FIRST BOUNDARY VALUE PROBLEM FOR ELLIPTIC EQUATIONS ON RECTANGULAR REGIONS

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الخلاصة :

يهتم هذا البحث بحلول مسألة « ديريشلية » للمعادلات الاهليليجية الخطية ، في المجال المستوي Ω الذي تحتوي حدودة على أركان . تعطي الشروط الكافية لجعل الحل من المرتبة $C_{2+\alpha}(\overline{\Omega})$: $0 < \alpha < 1$.

ABSTRACT

This paper is concerned with solutions of the Dirichlet problem for linear elliptic equations in sectionally smooth plane domains. Conditions sufficient for the solutions to be of class $C_{2+\alpha}$ up to the boundary are given.

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1. INTRODUCTION

We consider here the first boundary value problem for linear second order elliptic differential equations in a plane domain Ω with corners on its boundary. In [2, 3] it was shown that small angles on the boundary correspond to better differentiability properties of the solutions. In [6] the following result was obtained

Theorem 1. Let Ω be a rectangle and consider the Dirichlet problem

$$\Delta u = f \text{ in } \Omega, \qquad u = \psi \text{ on } \partial\Omega.$$

Assume that $f \in C_\alpha(\bar{\Omega})$ and that ψ is continuous on $\partial\Omega$ and belongs to $C_{2+\alpha}$ on the open sides of the rectangle. If the compatibility conditions at the four corners are satisfied, then $u \in C_{2+\alpha}(\bar{\Omega})$.

In the present paper we extend this result to the case of second order linear elliptic equations. In Section 2 we state the problem and the main result. Section 3 contains a study of the problem in a circular sector. The proof of the general case follows from this special one. In Section 4 we state a theorem concerning domains with more than one corner. We introduce here some symbols used in the paper. $C_{m+\alpha}(\Omega)$ ($m \geqq 0$ an integer, $0<\alpha<1$) is the space of functions having in Ω, m continuous partial derivatives and the derivatives of order m satisfy in Ω a Hölder condition with exponent α.

$D^k W$ is any kth partial derivative of W. $H_\mu^\Omega(D^k W)$ is the Hölder coefficient of $D^k W$ in Ω.

$$\|W\|_0^\Omega = \max_{x\in\Omega} |W(x)|$$

$$\|W\|_{2+\mu}^\Omega = \sum_{k=0}^{2} \|D^k W\|_0^\Omega + H_\mu^\Omega(D^2 W)$$

2. THE PROBLEM: MAIN RESULT

Let $\Omega \subset R^2$ be a simply connected domain whose boundary Γ is of class $C_{2+\alpha}$, $0<\alpha<1$, except at a point $P\in\Gamma$ where Γ has a corner. In Ω we consider the Dirichlet problem

$$Lu = a_{ij}(x)u_{ij} + a_i(x)u_i + a(x)u = f(x) \qquad (1)$$

$$u_{|\Gamma} = \psi, \qquad (2)$$

where $x=(x_1, x_2)$, $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$, $i, j = 1, 2$ and we use the summation convention. We assume that (1) is uniformly elliptic and that the functions a_{ij}, a_i, a, and f belong to $C_\alpha(\bar{\Omega})$ in (1) and $\psi \in C_0(\Gamma) \cap C_{2+\alpha}(\Gamma\backslash\{P\})$ in (2). Under these assumptions it is known that $u \in C_{2+\alpha}(\Omega_1) \cap C_0(\bar{\Omega})$, where Ω_1 is any compact subregion of $\bar{\Omega}$ with positive distance from the corner (*cf.* [1]). To investigate the behavior of the solutions near the corner point we proceed as follows. Without loss of generality, assume that the corner point is at the origin and that the two curves bounding the corner are represented by $x_1 = g_2(x_2)$ and $x_2 = g_1(x_1)$ where $g_1(0) = g_2(0) = g_1'(0) = 0$ and $g_2'(0) = \cot\gamma$. We transfer the equation

$$a_{ij}(0,0)u_{ij} = 0$$

to canonical form by applying the transformation

$$y_1 = \frac{1}{\Delta\sqrt{\alpha_{11}}}[\alpha_{12}(x_1 - g_2(x_2)) + \alpha_{11}(x_2 - g_1(x_1))]$$

$$y_2 = \frac{1}{\sqrt{\alpha_{11}}}(x_1 - g_2(x_2)),$$

$$\alpha_{11} = a_{11}(0,0) - 2g_2'(0)a_{12}(0,0) + g_2'^2(0)a_{22}(0,0),$$

$$\alpha_{22} = a_{22}(0,0)$$

$$\alpha_{12} = a_{22}(0,0)g_2'(0) - a_{12}(0,0),$$

$$\Delta = [a_{11}(0,0)a_{22}(0,0) - a_{12}^2(0,0)]^{\frac{1}{2}}.$$

The angle at the corner after transformation is given by

$$\omega = \arctan\{[a_{11}(0)a_{22}(0) - a_{12}^2(0)]^{\frac{1}{2}} / [a_{22}(0)\cot\gamma - a_{12}(0)]\} \qquad (3)$$

where γ is the angle at the corner of Γ. Note that $\omega = \gamma$ if the leading part of (1) is the Laplacian. In [2] it was proved that if $\omega < \pi$ then $u \in C_\nu(\bar{\Omega})$ where $1<\nu<2$. As was mentioned before, in the case of $C_{2+\alpha}$-boundaries with $\psi \in C_{2+\alpha}(\Gamma)$ we have $u \in C_{2+\alpha}(\bar{\Omega})$. In [3] we have proved that if $\omega < \pi/(2+\alpha)$ then also in this case $u \in C_{2+\alpha}(\bar{\Omega})$. We now generalize Theorem 1 as follows.

Theorem 2. Let $\Omega \subset R^2$ be a simply-connected bounded domain whose boundary Γ is of class $C_{2+\alpha}$, except at a point $P\in\Gamma$ where Γ has a corner with angle γ, $0<\gamma<2\pi$. Assume that (1) is uniformly elliptic and that a_{ij}, a_i, a, and f belong to $C_\alpha(\bar{\Omega})$ in (1) and $\psi(s) \in C_0(\Gamma) \cap C_{2+\alpha}(\Gamma\backslash\{P\})$ in (2). If $\omega = \pi/2$ in (3), then any solution u of (1), (2) in Ω satisfies $u \in C_{2+\alpha}(\bar{\Omega})$, provided that the compatibility conditions at the corner point are satisfied.

This theorem is the main result of the paper. As was mentioned before, the solution of the given problem belongs to $C_{2+\alpha}(\bar{\Omega}\backslash\Omega_1)$, where Ω_1 is any small neighborhood of the corner. Thus to prove Theorem 2 it is sufficient to show that $u \in C_{2+\alpha}(\bar{\Omega}_1)$. As shown in [2], for this purpose, it is sufficient to consider (1), (2) in a circular sector with angle ω at 0 where $a_{ij}(0)=\delta_{ij}$ in (1). This will be done in the following section.

3. THE PROBLEM IN THE SECTOR CASE

We follow here the notations of [2, Section 4] with $\omega=\pi/2$. Let Ω_σ be the sector

$$\Omega_\sigma=\{(r,\theta)|0<r<\sigma,\ \beta<\theta<\pi/2+\beta\}$$

where $\sigma<1$, (r,θ) are the polar coordinates of the point (x_1,x_2) and $\beta>0$ satisfies $\beta<\pi/4$. We now state the main result of this section.

Theorem 3. Let w be a bounded solution of (1), (2) in Ω_σ, where L in (1) is uniformly elliptic and $a_{ij}(0)=\delta_{ij}$. Let a_{ij}, a_i, a, and f belong to $C_\alpha(\bar{\Omega}_\sigma)$ and $\psi(r,\theta)$ is continuous at the corner of the sector and belongs to $C_{2+\alpha}$ on the lines $\theta=\beta$ and $\theta=\pi/2+\beta$. If $f(0)=0$ then $w \in C_{2+\alpha}(\bar{\Omega}_{r_0})$ where $2r_0<\sigma$.

We prove this theorem using Theorem 1 and some results of [2]. We note that $f(0)=0$ is the only compatibility condition needed. In [2, Theorem 4] it was proved that under the assumptions of Theorem 3 we have $w \in C_\nu(\bar{\Omega}_{r_0})$, where $1<\nu<\pi/(\omega+2\beta)<2$. In our case $\omega=\pi/2$. Thus for any arbitrarily small $\varepsilon>0$ we can take $\nu=2-\varepsilon$ by choosing $\beta<\varepsilon\pi/4(2-\varepsilon)$. Without loss of generality we may assume that

$$\psi(0,\beta)=\frac{d\psi}{dr}(0,\beta)=\frac{d\psi}{dr}(0,\pi/2+\beta)=0 \qquad (4)$$

cf. [2, Section 4]. Under the assumptions of Theorem 3 with (4) being satisfied, it was proved in [2] that in $\bar{\Omega}_{2r_0}$ we have

$$|D^k\, w(x)| \leqq M_k\, r^{2-k-\varepsilon}, \qquad k=0,\,1,\,2. \qquad (5)$$

It was also proved in [2] that the second derivatives of w may have 'pole-like' singularities at the corner point, nevertheless by multiplying these derivatives by $r^\tau, 0<\tau<1$, we get Hölder-continuous functions. We now extend this result by replacing r^τ with any $O(r^\tau)$-function.

Lemma 1. Let $v(x)$ be a solution of (1), (2) in Ω_σ. Assume that all the assumptions of Theorem 3 are satisfied and that in Ω_σ we have $|D^k v(x)| \leqq N_k r^{2-k-\eta}$, $k=0,1,2$ where $0 \leqq \eta<1$. If $h(x) \in C_\tau(\bar{\Omega}_\sigma)$, $\eta \leqq \tau<1$ and $h(0)=0$, then $h(x)D^2 v(x) \in C_\mu(\bar{\Omega}_{r_0})$, where $\mu=\min(\alpha,\tau-\eta)$.

Proof. Consider in Ω_{r_0} any two points $P(r_1,\theta_1)$ and $Q(r_2,\theta_2)$. To prove the lemma we have to show that

$$|h(P)\ D^2 v(P)-h(Q)\ D^2 v(Q)|/\overline{PQ}^\mu \leqq H, \qquad (6)$$

for some finite $H>0$. Let $0 \leqq r_2 \leqq r_1 \leqq r_0$. We consider separately the two cases $r_2 \leqq r_1/2$ and $r_2>r_1/2$. In the first case we have $\overline{PQ} \geqq r_1/2$ and since

$$|D^2\, v(x)| \leqq N_2\, r^{-\eta} \qquad (7)$$

and

$$|h(x)| \leqq N_3\, r^\tau, \qquad (8)$$

we can easily obtain (6). Consider now the case $r_2>r_1|2$. We introduce the mapping

$$x=\xi y \qquad (9)$$

where $\xi=2r_1/r_0$, $y=(y_1,y_2)$. This transformation takes

$$\Omega_0=\{(r,\theta)|r_1/2 \leqq r \leqq r_1,\ \beta \leqq \theta \leqq \pi/2+\beta\}$$

to

$$\Omega_1=\{(\rho,\theta)|r_0/4 \leqq \rho \leqq r_0/2,\ \beta \leqq \theta \leqq \pi/2+\beta\}$$

where $\rho=r/\xi$. In [2, Theorem 4] it was shown that the transformed function $v_1(y)=v(\xi y)$ satisfies in Ω_1

$$\|v_1\|_{2+\alpha}^{\Omega_1} \leqq \delta r_1^{2-\eta}, \qquad (10)$$

Now, for any $\mu \leqq \alpha$ we have

$$\xi^{2+\mu} H_\mu^{\Omega_0}(D^2\ v)=H_\mu^{\Omega_1}(D_1^2\ v_1) \leqq \delta_1 r_1^{2-\eta}, \qquad (11)$$

where D_1 is the partial derivative corresponding to D. Thus

$$H_\mu^{\Omega_0}(D^2\ v) \leqq \delta_2 r_1{}^{-\mu-\eta} \qquad (12)$$

We now prove (6) in the case $r_2>r_1/2$. From (7), (8) and (12) we obtain

$$\begin{aligned}
&|h(P)\ D^2\ v(P)-h(Q)\ D^2\ v(Q)|/\overline{PQ}^\mu\\
&\leqq |h(P)|\ |D^2\ v(P)-D^2\ v(Q)|/\overline{PQ}^\mu\\
&+|D^2\ v(Q)|\{|h(P)-h(Q)|/\overline{PQ}^\tau\}^{\mu/\tau}|h(P)-h(Q)|^{1-\mu/\tau}\\
&\leqq N_3 r_1^\tau \delta_2 r_1^{-\mu-\eta}+N_2 r_2^{-\eta} N_4\{2N_3 r_1^\tau\}^{\eta/\tau} \leqq H,
\end{aligned}$$

since $h \in C_\tau$ and $r_1/2<r_2 \leqq r_1$. This completes the proof of the lemma.

We now prove Theorem 3 assuming that (4) holds.

Proof of Theorem 3. Under the assumptions of the theorem and assuming that (4) holds we have the estimates

$$|D^2 w| \leqq M_2 r^{-\varepsilon} \tag{13}$$

and $w \in C_{2-\varepsilon}$. *cf.* [2, Theorems 3,4]. Here $C_{2-\varepsilon}$ means $C_{2-\varepsilon}(\bar{\Omega}_{r_0})$ and similarly we omit $\bar{\Omega}_{r_0}$ in the present proof. We write equation (1) as follows

$$\Delta w = f_1 \equiv f - aw - a_i w_i - (a_{ij} - \delta_{ij}) w_{ij}. \tag{14}$$

Since f, a, and a_i belong to C_α and $w \in C_{2-\varepsilon}$ with arbitrarily small $\varepsilon > 0$, then the first three terms on the right-hand-side of (14) belong to C_α. Using Lemma 1 with $h(x) = a_{ij}(x) - \delta_{ij}$, $v = w$, $\tau = \alpha$ and $\eta = \varepsilon$ we conclude that $(a_{ij} - \delta_{ij})\, w_{ij} \in C_{\alpha-\varepsilon}$. Thus $f_1 \in C_{\alpha-\varepsilon}$ in (14). Theorem 1 then gives $w \in C_{2+\alpha-\varepsilon}$. To prove that $w \in C_{2+\alpha}$ we proceed as follows. We write

$$w = w_0 + v, \tag{15}$$

where

$$w_0(x) = w(0) + x_i w_i(0) + (1/2!) x_i x_j w_{ij}(0).$$

It is clear that $v \in C_2$ and that $D^k v(0) = 0$, $k = 0, 1, 2$. Thus $|D^k v(x)| \leqq N_k r^{2-k}$, $k = 0, 1, 2$. In Ω_{r_0} the function $v(x)$ satisfies the equation

$$Lv = f_2 \equiv f - Lw_0 \in C_\alpha, \tag{16}$$

cf. (1), and coincides on $\theta = \beta$ and $\theta = \pi/2 + \beta$ with a function $\chi(r)$ where $\chi(0) = \chi'(0) = \chi''(0) = 0$. We now show that $v \in C_{2+\alpha}$. We write (16) as follows

$$\Delta v = f_3 \equiv f_2 - av - a_i v_i - (a_{ij} - \delta_{ij}) v_{ij} \tag{17}$$

The first three terms on the right-hand-side of (17) belong to C_α. Applying again Lemma 1 on $h(x) = a_{ij}(x) - \delta_{ij}$ and v_{ij} with $\tau = \alpha$ and $\eta = 0$ we obtain $(a_{ij} - \delta_{ij}) v_{ij} \in C_\alpha$. Thus $f_3 \in C_\alpha$ and using Theorem 1 we obtain $v \in C_{2+\alpha}$. From (15) we finally get $w \in C_{2+\alpha}(\bar{\Omega}_{r_0})$. This completes the proof of the theorem.

4. THE GENERAL CASE

As was mentioned in Section 2, Theorem 2 follows from Theorem 3. Using Theorem 7.3 in [1, p. 668], as well as [3] and Theorem 2 of the present paper we obtain the following theorem.

Theorem 4. Let $\Omega \subset R^2$ be a bounded plane domain whose boundary Γ consists of a finite number of curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, $k \geqq 2$ belonging to $C_{2+\alpha}$. Suppose that Γ_i and Γ_{i+1} intersect at O_i making an angle γ_i, $0 < \gamma_i < 2\pi$. Assume that u satisfies (1), (2) in Ω where $\psi \in C_{2+\alpha}(\Gamma \backslash \cup O_i) \cap C_0(\Gamma)$. If L is uniformly elliptic and a_{ij}, a_i, a, and f belong to $C_\alpha(\bar{\Omega})$, then $u \in C_{2+\alpha}(\bar{\Omega}_1)$, where Ω_1 is a compact subdomain of $\bar{\Omega}$ with positive distances from those corners satisfying neither $\omega_i < \pi/(2+\alpha)$ nor $\omega_i = \pi/2$. In the neighborhood of such a corner we have $u \in C_{\pi/\omega_i - \varepsilon}$. $\varepsilon > 0$ is arbitrarily small.

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