DECOMPOSITION OF MEASURES ORTHOGONAL TO $H^{\infty}(D)$

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الخلاصة :

نفرض أن D هو نطاق محدد في المسنوي المركب ، أن $H^{\infty}(D)$ هو النظام الجبري للدوال التحليلية المحددة والمعرفة على D وأن $M(D)$ هو فراغ 'Maximal Ideal' لـ $H^{\infty}(D)$ ونفرض أن $R(\bar{D})$ هو النظام الجبري للدوال المتصلة والمعرفة على $\bar{D}$ والتي يمكن تقريبها بدوال قياسية أقطابها خارج $\bar{D}$. نبرهن في نذا البحث نظرية مشابهة 'Bishop's Splitting Lemma' لـ $H^{\infty}(D)$ شرط أن يوجد لكل $Z \in D$ 'dominant representing measure' وأن يكون $R(\bar{D})$ 'boundedly pointwise dense' في $H^{\infty}(D)$

ABSTRACT

Let D be a bounded domain in the complex plane, $H^{\infty}(D)$ the algebra of bounded analytic functions on D and $M(D)$ its maximal ideal space, let $R(\bar{D})$ be the algebra of all continuous functions on $\bar{D}$ which can be approximated uniformly on $\bar{D}$ by rational functions with poles off $\bar{D}$. In this paper we prove a theorem similar to Bishop's Splitting Lemma for $H^{\infty}(D)$ provided that every $x \in D$ has a dominant representing measure and $R(\bar{D})$ is boundedly pointwise dense in $H^{\infty}(D)$.

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DECOMPOSITION OF MEASURES ORTHOGONAL TO $H^\infty(D)$

INTRODUCTION:

Let D be a bounded domain in the complex plane and let $H^\infty(D)$ denote the Banach algebra of bounded analytic functions on D and $M(D)$ its maximal ideal space. $R(\bar{D})$ will denote the algebra of all functions in $C(\bar{D})$ which can be approximated uniformly on $\bar{D}$ by rational functions with poles off $\bar{D}$. Bishop's Splitting Lemma [7] states that if μ is a Borel measure on $\bar{D}$ which is orthogonal to $R(\bar{D})$ and $U_1, U_2, \ldots, U_n$ form an open cover for $\bar{D}$ then there exist measures $\mu_1, \mu_2, \ldots, \mu_n$ on $\bar{D}$ such that each μ_1 is orthogonal to $R(\bar{D})$, $\overline{\text{supp}}\ \mu_i \subseteq U_i$, and $\mu = \Sigma_{i=1}^n \mu_1$. In this paper we prove a similar result for measures on $M(D)$ orthogonal to $H^\infty(D)$. But D will not be an arbitrary domain; it will be a domain such that every $z \in D$ has a dominant representing measure and $R(\bar{D})$ is boundedly pointwise dense in $H^\infty(D)$. At first glance this might seem a strong restriction, but in fact it is not and to the contrary these domains cover the ones needed for the Corona problem [5], because any Δ-domain [3] satisfies both of these conditions and [1] proving the Corona problem for Δ-domains is equivalent to proving it for general domains see [1] and [6] for details).

NOTATIONS, DEFINITIONS, AND SOME PRELIMINARIES

Throughout this paper D will denote a bounded domain in the complex plane, $H^\infty(D)$ the Banach algebra of founded analytic functions of D, and $M(D)$ its maximal ideal space. $R(\bar{D})$ will denote the algebra of all continuous functions on $\bar{D}$ which can be approximated uniformly on $\bar{D}$ by rational functions with poles off $\bar{D}$. All measures considered in this paper are regular Borel measures.

If A is a function algebra and $M(A)$ its maximal ideal space, then the pseudo-hyperbolic distance between two points ψ and ϕ in $M(A)$ is defined by $\rho(\psi, \phi) = \sup\{\psi(f): f \in A,\ \|f\| \leq 1;\ \phi(f) = 0\}$. The relation $\rho(\psi, \phi) < 1$ is an equivalence relation and the equivalence classes are the Gleason parts of $M(A)$ [11].

A measure m on ∂D is called a representing measure for $z \in \bar{D}$ with respect to $R(\bar{D})$ if $f(z) = \int f\, dm$, $\forall f \in R(\bar{D})$. It is called a dominant representing measure if whenever m' is another representing measure for z then $m' \ll m$ (m' is absolutely continuous with respect to m).

If m, μ are two measures such that $m \ll \mu$ and $\mu \ll m$ we will write $m \sim \mu$.

The closed support of a measure μ will be denoted by $\overline{\text{suppl}}\ \mu$.

By λ_z we will denote the harmonic measure on ∂D, for all $z \in D$. We will say $R(\bar{D})$ is boundedly pointwise dense in $H^\infty(D)$ if $\forall f \in H^\infty(D)$ there exists a sequence $\{f_n\} \subseteq R(\bar{D})$ such that $f_n(z) \to f(z)$, $\forall z \in D$, and $\|f\| \leq M$ for all n, and for some positive real number M.

For a function $f \in H^\infty(D)$, $\hat{f}$ will denote its Gelfand transform i.e. $\hat{f}$ is defined on $M(D)$ by $\hat{f}(\phi) = \phi(f)$. In [5] it was proved that if Z is defined on D by $Z(\lambda) = \lambda$ then $\hat{Z}(M(D)) = \bar{D}$, and $\hat{Z}$ is a homeomorphism on D.

If μ is a measure orthogonal to $R(\bar{D})$ (to $H^\infty(D)$) we will write $\mu \perp R(D)$ $(\mu \perp H^\infty(D))$.

If μ is a measure on $M(D)$, $\bar{\mu}$ will be the measure defined on $\bar{D}$ by $\bar{\mu}(E) = \mu(\hat{Z}^{-1}(E))$ [2].

If A is a function algebra on X, $p \in X$ is called a peak point if there exists $f \in A$ such that $f(p) = 1$ while $|f(y)| < 1$, $\forall y \in X$, $y \neq p$.

By the point mass measure for $x \in X$ we mean the measure δ_x defined by

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Lemma 1. If $z_0 \in D$ has a dominant representing measure m_{z_0} with respect to $R(\bar{D})$, then every point $z \in D$ has a dominant representing measure m_z and $m_z \sim m_{z_0}$.

Proof. Let $z \in D$. D is contained in a single Gleason part [4], so z and z_0 are in the same Gleason part. By [11] there exists a representing measure m_z for z with respect to $R(\bar{D})$ such that $m_{z_0} \ll m_z$. Similarly there exists a representing measure μ_{z_0} for z_0 such that $m_z \ll \mu_{z_0}$, but $\mu_{z_0} \ll m_{z_0}$ since m_{z_0} is a dominant representing measure, so $m_z \ll \mu_{z_0}$ hence $m_z \sim m_{z_0}$.

Let v_z be any representing measure for z. By the same theorem in [11], there exists v_{z_0}, a representing measure for z_0, such that $v_z \ll v_{z_0}$, but $v_{z_0} \ll m_{z_0} \ll m_z$ so $v_z \ll m_z$ which implies that m_z is a dominant representing measure.

Lemma 2. Let μ be a measure on D. Assume for some $z_0 \in D$ there exists a dominant representing measure m_{z_0}. Then there exists a measure μ^* with supp $\mu^* \subseteq \partial D$ such that $\mu^* \ll m_{z_0}$ and

$$\int_D f\,d\mu = \int_{\partial D} f\,d\mu^* \qquad \forall f \in R(D)$$

Proof. We use the sweeping method [10], i.e. for any Borel set $E \subseteq \partial D$, we define $\mu^*(E) = \int \lambda_z(E)d\mu(z)$, where λ_z is the harmonic measure on ∂D for $z \in D$. So $\mu^*(E) = \int_{\partial D} X_E d\mu^* = \int_{\partial D}(\int_D X_E d\lambda_z)d\mu(z)$ where X_E is the characteristic function on E.

Now for any $f \in C(\bar{D})$ we have

$$\int f\,d\mu^* = \int_D \left(\int_{\partial D} f(w)d\lambda_z(w)\right)d\mu(z).$$

If $f \in R(\bar{D})$ then $\int_{\partial D} f(w)d\lambda_z(w) = f(z)$ because λ_z is a representing measure with respect to $R(\bar{D})$ for z (see [7], p. 226]). Hence for such f we have:

$$\int f\,d\mu^* = \int_D \left(\int_{\partial D} f(w)d\lambda_z(w)\right)d\mu(z) = \int_D f\,d\mu,$$

and $\mu^* \ll \lambda_z \ll m_z \ll m_{z_0}$.

Lemma 3. Assume there exists $z \in D$ with dominant representing measure m with respect to $R(\bar{D})$. If $\mu \perp R(\bar{D})$ then $\exists h \in L^1(m)$ such that $\mu = hm$.

Proof. Since $\mu \perp R(\bar{D})$, $\mu = \Sigma h_i m_i$ where $h_i \in L^1(m_i)$ and each m_i is a representing measure for some point $z_i \in D$ such that $\overline{\text{suppl } m_i} \subseteq \partial D$ and $h_i m_i \perp R(\bar{D})$. In fact we also have $\|\mu\| = \Sigma \|h_i m_i\|$. (See [10] for a proof.) Write $\mu = h_1 m_1 + \Sigma_J h_j m_j$, where z_1 is any point in the Gleason part which contains D, and $J = \{j : z_j$ is not in the Gleason part which contains $D\}$, because $\mu \perp R(D)$. So $v \perp R(D)$ and applying Lemma 3 to v we get $v = h_1 m$ where h_1 is in $L^1(m)$. But since $\mu_2^* \ll m$ then $\exists h_2 \in L^1(m)$ such that $\mu_2^* = hm$ so $\mu_1 = v - \mu_2^* = h_1 m - h_2 m = (h1^1 - h1^2)m = hm$ and $h \in L^1(m)$.

From this we get $\mu = hm + \mu_2$.

We are in a position now to prove the main theorem.

Theorem. Let D be a domain which satisfies the following:

(1) there exists $z \in D$ with dominant representing measure m; (2) $R(\bar{D})$ is boundedly pointwise dense in $H^\infty(D)$.

Let $U_1, U_2, \ldots, U_n$ be an open cover for $\bar{D}$, and let μ be a measure on $M(D)$ such that $\mu \perp H^\infty(D)$. Then there exist measures $\mu_1, \mu_2, \ldots, \mu_n, \sigma$ on $M(D)$ such that:

(1) $\mu = \sum_{i=1}^{n} \mu_i + \sigma$;

(2) $\mu_i \perp H^\infty(D)$ for each $i = 1, 2, \ldots, n$;

(3) $\bar{\sigma} = 0$; and (4) $\overline{\text{supp } \mu_i} \subseteq \hat{Z}^{-1}(U_i)$.

Then by [8] for all $j \in J$, z_j is a peak point and hence the only representing measures for z_j are the point mass measures.

Now $m_1 \ll$ because m is a dominant representing measure hence there exists $g_1 \in L^1(m)$ such that $m_1 = g_1 m$. Then $\mu = h_1 g_1 m + \Sigma_J h_j \delta_j$ where δ_j is the point mass measure for z_j.

Since $\mu \perp R(\bar{D})$, $\mu|_{\{z_j\}} \perp R(\bar{D})$ [7]; so $h_j \delta_j = 0$, for every $j \in J$, hence $\mu = h_1 g_1 m = hm$. Clearly $h \in L^1(m)$.

Lemma 4. Assume μ is a measure on $\bar{D}$, $\mu \perp R(\bar{D})$. If there exists a dominant representing measure m for some $z \in D$ with respect to $R(\bar{D})$ then $\mu = hm + \mu_2$ where $h \in L^1(m)$ and supp $\mu_2 \subseteq D$.

Proof. Write $\mu = \mu_1 + \mu_2$ where $\mu_1 = \mu|_{\partial D}$ and $\mu_2 = \mu|_D$. Then by Lemma 2 there exists a measure μ_2^* such that $\overline{\text{supp } \mu_2^*} \subseteq \partial D$, $\mu_2^* \ll m$, $\int f d\mu_2 = \int f d\mu_2^*$ $\forall f \in R(\bar{D})$. Let $v = \mu_1 + \mu_2^*$ then $\overline{\text{supp } v} \subseteq \partial D$ and $\forall f \in R(\bar{D})$ we have $\int f dv = \int f d\mu_1 + \int f d\mu_2^* = \int f d\mu_1 + \int f d\mu_2 = \int f d\mu = 0$.

Proof. Since $\mu \perp M^\infty(D)$ we have [2] that $\bar{\mu} \perp R(\bar{D})$. By Bishop's Splitting Lemma [7] there exists measures $v_1, v_2, \ldots, v_n$ on $\bar{D}$ such that $v_i \perp R(\bar{D})$, $\overline{\text{supp } v_i} \subseteq U_i$ and $\bar{\mu} = \Sigma_{i=1}^n v_i$.

By Lemma 4 there exists $h_i \in L^1(m)$ and a measure τ_i on D such that $v_i = h_i m + \tau_i$, for every $i = 1, 2, \ldots, n$.

By theorem 11.1 in [7] or [9:8.1], we have $H^\infty(D)$ is isometrically isomorphic to $H^\infty(m)$, so there exists a canonical lift of m to $M(D)$, denote it by $\hat{m}$, see [5]. Also if $\{f_n\} \subset R(\bar{D})$ and f_n converges to $f \in H^\infty(D)$ pointwise boundedly then $\int_{M(D)} \hat{f}_n d_\mu \to \int \hat{f} d\mu$ for any $\mu \ll \hat{m}$. Now since $\hat{Z}$ is a homeomorphism over D, τ_i has a natural lift $\hat{\tau}_i$ to $M(D)$.

Let $\mu_i = \hat{h}_i \hat{m} + \hat{\tau}_i$ so $\bar{\mu}_i = v_i$, where $\hat{h}_i$ is the Gelfand transform of h_i to $M(L^\infty(m))$. Clearly $\overline{\text{supp } \mu_i} \subseteq \hat{Z}^{-1}(U_i)$ which proves (4). To prove (2): let $f \in H^\infty(D)$. Then there exists a bounded sequence $\{f_n\} \subseteq R(\bar{D})$ such that $f_n(z) \to f(z)$ for all $z \in D$. So

$$\begin{aligned}\int_{M(D)} \hat{f} d\mu_i &= \lim \int_{M(D)} \hat{f}_n \hat{h}_i d\hat{m} + \lim \int_{M(D)} \hat{f}_n d\hat{\tau}_i \\ &= \lim \int_{M(D)} \hat{f}_n \hat{h}_i d\hat{m} + \lim \int_D f_n d\tau_i \\ &= \lim \int_{\bar{D}} f_n h_i dm + \lim \int_D f_n d\tau_i\end{aligned}$$

$$= \lim \int_{\bar{D}} f_n \mathrm{d}\bar{\mu}_i = \lim \int_{\bar{D}} f_n \mathrm{d}\nu_i = 0.$$

To prove (1) and (3): Let $\sigma = \mu - \Sigma_{i=1}^{n} \mu_i$. Then for any $E \subseteq \bar{D}$ we have

$$\bar{\sigma}(E) = \sigma(\hat{Z}^{-1}(E)) = (\mu - \sum_{i=1}^{n} \mu_i)(\hat{Z}^{-1}(E))$$

$$= \mu(\hat{Z}^{-1}(E)) - \sum_{i=1}^{n} \mu_i(\hat{Z}^{-1}(E))$$

$$= \bar{\mu}(E) - \sum_{i=1}^{n} \nu_i(E) = 0$$

So $\mu = \Sigma_{i=1}^{n} \mu_i + \sigma$ and $\bar{\sigma} = 0$.

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