximates $y(z)$ for any real or complex values of z.

It is to be noted that if we put $z=1$ in Equation (12), then we get the usual polynomial approximation for the function $y(x)$. It is also interesting to note that the rational approximation obtained by the above method and Lanczos's method [4] have been found not to be the same. While it has been reported by Fox [5] that in the case of polynomial approximation, the two methods yield the same approximation.

4. NUMERICAL EXAMPLES

Example 1

Consider the differential equation

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + 16xy = 0; \quad y(0)=1, \quad y'(0)=0. \tag{13}$$

First, we observe from the differential equation, and the boundary condition $y'(0)=0$, that the solution sought is even. Let $x \to xz$, $R(z,x)=Y(zx)$, and $-1 \leqslant x \leqslant 1$. Then Equation (13) takes the form

$$xR''(z,x)+R'(z,x)+16z^2xR(z,x)=0$$
$$R(z,0)=1, \quad R'(z,0)=0 \tag{14}$$

where primes denote derivatives with respect to x, and z is a parameter. Also, let

$$Y(zx)=R(z,x)=\tfrac{1}{2}a_0(z)+\sum_{n=1}^{\infty} a_n(z)T_n(x). \tag{15}$$

Comparing coefficients of $T_n(x)$ in the expansion of the terms of the equation, we have

$$C_n(xR'')+C_n(R')+16z^2C_n(xR)=0.$$

Therefore, from Equation (10) we obtain

$$\tfrac{1}{2}\left(a_{n+1}^{(2)}+a_{n-1}^{(2)}\right)+a_n^{(1)}+8z^2(a_{n+1}+a_{n-1})=0. \tag{16}$$

Making use of Equation (6), Equation (16) simplifies to

$$a_{n-2}=a_{n+2}-\frac{n}{8z^2}\left(a_{n+1}^{(1)}+a_{n-1}^{(1)}\right). \tag{17}$$

This and the equation

$$a_{n-1}^{(1)}=a_{n+1}^{(1)}+2na_n \tag{18}$$

can be used alternatively to get a rational approximate solution. Taking $a_{10}=1$ with $a_{12}=a_{14}=\ldots=a_{11}^{(1)}=a_{13}^{(1)}=\ldots=0$, we obtain the trial solution given in Table 1. We find that when this trial solution has been computed, all Equations (17) and (18) have been satisfied to a certain accuracy. Thus, only the condition $R(z,0)=1$ remains to be satisfied, i.e.

$$\tfrac{1}{2}a_0-a_2+a_4-a_6+a_8-a_{10}=1. \tag{19}$$

This is done by multiplying the trial solution by a factor $\alpha(z)$. The trial solution yields

$$\tfrac{1}{2}a_0-a_2+a_4-a_6+a_8-a_{10}=-(3z^{10}+17.5z^8$$
$$+112z^6+648z^4+2{,}880z^2+7{,}200)/z^{10}$$

and must be multiplied by the reciprocal of this quantity. Thus we have

$$R(z,x)=\frac{1}{\alpha(z)}\left[\tfrac{1}{2}a_0(z)+\sum_{n=1}^{5} a_{2n}(z)T_{2n}(x)\right] \tag{20}$$

where the values of a_n are given in Table 1 and $\alpha(z)$ is given by

$$\alpha(z)=-(3z^{10}+17.5z^8+112z^6+648z^4$$
$$+2{,}880z^2+7{,}200)/z^{10}.$$

Finally, put $x=1$. Equation (20) yields the sought-for rational approximation in the form

$$R(z,1)=y(z)$$
$$=\frac{(-3z^{10}+241.5z^8-3{,}760z^6+17{,}928z^4-25{,}920z^2+7{,}200)}{(3z^{10}+17.5z^8+112z^6+648z^4+2{,}880z^2+7{,}200)}. \tag{21}$$

Table 1. Computation of Trial Solution for Example 1

n	Trial		1	z^2	z^4	z^6	z^8
	$z^{10-n}a_n(z)$	$z^{9-n}a_n^{(1)}(z)$					
0	$z^{10}a_0$		−14,400	+23,040	−11,376	+1,728	−81
1		$z^8a_1^{(1)}$	+57,600	−63,360	−18,624	−1,680	+36
2	z^8a_2		+14,400	−8,640	+1,536	−72	+1
3		$z^6a_3^{(1)}$	−28,800	+12,480	−1,392	+32	
4	z^6a_4		−3,600	+960	−64		
5		$z^4a_5^{(1)}$	+4,800	−880	+32		
6	z^4a_6		+400	−40	+1		
7		$z^2a_7^{(1)}$	−400	+20			
8	z^2a_8		−25				
9		$a_9^{(1)}$	+20				
10	a_{10}		+1				

Note. $z^6a_4=-3{,}600+960z^2-64z^4$ and $z^4a_5^{(1)}=4{,}800-880z^2+32z^4$.

As a check we find that $y(1)=-0.39717$, $y(\frac{1}{2})=0.22389$ which are in agreement with the analytical values $J_0(4)=-0.39715$ and $J_0(2)=0.22389$.

It is to be noted here that the polynomial Chebyshev approximation can be obtained directly from Equation (20) by taking $z=1$. In this case

$$\begin{aligned}Y(x)=R(1,x)=&0.0502T_0(x)-0.6653T_2(x)\\&+0.2490T_4(x)-0.0332T_6(x)\\&+0.0023T_8(x)-0.0001T_{10}(x),\end{aligned}$$

which is in agreement with that obtained by Clenshaw [6].

Example 2

Suppose we want to find a rational approximation for the function e^{x^2}, $x\in[-1,1]$. This function satisfies the differential equation

$$y'-2xy=0, \quad y(0)=1. \tag{22}$$

If $x\to xz$, $R(z,x)=y(zx)$, and $-1\leqslant x\leqslant 1$, then Equation (22) takes the form

$$\frac{dR}{dx}-2z^2xR=0, \quad R(z,0)=1. \tag{23}$$

Also, let $R(z,x)$ be given by Equation (15). Comparing the coefficients of $T_n(x)$ in the expansion of the terms of Equation (23), we have

$$C_n(R')-2z^2C_n(xR)=0.$$

Therefore, from Equation (8) we get

$$a_n^{(1)}-z^2(a_{n-1}+a_{n+1})=0.$$

With this equation in the form

$$a_{n-1}=\frac{1}{z^2}a_n^{(1)}-a_{n+1} \tag{24}$$

and Equation (18) we can readily compute a_n and $a_n^{(1)}$. Since e^{x^2} is an even function, $a_{2n+1}=a_{2n}^{(1)}=0$ for all n.

As a starting point we have taken $a_{12}=1$, $a_{14}=a_{16}=\ldots=a_{13}^{(1)}=a_{15}^{(1)}=\ldots=0$. With these starting values, Equations (24) and (18) can be used to compute $a_n, a_n^{(1)}$ for all $n<12$. The complete computation is shown in Table 2. These values of a_n have to be multiplied by a factor $1/\alpha(z)$ which is determined from the as-yet-unsatisfied boundary condition. This gives

$$\begin{aligned}\alpha(z)=&\tfrac{1}{2}a_0(z)-a_2(z)+a_4(z)-a_6(z)\\&+a_8(z)-a_{10}(z)+a_{12}(z)\end{aligned}$$

from which we find

$$\begin{aligned}\alpha(z)=&(6.5z^{12}-182z^{10}+2{,}912z^8-29{,}952z^2\\&+199{,}680z^4-798{,}720z^2+1{,}474{,}560)/z^{12}.\end{aligned}$$

Table 2. Computation of Trial Solution for Example 2

n	Trial		1	z^2	z^4	z^6	z^8	z^{10}	z^{12}
	$z^{12-n}a_n(z)$	$z^{11-n}a_n^{(1)}(z)$							
0	$z^{12}a_0$		2,949,120	−122,880	153,600	−6,144	1,152	−36	1
1		$z^{10}a_1^{(1)}$	2,949,120	+614,400	122,880	+9,216	576	+12	
2	$z^{10}a_2$		737,280	−30,720	15,360	−576	48	−1	
3		$z^8a_3^{(1)}$	737,280	+61,440	11,520	+384	16		
4	z^8a_4		92,160	−3,840	960	−32	1		
5		$z^6a_5^{(1)}$	92,160	+3,840	640	+8			
6	z^6a_6		7,680	−320	40	−1			
7		$z^4a_7^{(1)}$	7,680	+160	20				
8	z^4a_8		480	−20	1				
9		$z^2a_9^{(1)}$	480	+4					
10	z^2a_{10}		24	−1					
11		$a_{11}^{(1)}$	24						
12	a_{12}		1						

Note. $z^8a_4(z)=92{,}160-3{,}840z^2+960z^4-32z^6+z^8$ and $z^6a_5^{(1)}(z)=92{,}160+3{,}840z^2+640z^4+8z^6$.

Thus we have

$$R(z,x)=\frac{1}{\alpha(z)}\left[\tfrac{1}{2}a_0(z)+\sum_{n=1}^{6}a_{2n}(z)T_{2n}(x)\right]. \quad (25)$$

Put $x=1$, and Equation (25) yields the required rational approximation for e^{z^2} in the form

$$e^{z^2}=R(z,1)=P(z)/Q(z) \quad (26)$$

where

$$P(z)=(0.5z^{12}+42z^{10}+1{,}120z^8+16{,}128z^6+138{,}240z^4+675{,}840z^2+1{,}474{,}560)$$
$$Q(z)=(6.5z^{12}-182z^{10}+2912z^8-29{,}952z^6+199{,}680z^4-798{,}720z^2+1{,}474{,}560).$$

As a check we find $R(1,1)=2.718282=e$. Again the usual polynomial Chebyshev approximation can be obtained from Equation (25) by taking $z=1$. In this case

$$e^{x^2}=R(1,x)=1.753388T_0(x)+0.850392T_2(x)+0.105209T_4(x)+0.008722T_6(x)+0.000543T_8(x)+0.000027T_{10}(x)+0.00001T_{12}(x). \quad (27)$$

Relation (27) is in agreement with that obtained by Clenshaw [2].

Example 3

Consider the exponential integral

$$E_i(t)=\int_t^{\infty}\frac{e^{-t}}{t}dt. \quad (28)$$

For large values of t, the following asymptotic expansion holds

$$E_i(t)\simeq\frac{e^{-t}}{t}\left(1-\frac{1}{t}+\frac{2!}{t^2}-\frac{3!}{t^3}+\ldots\right).$$

The function $E_i(t)$ is an important transcendental function not only in the real range but everywhere in the complex domain. Let us now write

$$E_i(t)=\frac{e^{-t}}{t}y(x), \quad x=\frac{1}{t}, \quad (29)$$

then the function $y(x)$ satisfies the differential equation

$$x^2y'+(1+x)y=1. \quad (30)$$

A knowledge of the function $y(x)$ in the range $0\leqslant x\leqslant 1$ enables us to find the values of $E_i(t)$ in the range $1\leqslant t\leqslant\infty$. We therefore solve Equation (30) in the range $0\leqslant x\leqslant 1$, using the shifted Chebyshev polynomial $T_n^*(x)$ defined by

$$T_n^*(x)=\cos[n\cos^{-1}(2x-1)].$$

In the range $0\leqslant x\leqslant 1$, the coefficients of the expansion of $y^{(s)}(x)$ are denoted by $A_n^{(s)}$, and the coefficient of $T_n^*(x)$ in the expansion of $y(x)$ by $C_n^*(y)$ for $n>0$. Again $C_0^*(y)$ denotes twice the coefficient of $T_0^*(x)$. The equations corresponding to Equations (6) and (10) are then given by

$$4nA_n^{(s)}=A_{n-1}^{(s+1)}-A_{n+1}^{(s+1)} \quad (31)$$

$$C_n^*(x^ky)=\frac{1}{2^k}\sum_{j=0}^{k}\binom{2k}{j}A_{n-k+j}. \quad (32)$$

Now, let $x\to xz$, $R(z,x)=Y(zx)$, and $0\leqslant x\leqslant 1$. Then Equation (30) transforms to

$$zx^2\frac{dR}{dx}+(1+zx)R=1. \quad (33)$$

Here the function $R(z,x)$ has the expansion

$$R(z,x)=\tfrac{1}{2}A_0(z)T_0^*(x)+\sum_{n=1}^{\infty}A_n(z)T_n^*(x). \quad (34)$$

Comparing the coefficients of $T_n^*(x)$ in Equation (33) yields

$$\frac{z}{16}(A_{n+2}^{(1)}+4A_{n+1}^{(1)}+6A_n^{(1)}+4A_{n-1}^{(1)}+A_{n-2}^{(1)})+A_n+\frac{z}{4}(A_{n+1}+2A_n+A_{n-1})=C_n^*(1). \quad (35)$$

The right-hand side is of course zero for all n except $n=0$. Using Equation (31) to eliminate $A_{n+2}^{(1)}$ and $A_{n-2}^{(1)}$ we have

$$z(A_{n+1}^{(1)}+2A_n^{(1)}+A_{n-1}^{(1)})+nz(A_{n-1}-A_{n+1})+2(z+2)A_n=4C_n^*(1). \quad (36)$$

This equation and Equation (31) are used to obtain the coefficients by recurrence from an arbitrary starting value in the usual way. For $n>0$ the equations, in the form in which they are used, are

$$A_{n-1}=A_{n+1}-\frac{1}{n}\left[A_{n+1}^{(1)}+2A_n^{(1)}+A_{n-1}^{(1)}+2\left(\frac{z+2}{z}\right)A_n\right], \quad (37)$$

$$A_{n-1}^{(1)}=A_{n+1}^{(1)}+4nA_n. \quad (38)$$

It is to be noted that when A_0 has been obtained, Equation (36) with $n=0$ has not yet been satisfied. If we denote the left-hand side of this equation by L, we have

$$L=2[z(A_0^{(1)}+A_1^{(1)})+(z+2)A_0].$$

All the equations that have been used in computing the trial solution are homogeneous, and we can therefore arrange for the whole set to be satisfied by multiplying the trial solution by the factor $1/\alpha(z)$, where

$$\alpha(z)=\frac{L}{4C_0^*(1)}=[z(A_0^{(1)}+A_1^{(1)})+(z+2)A_0]/4.$$

The coefficients so obtained satisfy all the Equations (36) and are thus the coefficients in the required Chebyshev expansion. The computational details, starting from a trial solution, $A_{10}=10$, $A_{11}=A_{12}=\ldots=A_{10}^{(1)}=A_{11}^{(1)}=\ldots=0$ are given in Table 3.

From the trial solution, $\alpha(z)$ is found to be

$$\begin{aligned}\alpha(z)=(&102z^{11}+25{,}456z^{10}+518{,}716z^9+2{,}733{,}405z^8\\&+5{,}532{,}171z^7+5{,}232{,}886z^6+2{,}580{,}588z^5\\&+707{,}037z^4+112{,}625z^3+11{,}014z^2+664z+16)/4z^{10}.\end{aligned}$$

Thus we have

$$R(z,x)=\frac{1}{\alpha(z)}\left[\tfrac{1}{2}A_0(z)+\sum_{n=1}^{10}A_n(z)T_n^*(x)\right]. \quad (39)$$

Put $x=1$, Equation (39) yields the required rational approximation for the function $y(z)$ defined by Equation (29) in the form

$$y(z)=R(z,1)=P(z)/Q(z), \quad (40)$$

where

$$\begin{aligned}P(z)=4(&188z^{10}+21{,}584z^9+244{,}030z^8+746{,}049.5z^7\\&+897{,}087.5z^6+511{,}636z^5+153{,}215z^4\\&+25{,}710.5z^3+2{,}595.5z^2+162z+4),\\Q(z)=(&102z^{11}+25{,}456z^{10}+518{,}716z^9+2{,}733{,}405z^8\\&+5{,}532{,}171z^7+5{,}232{,}886z^6+2{,}580{,}588z^5\\&+707{,}037z^4+112{,}625z^3+11{,}014z^2+664z+16).\end{aligned}$$

As a check we find $E_i(1)=0.2193838$, the correct value being 0.2193839. Here it is to be noted that $y(0)=1$ which is exactly correct with the value of y at zero. Again the polynomial Chebyshev approximation is obtained from Equation (39) by taking $z=1$. This gives

$$\begin{aligned}y(x)&=R(1,x)\\&=0.757872T_0^*(x)-0.191887T_1^*(x)+0.037503T_2^*(x)\\&\quad-0.009074T_3^*(x)+0.002511T_4^*(x)-0.000764T_5^*(x)\\&\quad+0.000250T_6^*(x)-0.000087T_7^*(x)+0.000031T_8^*(x)\\&\quad-0.000011T_9^*(x)+0.000002T_{10}^*(x). \quad (41)\end{aligned}$$

The relation (41) is again in agreement with that obtained by Clenshaw [2].

5. POLYNOMIAL AND RATIONAL APPROXIMATION FROM LEGENDRE EXPANSION

We shall consider in this section the expansion of a function $f(x)$ in terms of the Legendre polynomials $P_n(x)$. Let

$$y(x)=\sum_{n=0}^{\infty}a_nP_n(x) \quad (42)$$

and for the s^{th} derivative of y

$$y^{(s)}(x)=\sum_{n=0}^{\infty}a_n^{(s)}P_n(x), \quad s=1,2,\ldots,m. \quad (43)$$

Then,

$$y^{(s+1)}=\sum_{n=0}^{\infty}a_n^{(s+1)}P_n(x).$$

On making use of the recurrence relation

$$P_n(x)=\frac{1}{2n+1}\frac{\mathrm{d}P_{n+1}(x)}{\mathrm{d}x}-\frac{1}{2n-1}\frac{\mathrm{d}P_{n-1}(x)}{\mathrm{d}x}$$

we get

$$y^{(s+1)}=\sum_{n=1}^{\infty}\left[\frac{a_{n-1}^{(s+1)}}{2n-1}-\frac{a_{n+1}^{(s+1)}}{2n+1}\right]\frac{\mathrm{d}P_n(x)}{\mathrm{d}x}. \quad (44)$$

On differentiating Equation (43), we find

$$y^{(s+1)}=\sum_{n=1}^{\infty}a_n^{(s)}\frac{\mathrm{d}P_n(x)}{\mathrm{d}x}. \quad (45)$$

Equations (44) and (45) give

$$a_n^{(s)}=\frac{a_{n-1}^{(s+1)}}{2n-1}-\frac{a_{n+1}^{(s+1)}}{2n+1}, \quad n\geqslant 1. \quad (46)$$

This equation is not as easy to use as Equation (6), since the coefficients on the right-hand side are functions of n. To simplify the computing, we define

$$a_n^{(s)}=(n+\tfrac{1}{2})b_n^{(s)}; \quad n\geqslant 0, \quad s=0,1,2,\ldots,m. \quad (47)$$

The Equation (46) then takes the simpler form

$$(2n+1)b_n^{(s)}=b_{n-1}^{(s+1)}-b_{n+1}^{(s+1)}, \quad n\geqslant 1. \quad (48)$$

Table 3. Computation of Trial Solution for Example 3

n	Trial: $z^{10-n}A_n(z)$	Trial: $z^{9-n}A_n^{(1)}$	1	z	z^2	z^3	z^4	z^5	z^6	z^7	z^8	z^9	z^{10}
0	$z^{10}A_0$	$z^9A_0^{(1)}$	+8 −8	+328 −316	+5,347 −4,900	+53,793 −47,280	+328,924 −269,564	+1,147,368 −842,860	+2,158,637 −1,357,480	+2,013,885 −1,029,716	+801,396 −319,444	+102,240 −31,864	+2,182
1	z^9A_1	$z^8A_1^{(1)}$	−2 +8	−79 +296	−1,222 +4,208	−11,748 +37,808	−66,438 +188,624	−203,011 +474,008	−308,273 +542,456	−201,878 +238,296	−43,112 +29,784	−1,571	
2	z^8A_2	$z^7A_2^{(1)}$	+1 −12	+37 −288	+524 −3,812	+4,688 −30,816	+23,095 −124,388	+55,975 −222,204	+58,284 −146,996	+19,884 −25,580	+1,164		
3	z^7A_3	$z^6A_3^{(1)}$	−1 +16	−24 +304	−316 +3,864	−2,538 +26,208	−10,010 +76,184	−16,710 +79,224	−9,136 +20,472	−859			
4	z^6A_4	$z^5A_4^{(1)}$	+1 −20	+19 −360	+240 −4,268	+1,608 −21,684	+4,456 −37,364	+4,011 −15,272	+622				
5	z^5A_5	$z^4A_5^{(1)}$	−1 +24	−18 +480	−212 +4,888	−1,045 +15,048	−1,623 +10,520	−436					
6	z^4A_6	$z^3A_6^{(1)}$	+1 −28	+20 −784	+201 −4,904	+579 −6,552	+291						
7	z^3A_7	$z^2A_7^{(1)}$	−1 +64	−28 +1,152	−171 +3,536	−180							
8	z^2A_8	$zA_8^{(1)}$	+2 −144	+36 −1,512	+98								
9	zA_9	$A_9^{(1)}$	−4 +400	−42									
10	A_{10}		+10										

Note. $z^2A_8(z)=2+36z+98z^2$ and $zA_8^{(1)}(z)=-(144+1512z)$.

Again, let $C_n(y)$ denote the coefficient of $P_n(x)$ in the expansion of y. Then

$$xy = \sum_{n=0}^{\infty} a_n x P_n(x)$$

$$= \sum_{n=0}^{\infty} \left[\frac{n}{2n-1} a_{n-1} + \frac{n+1}{2n+3} a_{n+1} \right] P_n(x)$$

on using the relation

$$xP_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x). \quad (49)$$

Thus,

$$C_n(xy) = \frac{n}{2n-1} a_{n-1} + \frac{n+1}{2n+3} a_{n+1}, \quad n \geq 0, \quad (50)$$

and in terms of the coefficients b_n, we find

$$C_n(xy) = \frac{n}{2} b_{n-1} + \frac{n+1}{2} b_{n+1}, \quad n \geq 0. \quad (51)$$

By continued application of Equation (50), we can find $C_n(x^2 y)$, $C_n(x^3 y)$, etc.

In general, Equations (46) and (48) are only valid for $n \geq 1$, since a_n and b_n have not yet been defined for negative values of n. (For the Chebyshev polynomials $T_n(x)$, $a_{-n} = a_n$ for all values of n.)

For an expansion in Legendre polynomials, we can give a meaning to a_{-n} and b_{-n} for $n = 1, 2, \ldots$. With $n = 0$, and from the recurrence relation (49), we see that $P_{-1}(x)$ is indeterminate. We define $P_{-1}(x) = -P_0(x)$, so we find that in general

$$P_{-n}(x) = -P_{n-1}(x).$$

For the coefficients $a_n^{(s)}$ we must have $a_{-n}^{(s)} = -a_{n-1}^{(s)}$, and from Equation (47) we deduce that $b_{-n}^{(s)} = b_{n-1}^{(s)}$ for $n = 0, 1, 2, \ldots$ and for all values of s.

To get a rational approximation from the Legendre expansion for the differential equation (1), the same procedure as given in Sections 2 and 3 can be followed.

In the following, we compute the polynomial and rational approximations for the solution of the differential equation of Example 2 using Legendre polynomials instead of Chebyshev polynomials. Let

$$y(zx) = R(z, x) = \sum_{n=0}^{\infty} a_n(z) P_n(x). \quad (52)$$

Comparing the coefficients of $P_n(x)$ in the expansion of the terms of the differential equation (23), we get

$$a_n^{(1)}(z) - 2z^2 \left[\frac{n}{2n-1} a_{n-1}(z) + \frac{n+1}{2n+3} a_{n+1}(z) \right] = 0$$

Table 4. Computation of Trial Solution for Example 2 in Terms of b_n and $b_n^{(1)}$ using Legendre Polynomials

n	Trial		1	z^2	z^4	z^6	z^8	z^{10}	z^{12}
	$z^{12-n}b_n(z)$	$z^{11-n}b_n^{(1)}(z)$							
0	$z^{12}b_0$		11,814,150	−2,149,507	589,981	−49,933	4,329	−158	4
1		$z^{10}b_1^{(1)}$	7,876,100	+667,289	211,214	+8,533	769	+12	
2	$z^{10}b_2$		1,575,220	−136,580	31,366	−1,588	88	−1	
3		$z^8b_3^{(1)}$	1,350,189	+54,384	16,473	+329	17		
4	z^8b_4		150,021	−9,111	1,479	−51	1		
5		$z^6b_5^{(1)}$	136,383	+3,162	778	+8			
6	z^6b_6		10,491	−510	50	−1			
7		$z^4b_7^{(1)}$	9,792	+138	21				
8	z^4b_8		576	−24	1				
9		$z^2b_9^{(1)}$	546	+4					
10	z^2b_{10}		26	−1					
11		$b_{11}^{(1)}$	25						
12	b_{12}		1						

Note. $z^4 b_8(z) = 576 - 24z^2 + z^4$ and $z^4 b_7^{(1)}(z) = 9{,}792 + 138z^2 + 21z^4$.

Table 5. Computation of Trial Solution for Example 2 in Terms of a_n and $a_n^{(1)}$ using Legendre Polynomials

n	Trial		1	z^2	z^4	z^6	z^8	z^{10}	z^{12}	$P_n(0)$
	$z^{12-n}a_n(z)$	$z^{11-n}a_n^{(1)}(z)$								
0	$z^{12}a_0$		59,207,075	−1,074,753.5	294,990.5	−24,966.5	2,164.5	−79	2	+1.000000
1		$z^{10}a_1^{(1)}$	11,814,150	+1,000,933.5	316,821	+12,799.5	1,153	+18		
2	$z^{10}a_2$		3,938,050	−341,450	78,415	−3,970	220	−2.5		−0.500000
3		$z^8a_3^{(1)}$	4,725,661.5	+190,344	57,655.5	+1,151.5	59.5			
4	z^8a_4		675,094.5	−40,999.5	6,655.5	−229.5	4.5			+0.375000
5		$z^6a_5^{(1)}$	750,106.5	+17,391	4,334	+44				
6	z^6a_6		68,191.5	−3,315	325	−6.5				−0.312500
7		$z^4a_7^{(1)}$	73,440	+1,035	157.5					
8	z^4a_8		4,896	−204	8.5					+0.273438
9		$z^2a_9^{(1)}$	5,187	+38						
10	z^2a_{10}		273	−10.5						−0.246094
11		$a_{11}^{(1)}$	287.5							
12	a_{12}		12.5							+0.225586

Note. $z^4a_8(z)=4{,}896-204z^2+8.5z^4$ and $z^4a_7^{(1)}=73{,}440+1{,}035z^2+157.5z^4$.

and in terms of the coefficients b_n, we find

$$\frac{2n+1}{2} b_n^{(1)}(z) - z^2 [n b_{n-1}(z) + (n+1) b_{n+1}(z)] = 0.$$

With this equation in the form

$$b_{n-1} = \frac{1}{2n}\left[\frac{2n+1}{z^2} b_n^{(1)} - 2(n+1) b_{n+1}\right] \tag{53}$$

and using Equation (48) with $s=0$ in the form

$$b_{n-1}^{(1)} = b_{n+1}^{(1)} + (2n+1) b_n \tag{54}$$

we can compute b_n, $b_n^{(1)}$ and hence a_n by the method of recurrence.

Taking $b_{12}=1$, with $b_{14}=b_{16}=\ldots=b_{13}^{(1)}=b_{15}^{(1)}=\ldots=0$, and rounding the coefficients of the powers of z in the other coefficients as they are calculated, we obtain the trial solution shown in Table 4. The coefficients $a_n(z)$ and $a_n^{(1)}(z)$ can now be computed by making use of Equation (47) with $s=0$ and $s=1$ respectively. These are given in Table 5. We find that when this trial solution has been computed, all Equations (53) and (54) have been satisfied to a certain accuracy. The as-yet-unsatisfied condition gives

$$\sum_{n=0}^{6} a_{2n}(z) P_{2n}(0) = \alpha(z)$$

from which

$$\alpha(z) = (5{,}907{,}075 - 3{,}043{,}778.5z^2 + 718{,}875.96z^4 - 100{,}858.66z^6 + 9{,}020z^8 - 499.59z^{10} + 14.7z^{12})/z^{12}.$$

Thus we have

$$y(zx) = R(z, x) = \frac{1}{\alpha(z)} \sum_{n=0}^{6} a_{2n}(z) P_{2n}(x). \tag{55}$$

Put $x=1$ and Equation (55) yields the required rational approximation for e^{z^2} obtained from the Legendre expansion in the form

$$e^{z^2} = R(z, 1) = P(z)/Q(z),$$

where

$$P(z) = 5{,}907{,}075 + 2{,}863{,}296.5z^2 + 628{,}635z^4 + 80{,}640.5z^6 + 6{,}431z^8 + 305.5z^{10} + 8z^{12}$$

and

$$Q(z) = 5{,}907{,}075 - 3{,}043{,}778.5z^2 + 718{,}875.96z^4 - 100{,}858.66z^6 + 9{,}020z^8 - 499.59z^{10} + 14.7z^{12}.$$

As a check we find $y(1) = 2.718281 = e$. The polynomial Legendre approximation is obtained from Equation (55) by taking $z=1$, which gives

$$y(x) = R(1, x) = 1.46265P_0(x) + 1.05198P_2(x) + 0.18354P_4(x) + 0.01868P_6(x) + 0.00135P_8(x) + 0.00008P_{10}(x). \tag{56}$$

The relation (56) is in agreement with that obtained by Elliott [7].

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