ON COMPLETE BOOLEAN ALGEBRA OF PROJECTIONS

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The aim of this note is to prove that a bounded Boolean algebra of projections on a weakly complete Banach space X can be embedded in a σ-complete Boolean algebra of projections on X.

1. NOTATIONS

Throughout this note, X will be a complex Banach space with a dual space X^*. The value of the functional x^* in X^* at x in X will be denoted by $\langle x, x^* \rangle$. We use $L(X)$ to denote the algebra of all linear bounded operators on X. The zero and identity operators in $L(X)$ will be denoted by 0 and I, respectively. $C(X)$ will be the algebra of all continuous, complex-valued functions on X. If A is a subset of X then $\mathrm{clm}(A)$ is the closed linear manifold spanned by A. The field of complex numbers will be denoted by **C**.

2. PRELIMINARIES

2.1. Definition

A Boolean algebra of projections on X is a commutative subset, **B**, of $L(X)$ such that

(1) $E^2 = E$ for each $E \in \mathbf{B}$;

(2) $0 \in \mathbf{B}$;

(3) If $E \in \mathbf{B}$ then $I - E \in \mathbf{B}$;

(4) If E, $F \in \mathbf{B}$ then $E \vee F = E + F - EF$ and $E \wedge F = EF$ are in **B**.

A Boolean algebra of projections, **B**, on X is said to be bounded if $||E|| \leq M$ for every E in **B**, where M is a real number.

2.2. Definition

A Boolean algebra **B** of projections on X is complete (σ-complete) if for every subset (sequence) $\{E_i\}$ in **B**, the greatest lower bound $\wedge E_i$ and the least upper bound $\vee E_i$ of $\{E_i\}$ exist in **B** and $(\vee E_i)X = \mathrm{clm}\{E_i X\}$, $(\wedge E_i)X = \cap E_i X$.

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2.3. Theorem

Let **B** be a σ-complete Boolean algebra of projections on X. Then the following statements are equivalent:

(1) **B** is complete.

(2) **B** is strongly closed.

(3) **B** is weakly closed.

Proof

(1)$\rightarrow$(2): see Reference [4] (Corollary 7, p. 2201).

(2)$\rightarrow$(3): Since **B** is strongly closed, $\mathbf{B}=\mathbf{B}^s$ (the strong closure of **B**). Since **B** is σ-complete then by Reference [4] (Theorem 27, p. 2218), $\mathbf{B}^s \supseteq \mathbf{B}^w$ (the weak closure of **B**). Since always $\mathbf{B}^s \subseteq \mathbf{B}^w$, it follows that $\mathbf{B}=\mathbf{B}^s=\mathbf{B}^w$. Hence **B** is weakly closed.

(3)$\rightarrow$(2): Since **B** is σ-complete, $\mathbf{B}^s=\mathbf{B}^w$. Since **B** is weakly closed, it follows that $\mathbf{B}=\mathbf{B}^w=\mathbf{B}^s$ which means that **B** is strongly closed.

(2)$\rightarrow$(1): Since **B** is σ-complete, then by Reference [1] (Theorem 2.7, p. 350), $\mathbf{B}^s$ is complete. Since **B** is strongly closed, $\mathbf{B}=\mathbf{B}^s$ which implies that **B** is complete.

2.4. Let **B** be a complete Boolean algebra of projections on X and let Ω be its Stone representation space [4], Theorem 1.12.1, p. 41). Then by Reference [4] (Exercise 16, p. 2225), Ω is extremely disconnected in the sense that the closure of every open subset is open. If Σ_Ω denotes the Borel field of Ω, then to each Borel set e_0 in Σ_Ω, corresponds a unique open-and-closed set e such that $(e_0\backslash e) \cup (e\backslash e_0)$ is a set of the first category. Moreover, each Borel function differs from a unique continuous function, on a Borel set of the first category. Now if e is an open-and-closed subset of Ω, we denote by $E(e)$ the element of **B** corresponding to e. This mapping is extended to the Borel field Σ_Ω by setting $E(e_0)=E(e)$ for each Borel set e_0, where e is the open-and-closed subset of Ω which differs from e_0 by a set of the first category, in the above sense. It follows from the definition of completeness and Reference [4] (lemma 4, pp. 2197–2198) that the vector and scalar-valued measures $E(\cdot)x$ and $<E(\cdot)x, x^*>$ associated with $x\varepsilon X$ and $x^*\varepsilon X^*$, respectively, are countably additive on Σ_Ω. By Theorem 2.2 of Reference [1], **B** is uniformly bounded. Comparison of the above with the definition of a spectral measure ([3], p. 119) shows that we may regard a complete Boolean algebra of projections as a spectral measure defined on the Borel field of the Stone representation space.

2.5. Theorem

A bounded Boolean algebra **B** of projections on a weakly complete Banach space X can be embedded in a σ-complete Boolean algebra of projections on X.

Proof. Let Ω be the Stone representation space of **B**. Let $K(\Omega)$ be the set of all characteristic functions of open-and-closed subsets of Ω, and let $\psi: K(\Omega)\rightarrow \mathbf{B}$ such that $\psi(K_e)=\mathbf{B}(e)$ be the representation isomorphism for **B**. Let $K^1(\Omega)$ be the algebra of all finite sums $\Sigma c_j K_{e_j}$ ($c\in\mathbf{C}$, e_j is open and closed in Ω), and let $\mathbf{B}^1$ be the corresponding algebra of sums $\Sigma c_j \mathbf{B}(e_j)$. Then ψ extends to an algebra isomorphism $\psi^1: K^1(\Omega)\rightarrow \mathbf{B}^1$ such that $\psi^1(\Sigma c_j K_{e_j})=\Sigma c_j \mathbf{B}(e_j)$ and ψ^1 is an isometry ([2], Theorem 2.1). Since Ω is totally disconnected, $K^1(\Omega)$ is norm dense in $C(\Omega)$. Hence ψ^1 can be extended to an isometric isomorphism (also denoted by ψ^1) $\psi^1: C(\Omega)\rightarrow L(X)$ and by Reference [4] (Theorem 4, p. 2184),

$$\psi^1(f)^* = \int_\Omega f(\lambda)E(\mathrm{d}\lambda), \quad (f\in C(\Omega))$$

where E is a spectral measure in X^* defined on the Borel sets in Ω. Let Σ be the algebra of Borel sets in Ω and let Σ_0 be the class of sets in Σ such that there is a spectral measure F defined by $F(e)^*=E(e)$ for every e in Σ_0. Then it is easy to show that Σ_0 is a field (or a Boolean algebra of sets).

Let $\{e_n\}$ be a sequence of sets in Σ_0 then

$$\langle x, E\left(\bigcup_{n=1}^{m} e_n\right) y\rangle = \langle x, F\left(\bigcup_{n=1}^{m} e_n\right)^* y\rangle$$

$$= \langle F\left(\bigcup_{n=1}^{m} e_n\right) x, y\rangle \text{ for } x\in X,\ y\in X^*.$$

Since $\langle x, E(\cup_{n=1}^m e_n)y\rangle$ is a Cauchy sequence, $\langle F(\cup_{n=1}^m e_n)x, y\rangle$ is a Cauchy sequence. Hence $F(\cup_{n=1}^m e_n)x$ is a weak Cauchy sequence. Since X is weakly complete, $F(\cup_{n=1}^m e_n)x$ is weakly convergent for each $x\in X$. Since $F(\cup_{n=1}^m e_n)$ is naturally ordered and uniformly bounded, then, by Reference [3] (Theorem 6.4, pp. 159–160), the sequence $\{F(\cup_{n=1}^m e_j)\}$ is convergent in the strong operator topology. Hence, by Reference [4] (Lemma 4, pp. 2197–2198), $\{F(e): e\in\Sigma_0\}$ is a σ-complete Boolean algebra of projections in $L(X)$ and **B** is embedded in it.

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