APPLICATION OF QUATERNION ALGEBRA TO THE THEORY OF SKEW FRICTION GEARS

Charles K. Wojcik

Department of Mechanical Engineering, University of Petroleum and Minerals, Dhahran, Saudi Arabia

الخلاصة :

يمكن عن طريق استخدام الجبر الرباعى التعامل مع الدوران المحدد في الفراغ بطريقة بسيطة وممتازة ، فمن أجل الحصول على كفاءة أعلى في نقل الحركة باستخدام التروس الخشنة المتزاوية ، يتعين تقليل الانزلاق ما بين نقط التلامس ، وتوضح هذه الدراسة أن الانزلاق يبلغ أدني قيمة له عندما يكون خط التلامس ما بين اثنين من هذه التروس ، خطا مستقيما يمكن تحديده بواسطة متجهات السرعة الزاوية لعمودى نقل الحركة الدورانية ونسبة سرعتيهما ، وهذا الخط يمكن أن يمثل بمتلجه ، وعن طريق استخدام المعامل الرباعى لدوران المتجهات فانه يمكن استخلاص المعادلات الاتجاهية لسطوح التروس الخشنة المتزاوية .

ABSTRACT

With the aid of quaternion algebra, finite rotations in space may be dealt with in a simple and elegant manner.

For higher efficiency of a skew friction-gear transmission, slippage between the points of contact should be minimized. It is shown here, that the slippage is minimized when the line of contact between two such gears is a straight line determined by the angular velocity vectors of the two rotating shafts and their speed ratio. This line can be represented by a vector. Using the quaternion operator for rotation of vectors, vector equations of the surfaces of skew friction gears are derived.

0377–9211/80/0105–0031$01.00

APPLICATION OF QUATERNION ALGEBRA TO THE THEORY OF SKEW FRICTION GEARS

INTRODUCTION

Quaternion algebra was developed by Sir William Rowan Hamilton [1] in 1843. He was trying, by analogy with the complex number representation of real two-dimensional vectors, to extend vector algebra to include multiplication and division of two vectors and to deal with rotations in three dimensions. He accomplished this by introducing a triple of ordered numbers that provides a three-dimensional analog of the complex number system.

Quaternion algebra has been used in many branches of science, such as geometry [2–4], vector analysis [3, 5, 6], and applied mechanics and physics [2, 7, 8]. More recently, dual quaternions have been applied to the analysis of space mechanisms [9–12]. In this paper, the quaternion operator for rotation of vectors is used to derive the vector equations of the surfaces of the skew friction gears.

DEFINITIONS AND BASIC OPERATIONS

A quaternion is a *hypercomplex number* of the form

$$Q = q_0 + q_i \mathbf{e}_i, \qquad (i = 1, 2, 3) \tag{1}$$

where q_0 and q_i are real numbers, while $\mathbf{e}_i$ are generalizations of the square root of -1 and can be considered as unit vectors associated with the given axes of a cartesian coordinate system. They obey the following rules of multiplication

$$\mathbf{e}^2 = -1, \quad \mathbf{e}_i \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k, \qquad (i \neq j) \tag{2}$$

where ε_{ijk} are the permutation symbols defined by the relationship

$$\varepsilon_{ijk} = \begin{Bmatrix} +1 \\ -1 \\ 0 \end{Bmatrix} \text{when } i, j, k \text{ is} \begin{Bmatrix} \text{an even} \\ \text{an odd} \\ \text{no} \end{Bmatrix} \text{permutation of } 1, 2, 3. \tag{3}$$

The *product* of two quaternions, Q and Q', is

$$QQ' = (q_0 + q_i \mathbf{e}_i)(q_0' + q_j' \mathbf{e}_j)$$
$$= q_0 q_0' - q_i q_i' + q_0 q_i' \mathbf{e}_i + q_i q_0' \mathbf{e}_i + \varepsilon_{ijk} q_i q_j' \mathbf{e}_k. \tag{4}$$

The product $Q'Q$ differs from QQ' only in the sign of the last term.

A quaternion may be considered as a sum of a scalar, q_0, and a vector whose components are q_i, i.e.

$$Q = q_0 + q_i \mathbf{e}_i = S(Q) + \mathbf{V}(Q). \tag{5}$$

The product of two quaternions QQ' may then be written

$$QQ' = SS' - \mathbf{V} \cdot \mathbf{V}' + S\mathbf{V}' + S'\mathbf{V} + \mathbf{V} \times \mathbf{V}'. \tag{6}$$

In the case when the scalar parts of the two quaternions are zeros, i.e. when the quaternions are solely vector quantities, we have

$$\mathbf{V}\mathbf{V}' = -\mathbf{V} \cdot \mathbf{V}' + \mathbf{V} \times \mathbf{V}' = S(QQ') + \mathbf{V}(QQ'). \tag{7}$$

Hence, multiplication of twovectors, $\mathbf{V}$ and $\mathbf{V}'$, yields a quaternion quantity whose scalar part is equal to the negative value of their dot product, and the vector part equal to their cross product.

The *conjugate* of a quaternion Q, written $\tilde{Q}$, is defined as

$$\tilde{Q} = S(Q) - \mathbf{V}(Q) = q_0 - q_i \mathbf{e}_i. \tag{8}$$

The *norm* of a quaternion Q, written $\|Q\|$, is the product of the quaternion and its conjugate

$$\|Q\| = Q\tilde{Q} = \tilde{Q}Q = q_i q_i, \qquad (i = 0, 1, 2, 3). \tag{9}$$

A quaternion whose norm is unity is called a *unit quaternion* which we shall denote by q. Any quaternion, Q, may be normalized to a unit quaternion

$$q = \frac{Q}{\sqrt{\|Q\|}} = \frac{Q}{|Q|}. \tag{10}$$

Since the norm of a unit quaternion is unity, it can be readily shown that

$$\tilde{q} = q^{-1}.$$

A unit quaternion, q, may be written in the following form

$$q = \cos\theta + \mathbf{s} \sin\theta \tag{12}$$

where:

$$\cos\theta = \frac{q_0}{|Q|}, \text{ and } \sin\theta = \frac{\sqrt{(q_i q_i)}}{|Q|}, \qquad (i = 1, 2, 3) \tag{13}$$

and when $q_i q_i \neq 0$, the unit vector

$$\mathbf{s} = \frac{q_i \mathbf{e}_i}{\sqrt{(q_i q_i)}}, \qquad (i = 1, 2, 3) \tag{14}$$

is referred to as the axis of the unit quaternion.

The unit vector, **s**, obeys the rule of multiplication stated for the unit vectors, $\mathbf{e}_i$, namely

$$\mathbf{s}^2=\mathbf{ss}=-1. \quad (15)$$

Applying this rule to higher powers of **s**, we have, in general

$$\mathbf{s}^n=(-1)^{n/2}, \text{ for } n \text{ even},$$
$$\mathbf{s}^n=(-1)^{(n-1)/2}\mathbf{s}, \text{ for } n \text{ odd}. \quad (16)$$

Since

$$\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots$$

and

$$\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots \quad (17)$$

using these expressions and Equation (16), we can write

$$q=1+\mathbf{s}\theta-\frac{\theta^2}{2!}-\mathbf{s}\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+\mathbf{s}\frac{\theta^5}{5!}-\ldots=\mathrm{e}^{\mathbf{s}\theta} \quad (18)$$

where e is the base of natural logarithms.

ROTATION OF A VECTOR

With the aid of quaternion algebra, finite rotations of a rigid body about a fixed point may be dealt with in a simple and elegant manner.

Consider a position vector, **r**, that intersects an axis, **s**, at point O, as shown in Figure 1. The rotation of that vector, through an angle 2θ, to a new position defined by **r**′, can be represented by the following quaternion operation

$$\mathbf{r}'=q\mathbf{r}q^{-1} \quad (19)$$

where q is the unit quaternion, as defined by Equation (12), whose unit vector **s** lies along the s-axis. Its direction is determined by the right-hand rule. The angle, θ, in Equation (12) is equal to one half of the angle of rotation.

Transformation of the vector **r** into the vector **r**′, by pure rotation, is an orthogonal transformation during which the norm of the vector is preserved. This can be readily verified by taking the norm of both sides of Equation (19)

$$\|\mathbf{r}'\|=\|q\mathbf{r}q^{-1}\|=\|q\|\;\|\mathbf{r}\|\;\|q^{-1}\|=\|\mathbf{r}\|. \quad (20)$$

In the following, we shall show that Equation (19) indeed represents the rotation of the vector from **r** to **r**′ position.

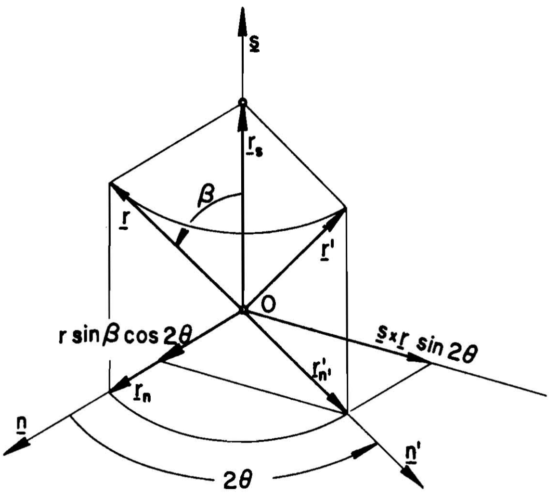

Figure 1. Rotation of a Vector in Space

Using in Equation (19) the definition of the unit quaternion and of its inverse, we can write

$$\mathbf{r}'=q\mathbf{r}q^{-1}=(\cos\theta+\mathbf{s}\sin\theta)\mathbf{r}(\cos\theta-\mathbf{s}\sin\theta)$$
$$=\mathbf{r}\cos^2\theta+\mathbf{sr}\sin\theta\cos\theta-\mathbf{rs}\sin\theta\cos\theta-(\mathbf{sr})\mathbf{s}\sin^2\theta. \quad (21)$$

Using the rules for multiplication of vectors (Equation (7)) and for triple-vector product, we can show that

$$\mathbf{r}'=\mathbf{r}\cos 2\theta+\mathbf{s}\times\mathbf{r}\sin 2\theta+\mathbf{s}(\mathbf{s}\cdot\mathbf{r})(1-\cos 2\theta). \quad (22)$$

Replacing $\mathbf{s}\cdot\mathbf{r}$ by $r\cos\beta$, where β is the angle formed by the two vectors, and replacing the vector **r** by its components, $\mathbf{s}r\cos\beta$ and $\mathbf{n}r\sin\beta$, we obtain

$$\mathbf{r}'=\mathbf{s}r\cos\beta+\mathbf{n}r\sin\beta\cos 2\theta+\mathbf{s}\times\mathbf{r}\sin 2\theta. \quad (23)$$

All the above vector components are shown in Figure 1. Since

$$\mathbf{n}r\sin\beta\cos 2\theta+\mathbf{s}\times\mathbf{r}\sin 2\theta=\mathbf{n}'r\sin\beta, \quad (24)$$

we can write

$$\mathbf{r}'=\mathbf{s}r\cos\beta+\mathbf{n}'r\sin\beta. \quad (25)$$

Therefore, the quaternion transformation, as defined by Equation (19), does indeed represent a rotation of a vector about the s-axis.

Using the exponential notation, rotation of a vector about an s-axis can be written

$$\mathbf{r}'=\mathrm{e}^{\mathbf{s}\theta}\mathbf{r}\mathrm{e}^{-\mathbf{s}\theta}. \quad (26)$$

A *special case* exists when vector **r** is rotated in a plane perpendicular to the s-axis.

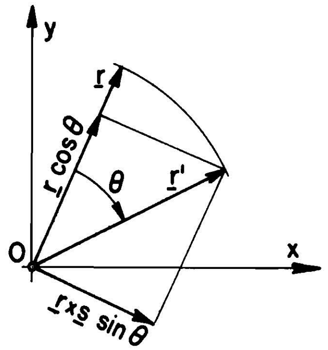

Figure 2. Rotation of a Vector in a Plane

In general, post-multiplication of a vector by a unit quaternion yields

$$\mathbf{r}q=\mathbf{r}(\cos\theta+\mathbf{s}\sin\theta)$$
$$=-(\mathbf{r}\cdot\mathbf{s})\sin\theta+\mathbf{r}\cos\theta+\mathbf{r}\times\mathbf{s}\sin\theta. \quad (27)$$

When vectors **r** and **s** are perpendicular, their dot-product is zero and the last equation takes the form

$$\mathbf{r}q=\mathbf{r}\cos\theta+\mathbf{r}\times\mathbf{s}\sin\theta=\mathbf{r}' \quad (28)$$

where $\mathbf{r}'$ is the vector obtained by rotating **r** through an angle θ in the c.w. direction, as shown in Figure 2. In a similar way, it can be shown that pre-multiplication of vector **r** by a unit quaternion results in rotation of that vector by the same angle, θ, but in the c.c.w. direction.

From the above discussion we conclude that, when a vector **r** and the *s*-axis are perpendicular, a simultaneous pre- and post-multiplication of a vector by a unit quaternion, or its inverse, leaves that vector unchanged.

Using the exponential form of notation for the unit quaternion, rotation of a vector **r**, perpendicular to the *s*-axis, can be written as follows

$$\mathbf{r}'=e^{s\theta}\mathbf{r}=\mathbf{r}e^{-s\theta} \quad (29)$$

where θ is the angle of rotation in the c.c.w. direction. The form of the above equation resembles the notation for the rotation of a vector in a complex plane.

SURFACES OF SKEW FRICTION GEARS

Consider a problem of transmitting motion between two skew shafts by means of friction gears. Let the axes of the two shafts, shown in Figure 3, be denoted by $\mathbf{s}_1$ and $\mathbf{s}_2$. Further, let the shafts rotate with uniform velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. Denote the distance between the axes, O_1O_2, by the vector **c**; consider an arbitrary point M to be a point of contact; point M' is its projection on the line O_1O_2. While considering any point of

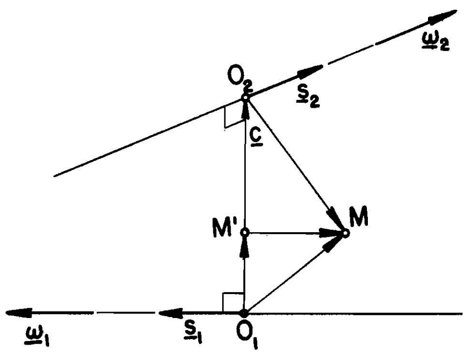

Figure 3. Skew Axes

contact between two bodies moving in a given reference system, it is important to distinguish three momentarily coinciding points, M_0—fixed in the reference frame, M_1—fixed in the body 1, and M_2—fixed in the body 2. Here, bodies 1 and 2 are the gears 1 and 2, respectively. With this clarification, we can write the expressions for the velocities of points M_1 and M_2 with respect to the frame

$$\mathbf{v}_{10}=\boldsymbol{\omega}_1\times\mathbf{O}_1\mathbf{M}=\boldsymbol{\omega}_1\times(\mathbf{O}_1\mathbf{M}'+\mathbf{M}'\mathbf{M})$$
$$=\boldsymbol{\omega}_1\times\mathbf{O}_1\mathbf{M}'+\boldsymbol{\omega}_1\times\mathbf{M}'\mathbf{M} \quad (30)$$

and

$$\mathbf{v}_{20}=-\boldsymbol{\omega}_2\times\mathbf{M}'\mathbf{O}_2+\boldsymbol{\omega}_2\times\mathbf{M}'\mathbf{M}. \quad (31)$$

The relative (slip) velocity of the points M_1 and M_2 is

$$\mathbf{v}_{12}=\mathbf{v}_{10}-\mathbf{v}_{20}$$
$$=(\boldsymbol{\omega}_1\times\mathbf{O}_1\mathbf{M}'+\boldsymbol{\omega}_2\times\mathbf{M}'\mathbf{O}_2)+(\boldsymbol{\omega}_1-\boldsymbol{\omega}_2)\times\mathbf{M}'\mathbf{M}. \quad (32)$$

Vectors, $\boldsymbol{\omega}_1\times\mathbf{O}_1\mathbf{M}'$ and $\boldsymbol{\omega}_2\times\mathbf{M}'\mathbf{O}_2$, are perpendicular to the line O_1O_2. If we attach these vectors to some point, say M, they will form a plane that is perpendicular to O_1O_2 and parallel to both $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. We observe that the vector $(\boldsymbol{\omega}_1-\boldsymbol{\omega}_2)\times\mathbf{M}'\mathbf{M}$ is perpendicular to that plane and to the vectors, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, and $\mathbf{M}'\mathbf{M}$.

For high efficiency of the gear transmission, the absolute value of the slip velocity $|\mathbf{v}_{12}|$ should be minimized. Since the vectors $(\boldsymbol{\omega}_1\times O_1M'+\boldsymbol{\omega}_2\times M'O_2)$ and $(\boldsymbol{\omega}_1-\boldsymbol{\omega}_2)\times\mathbf{M}'\mathbf{M}$ are perpendicular, we can write

$$\min|\mathbf{v}_{12}|=\min|(\boldsymbol{\omega}_1\times\mathbf{O}_1\mathbf{M}'+\boldsymbol{\omega}_2\times\mathbf{M}'\mathbf{O}_2)|$$
$$+\min|\boldsymbol{\omega}_1-\boldsymbol{\omega}_2)\times\mathbf{M}'\mathbf{M}|. \quad (33)$$

We observe that the second term in this equation becomes zero when vectors $(\boldsymbol{\omega}_1-\boldsymbol{\omega}_2)$ and $\mathbf{M}'\mathbf{M}$ are parallel. This establishes the direction of $\mathbf{M}'\mathbf{M}$. Now, we shall look for the location of point M' on O_1O_2 that will minimize the first term in the last equation.

Let $\mathbf{O_1M'} = \lambda \mathbf{c}$, where λ is some scalar quantity such that $0 < \lambda < 1$. Then the first term can be written

$$|(\boldsymbol{\omega}_1 \times \mathbf{O_1M'} + \boldsymbol{\omega}_2 \times \mathbf{M'O_2})| = |[\lambda\boldsymbol{\omega}_1 + (1-\lambda)\boldsymbol{\omega}_2] \times \mathbf{c}|. \quad (34)$$

Since $\mathbf{c}$ has a constant value and it is perpendicular to the vector $[\lambda\boldsymbol{\omega}_1 + (1-\lambda)\boldsymbol{\omega}_2]$, in order to minimize the above term, it is sufficient to minimize the magnitude of the vector

$$\lambda\boldsymbol{\omega}_1 + (1-\lambda)\boldsymbol{\omega}_2 = \mathbf{u} \quad (35)$$

which can be written

$$|\mathbf{u}| = \sqrt{[\lambda^2\omega_1^2 + (1-\lambda)^2\omega_2^2 + 2\lambda\omega_1\omega_2(1-\lambda)\cos\beta]} \quad (36)$$

where β is the angle formed by vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. For $|\mathbf{u}|$ to be a minimum, its derivative with respect to λ must be equal to zero. Differentiating Equation (36) with respect to λ and setting the result equal to zero, we have

$$\lambda\omega_1^2 - (1-\lambda)\omega_2^2 - (2\lambda-1)\omega_1\omega_2\cos\beta = 0 \quad (37)$$

Dividing the last equation by ω_2^2 and setting $\omega_1/\omega_2 = n$, we obtain

$$n^2\lambda - (1-\lambda) - (2\lambda-1)n\cos\beta = 0. \quad (38)$$

Solving for λ, we get

$$\lambda = \frac{1 - n\cos\beta}{n^2 + 1 - 2n\cos\beta}. \quad (39)$$

From the above discussion we conclude that the line of contact, for which the slippage is a minimum, is a straight line parallel to $(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)$ and intersecting O_1O_2 at point M' whose position on that line is defined by Equation (39).

If we define $\mathbf{r}_1$ to be a vector radius defining the position of an arbitrary point on that line, measured from the point O_1, then we can write a vector equation for that line

$$\mathbf{r}_1 = \lambda\mathbf{c} + \mu(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \quad (40)$$

where μ is any real number.

Rotation of the vector $\mathbf{r}_1$ about the s_1-axis, will generate a surface known as a hyperboloid of revolution. Using the quaternion notation, we can write the vector equation for that surface

$$\mathbf{R}_1 = e^{s_1\theta/2}\mathbf{r}_1 e^{-s_1\theta/2}. \quad (41)$$

A similar set of equations can be written for gear 2, namely

$$\mathbf{r}_2 = (\lambda-1)\mathbf{c} + \mu(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \quad (42)$$

and

$$\mathbf{R}_2 = e^{s_2\theta/2}\mathbf{r}_2 e^{-s_2\theta/2} \quad (43)$$

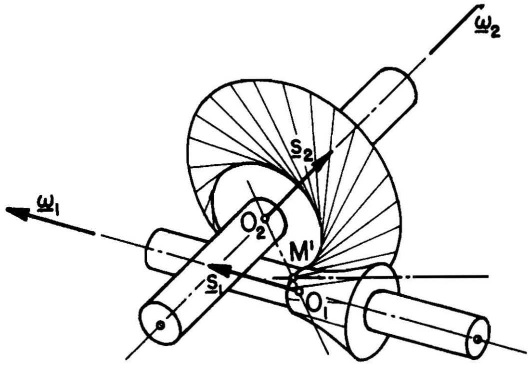

Figure 4. Surfaces of Skew Friction Gears

where the vectors, $\mathbf{r}_2$ and $\mathbf{R}_2$, are measured from the point O_2. The generated surfaces and the skew friction gears are shown in Figure 4.

The vector Equations (41) and (43) were derived for skew friction gears, they also represent the pitch surfaces of a pair of hypoid bevel gears.

REFERENCES

[1] W. R. Hamilton, 'On a New Species of Imaginary Quantities Connected with a Theory of Quaternions', *Dublin Proc.*, **2** (1843), pp. 424–434.

[2] W. R. Hamilton, *Elements of Quaternions*, Vol. 1 and 2. London, Longmans and Green, 1901.

[3] A. S. Hardy, *Elements of Quaternions.* Boston, Ginn and Co., 1887.

[4] P. Kelland and P. G. Taid, *Introduction to Quaternions.* London, McMillan, 1882.

[5] A. L. Baker, *Quaternions as the Result of Algebraic Operations.* New York, Van Nostrand, 1911.

[6] L. Brand, *Vector and Tensor Analysis.* New York, Wiley, 1947.

[7] O. F. Fisher, *Universal Mechanics and Hamilton's Quaternions.* Stockholm, Axion Institute, 1957.

[8] O. F. Fisher, *Five Mathematical Structural Models in Natural Philosophy with Technical Physical Quanternions.* Stockholm, Axion Institute, 1957.

[9] W. Blaschke, 'Anwendung Dualer Quaternionen auf Kinematik', *Ann. Acad. Scient. Fennicae*, Helsinki, A I 250/3 (1958).

[10] W. Blaschke, *Kinematik und Quaternionen.* Berlin, Deutscher Verlag der Wissenschaften, 1960.

[11] A. T. Yang, 'Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms', *Doctoral Dissertation*, Columbia University, 1963.

[12] C. K. Wojcik, 'Contributions to the Analysis and Synthesis of Mechanisms', *Doctoral Dissertation*, U. C. L. A., 1964.

Paper Received 16 October 1978; Revised 16 September 1979.