SOME RESULTS ON HEREDITARY COMPLETE SPACE

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الخلاصـة :

سنعطي في هذا البحث مثالا لفضاء ورِيث تام ولكنه ليس B_r تام . وكذلك سنبرهن ان فضائي داله شوارتس الفاحصة والموزعة غير متممة .

ABSTRACT

In this paper we will give an example of a space which is hereditary complete but not B_r-complete; also, we prove that the spaces of Schwartz test functions and distributions are not complemented.

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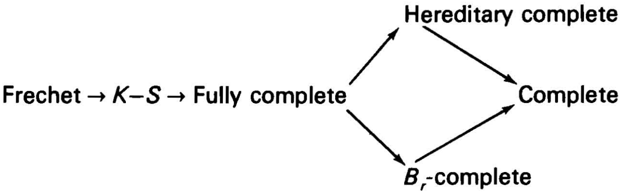
INTRODUCTION

K–S space, fully complete, B_r-complete, hereditary complete and S_r-complete spaces have been studied by many authors; see for example references [6, 4, 7, 1, 3]. Moreover, the following chain of implications is well-known; see reference [1] for details.



It is not known if a B_r-complete space is fully complete. Dostal [1; p. 27] mentioned that it has been claimed that the space of distributions D' is B_r-complete. It was known that D' is not hereditary complete, see Reference [8], and so it is not fully complete.

This example seemed to end a long search for a space which is B_r-complete but not fully complete. However, in 1974 Valdivia [10] proved that D' is not B_r-complete and so the problem surfaced again. Now, recall from Reference [3] that a complete locally convex space E is S_r-complete if and only if for any closed subspace M there is a continuous projection from E onto M. This is equivalent to the following statement; For any closed subspace M there is a closed subspace N such that N is a topological complement of M.

Recall the following definitions; see Reference [1; p. 25]. $U(E)$ will denote the family of all neighborhoods of the origin in a locally convex space E. E' is the set of all continuous linear functionals on E. u^0 will denote the polar of u. Let E be a locally convex space and Q a subset of E'. Then Q is said to be almost closed if for each $u \in U(E)$, the set $Q \cap u^0$ is closed in the relative $\sigma(E', E)$-topology on u^0.

E is called B_r-complete, if every subspace $Q \subseteqq E'$ which is dense in E' in the topology $\sigma(E', E)$ is almost closed.

E is fully complete, if every subspace Q of E' is almost closed.

E is K–S, if every convex subset Q of E' is almost closed.

E is hereditary complete, if E/N is complete for an arbitrary closed subspace N of E.

The proof of this paper's results depends mainly on the following Lemma:

Lemma. Every S_r-complete is hereditary complete.

Proof. See Appendix.

Example of a hereditary complete space which is not B_r-complete.
We consider the space ϕ_d which is the direct sum of d complex plans with $d \geqq 2\aleph_0$. Every closed subspace of ϕ_d must have a topological complement, for proof see Reference [5; p. 431], so ϕ_d is S_r-complete. Making use of the lemma we see that ϕ_d is hereditary complete.

ϕ_d is not B_r-complete for $d \geqq 2\aleph_0$ for proof see Reference [2].

Theorem. The Schwartz test functions D and distributions D' are not complemented, that is, are not S_r-complete.

Proof. Since D is not hereditary complete, see Reference [9], so by the lemma it is not S_r-complete which means there is a closed subspace of D which has no topological complement, so D is not complemented. Similarly D' is not complemented because it is not hereditary complete, see Reference [7].

APPENDIX

(The lemma is proved in Reference [3]; for this article to be self-contained, we copy the proof.)

The proof depends on the following two propositions:

Proposition *1.* Let E be an S_r-complete space; if M is a closed subspace of E, then M is S_r-complete.

Proof. Let N be any closed subspace of M; then N is a closed subspace of E. Since E is S_r-complete, then there is a continuous projection

$P: E \to N$ and so its restriction to
M, $P' = P_{1_M}: M \to N$
$$x \to P(x)$$

is a continuous projection from M onto N; thus, M is S_r-complete.

Proposition 2. Let E be an S_r-complete space; then for any closed subspace M the quotient E/M is S_r-complete.

Proof. Since M is closed and E is S_r-complete, there is a continuous projection

$$P: E \rightarrow M.$$

Let $N = \ker(P)$ and I the identity map; then N is a closed subspace of E and $P' = I - P: E \rightarrow N$ is a continuous projection with kernel M. Therefore E/M is topologically isomorphic to N. But by proposition 1, N is S_r-complete so E/M is S_r-complete.

Proof of the Lemma. If E is S_r-complete, then by proposition 2, E/M is S_r-complete for any closed subspace M, and thus E/M is complete; this means E is hereditary complete.

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