FACTORIZATION OF A POLYNOMIAL MATRIX WITH RESPECT TO THE UNIT CIRCLE

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1. INTRODUCTION

Matrix factorization problems appear in many Systems and Control theory problems. For example, in linear optimal control an $m \times m$ matrix:

$$B(z) = B_0 z^n + B_1 z^{n-1} + \cdots + B_{n-1} z + B_n + B_{n-1}^T z^{-1} + \cdots + B_0^T z^{-n} , \tag{1}$$

needs to be factorized such that:

$$B(z) = H^*(z) H(z) , \tag{2}$$

where

$$H(z) = H_0 + H_1 z + \cdots + H_n z^n , \tag{3}$$

with $H^{-1}(z)$ having no poles inside the unit circle [1]. In the above expressions, ^{T} stands for the operation of transposition, index * corresponds to the operation of transposition and change of argument from z to z^{-1} (for example in (3), $H^*(z) = H_0^T + H_1^T z^{-1} + \cdots + H_n^T z^{-n}$). We sometimes omit the argument z for brevity. There are several numerical algorithms for finding the polynomial $H(z)$ (see, for example, [1–4]). In Game Theory and H_∞ optimal control problems (see [5–7]) the so called J-spectral factorization problem is obtained, which is given as:

$$B(z) = H^*(z) T H(z) , \tag{4}$$

where the symmetrical matrix T may have eigenvalues of different signs.

In this paper a new computationally reliable algorithm for J-spectral factorization is presented.

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2. MAIN RESULT: A NEW ALGORITHM FOR FACTORIZATION OF POLYNOMIAL MATRICES

In this section we give a new algorithm for factorization of polynomial matrices. The following theorem gives the main result of the paper.

Theorem 1 *The polynomial matrix given by (1) can be factorized as:*

$$B(z) = \left[D^* + N^*\psi^T S\Gamma[B_\phi + \Gamma^T S\Gamma]^{-1}\right][B_\phi + \Gamma^T S\Gamma]\left[D + [B_\phi + \Gamma^T S\Gamma]^{-1}\Gamma^T S\psi N\right] , \tag{5}$$

where:

$$D = I + z^n B_\phi^{-1} B_0, \quad N^T = [Iz,\, Iz^2,\, \cdots,\, Iz^n] \tag{6}$$

$$B_\phi = B(1) = B_0 + B_1 + \cdots + B_n + \cdots + B_1^T + B_0^T \tag{7}$$

$$\psi = \begin{pmatrix} 0 & 0 & 0 & \cdots & -B_\phi^{-1}B_0 \\ I & 0 & 0 & \cdots & . \\ 0 & I & 0 & \cdots & . \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & I & \cdots & 0 \end{pmatrix}, \quad \Gamma^T = [I\ 0, \ldots, 0]; \tag{8}$$

where each block of Ψ has dimension $m \times m$, I is the identity matrix with appropriate dimension. The $mn \times mn$ matrix S is the stabilizing solution of discrete algebraic Riccati equation (ARE):

$$S = \psi^T S\psi - \psi^T S\Gamma(B_\phi + \Gamma^T S\Gamma)^{-1}\Gamma^T S\psi + R\,, \tag{9}$$

where

$$R = \begin{pmatrix} \bar{B} & B_{n-1} & \cdots & B_1 \\ B_{n-1}^T & \bar{B}\ 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ B_1^T & 0 & \cdots & 0\ \bar{B} \end{pmatrix}, \ \bar{B} = \frac{1}{n}\left(B_n - B_\phi - B_0^T B_\phi^{-1} B_0\right) , \tag{10}$$

such that:

$$(I - \Gamma(B_\phi + \Gamma^T S\Gamma)^{-1}\Gamma^T S)\psi \tag{11}$$

has all of its eigenvalues inside the unit circle. Comparing (4) and (5) we get:

$$H(z) \ = \ D + \left(B_\phi + \Gamma^T S\Gamma\right)^{-1}\Gamma^T S\psi N \tag{12}$$

$$T \ = \ B_\psi + \Gamma^T S\Gamma\,. \tag{13}$$

Note that in the case of $S > 0$, the sign of matrix $B_\phi + \Gamma^T S\Gamma$ is the same as the sign of the matrix B_ϕ [6].

3. PROOF OF THE MAIN THEOREM

The proof that relation (5) gives the desired factorization consists of two steps:

1. The formal verification of the equivalency of (1) and (5).
2. Localization of the spectrum of the polynomial $H(z)$ defined by (12).

The proof of Step 1 is obtained by direct verification. From (9) we have:

$$\psi^T S\Gamma[B_\phi + \Gamma^T S\Gamma]^{-1}\Gamma^T S\psi = R - S + \psi^T S\psi \ ,$$

and

$$\psi N + \Gamma D = z^{-1} N \ .$$

Substituting these relations in (5) we obtain the polynomial:

$$B(z) = D^* B_\phi D + N^* RN = B_0 z^n + B_1 z^{n-1} + \cdots + B_{n-1} z + B_n + B_{n-1}^T z^{-1} + \cdots + B_0^T z^{-n} \ ,$$

which establishes the equality of (5) and (1).

To prove the second step, we show that the determinant of polynomial $H(z)$ has no zeros inside the unit circle. Let $[S_1, \cdots, S_n] = \Gamma^T S$, where blocks S_i, $i = 1, 2, \cdots, n$ have dimensions $m \times m$. From (12), we have:

$$H(z) = I + H_n z^n + \cdots + H_1 z \ , \tag{14}$$

where:

$$H_i = T^{-1} S_{i+1}, \quad i = 1, 2, \cdots, n-1 \tag{15}$$

$$H_n = B_\phi^{-1} B_0 - T^{-1} S_1 B_\phi^{-1} B_0 \ , \tag{16}$$

where the matrix T is defined by Equation (13). Rewrite (14) in the following form:

$$H(z) = z^n \bar{H}(s), \quad \bar{H}(z) = Is^n + H_1 s^{n-1} + \cdots + H_n \ , \tag{17}$$

with $s = z^{-1}$. From [8] (Lemma 6.3-20), the matrix Ω, which is the linearization of the polynomial $\bar{H}(s)$, is given as:

$$\Omega = \begin{pmatrix} -H_1 & -H_2 & \cdots & -H_n \\ I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix} , \tag{18}$$

which is equal to (11). Eigenvalues of matrix given by (11) lie inside the unit circle since the zeros of the determinant of polynomial $\bar{H}(s)$ also lie inside the unit circle. Therefore, the matrix $H^{-1}(z)$ has no poles inside the unit circle.

4. EXAMPLES

In this section we give two examples to illustrate the results. In Section 4.1 an example from [1] is studied. In Section 4.2, a modification of Example 1 is studied to illustrate J-Spectral factorization.

4.1. Example 1

Consider the example reported in [1]. The polynomial $B(z)$ is given as:

$$B(z) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} z + \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} z^{-1}. \tag{19}$$

Following (9) and (12) we have:

$$S = \begin{bmatrix} -0.25 & -0.75 \\ -0.75 & 1.75 \end{bmatrix}, \quad T = \begin{bmatrix} 0.75 & -0.75 \\ -0.75 & 4.75 \end{bmatrix} \tag{20}$$

$$H(z) = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{21}$$

In this case there is a factorization of the form given by (2). Since the matrix T is positive definite, using Cholesky decomposition, we have:

$$T = L^T L = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 2 \end{bmatrix}. \tag{22}$$

Finally:

$$B(s) = \tau^*(z)\tau(z)\ , \tag{23}$$

where

$$\tau(z) = LH(z) = \begin{bmatrix} 0 & 0 \\ 1/2 & -1/2 \end{bmatrix} z + \begin{bmatrix} \sqrt{3}/2 & -\sqrt{3}/2 \\ 0 & 2 \end{bmatrix}.$$

This result is the same as the one reported in [1].

4.2. Example 2

We modify Example 1 for the purpose of illustration of the entire procedure of J-spectral factorization. Consider the case where the matrix T has eigenvalues of different signs. Let the polynomial (1) have the following form:

$$B(z) = \tau(z)J\tau^*(z)\ , \tag{24}$$

where $J = diag(I, -I)$ and the polynomial $\tau(z)$ is as defined above. We have:

$$B(z) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} z + \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & -4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z^{-1}. \tag{25}$$

Using (9) and (12), we get:

$$S = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} T = \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & -4 \end{bmatrix} \tag{26}$$

$$H(z) = \begin{bmatrix} 0 & \sqrt{3}/3 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{27}$$

In this example the matrix T has eigenvalues of different signs and hence may be represented in the form:

$$T = L^T J L, \quad L = \lambda U, \quad \lambda = diag(0.8036, 2.1555) \tag{28}$$

$$U = \begin{bmatrix} 0.937 & 0.3493 \\ -0.3493 & 0.937 \end{bmatrix}, \quad L = \lambda U = \begin{bmatrix} 0.753 & 0.2807 \\ -0.753 & 2.0196 \end{bmatrix} . \tag{29}$$

Finally, J-spectral factorization of the polynomial $B(z)$ is obtained as $B(z) = [LH(z)]^* JLH(z)$.

5. CONCLUSION

A new algorithm for the J-spectral factorization of polynomial matrices with respect to the unit circle is presented. The algorithm is based on construction of a stabilizing solution for the algebraic Riccati equation. Two examples are given to illustrate the results.

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