ON MINIMIZING CONTROL COSTS IN A COMMUNICATION NETWORK

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The present note deals with the problem of finding ways of minimizing the number of control points and their associated costs in a communication network, i.e. a network used for transporting or transmitting goods or information from one point to another.

The graphic representation of such a network can be somewhat simplified. A communication network must have a path leading from any node to any other one. There is no need to represent those paths on the network. It is sufficient to represent it as a set of nodes connected by arcs as follows: There should be an oriented arc between two nodes (i, j) if there exists at least one path in the network using a direct connection from i to j.

Control of the network is required only to make sure that the correct dispatching takes place. Two problems can be formulated:

a) Find a solution, i.e. a set of control points minimizing the number of such control nodes,
b) Among all the possible solutions to problem A, find the one(s) having the smallest control cost.

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Algorithms for solving these two problems will be based on the two following lemmas:

Lemma I. Either the communication network is a closed chain† or it does not contain any such closed chain.

Lemma II. Any node with two or more successors must be a control node. Of two consecutive nodes, one of them must be labeled.

Lemma III. For any tree‡, an optimal solution to problem A is derived by labeling 0 (i.e. no control station at that node) any origin§ of the tree, 1 (i.e. making them a control node) their successors and removing the corresponding nodes and arcs from the network.

Lemma IV. The solution defined in Lemma III is also an optimal solution to problem B when all the branches of the tree¶ contain an odd number of nodes.

† A closed chain (or loop) is a set of nodes $a, b, c, \ldots, n$ such that a has only one successor b, b has only c as successor and n has a as successor.

‡ A tree is a connected set of nodes which contains no closed chain and where no node has more than one successor.

§ Origins of a tree are any nodes of this tree without predecessors.

¶ A branch of a tree is a set of nodes consisting of an origin and all its successors; the branch is odd (even) if the number of its nodes is odd (even).

Lemma V. If, in a tree, a node is such that the subtree made of this node and its ancestors consists of even branches intersecting at that node, such a node must be labeled 1 and the labels must alternate inside the subtree. (This is valid for both problems A and B.)

Lemma VI. For problem B, if there is in a tree an origin such that its associated cost is larger than the cost associated with its successor, such an origin must be labeled 0 and its successor 1 (this can also be applied to the extremity of an open chain and to its predecessor).

These lemmas allow one to formulate the following algorithms for each of the problems:

1. PROBLEM A

If the network is a loop, label it by alternating 0 and 1's, starting the labeling at any node. If not, then label 1 any node with two or more successors, eliminate those nodes and the corresponding arcs from the network. Label 0 any disconnected node. The remaining network consists of one or more trees or simple open chains for which one can apply Lemma III, i.e., one can label 0 any origin and 1 its successor, eliminate them from the network and continue this process until no nodes remain to be labeled.

2. PROBLEM B

When the network is a loop, select the cheapest among the possible solutions to problem A, 2 if the loop is even, $(2n-1)$ if the loop contains $(2n-1)$ nodes. Otherwise, proceed as for problem A, by labeling 1 and removing from the network all the nodes with two or more successors and labeling 0 any disconnected node.

Apply lemmas IV, V, VI whenever it is feasible and remove the corresponding nodes from the network. For any even open chain, compare the cost of all the feasible solutions to problem A (the number of such solutions $f(\cdot)$ for a chain with $2n$ nodes is given by: $f(2n)=2+f(2n-4)$ with $f(0)=1$ and $f(2)=2$) and select the cheapest one.

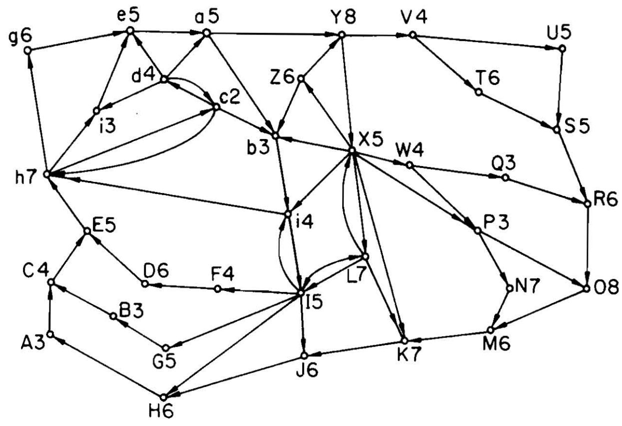
The repeated application of these steps should make a sensible reduction in the size of the remaining subnetworks.

For any such subnetwork, use the first algorithm to compute the minimum number of control nodes needed. Then select the intersection nodes nearest the destination (=last node) of that tree and use a branch and bound technique by creating two possible sets of solutions obtained by labeling that node either 0 or 1. Bounds can be used on the numbers of nodes needed (one knows the number of nodes required) and on the cost of the solution as one continues to find cheaper and cheaper solutions to problem A.

These algorithms work very fast—even by hand—and allow one to work on very large networks.

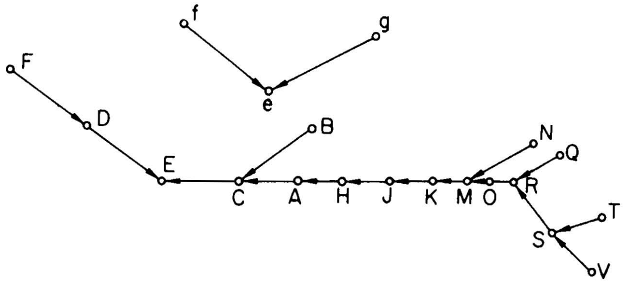
3. NUMERICAL EXAMPLE

Consider the following 35-node network where the numbers adjacent to each node represent the corresponding cost of each node. This network has been drawn with oriented arcs indicating the flows of communications.

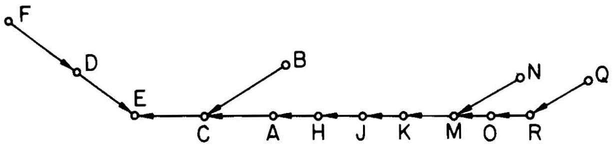


Each letter represents a node; the solution to problem B is derived as follows:

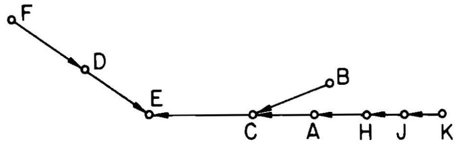
a) The nodes with two or more successors are labeled 1 (=control station) and disappear from the network, i.e. nodes I, L, P, V, W, X, Y, Z, a, c, d, h, i, for a total cost of 64.
b) Node b is then disconnected and must be labeled 0. The residual network is then:



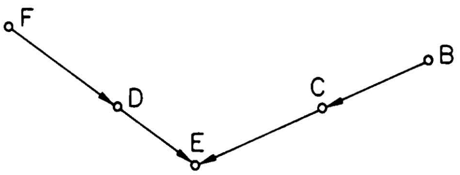
c) By Lemma V, we label g=f=0, e=1 and T=V=0, S=1. So far 15 nodes have control stations for a total cost of 74. The residual network is:



d) Two new applications of Lemma V lead to M=R=1; N=Q=0. So far there are 17 control nodes for a cost of 86. The remaining network is:



e) Two successive applications of Lemma VI imply that: 1) K=0 and J=1, 2) H=0 and A=1. This brings the number of control nodes to 19 for a total cost of 95. The last subset of nodes is:



f) Lemma IV implies that D=C=1 and F=B=E=0. This ends the algorithm and gives an optimal solution consisting of 21 nodes with control stations at a cost of 105.

This solution is optimal for both problems A and B. Some minor modifications of the network such as the removal of the arc (P–O) would dramatically alter the iterations. The algorithm is thus fairly dependent upon the structure of the network.

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