THE FINITE ELEMENT METHOD APPLIED TO LARGE ELASTO–PLASTIC DEFORMATIONS OF SOLIDS

George Z. Voyiadjis

Department of Civil Engineering, University of Petroleum and Minerals, Dhahran, Saudi Arabia

الخلاصة

في بحث سابق (مرجع رقم ١) اقترحت مجموعة من المعادلات التكوينية الخاصة بالمعادن في مرحلة المرونة اللدونة المعرضة لدرجات كبيرة من الاجهاد — وفي هذا البحث تم تطبيق طريقة (Finite Element) لدراسة الازاحة والجهد والاجهاد الناتج في مثل هذه المعادن في حالة الاجهاد المستوى .

ABSTRACT

The constitutive equations describing the behavior of large elasto–plastic deformations of solids have been proposed in a previous paper [1]. The problem of analysis of displacements, stresses and strains in elements made of this material subject to arbitrarily large deformations under the conditions of plane strain has been formulated in terms of the finite element method.

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1. INTRODUCTION

In this paper the finite element method is used in the case of problems involving large elasto–plastic deformations of materials. Such materials display an initial linear elasticity which is followed, at increasing loading, by plastic strains with no rate, or viscous effects. The topics included in this paper are:

1. The finite element formulation of the analysis of two-dimensional problems.
2. Methods of solution of the equations resulting from the use of the finite element method.

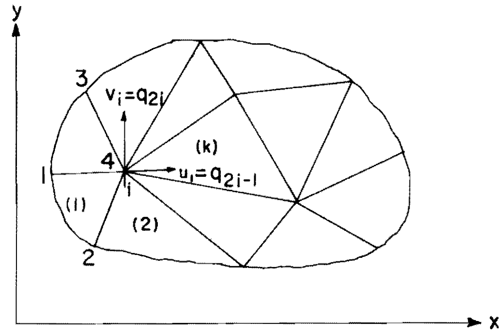

Figure 1. Typical Element Mesh; Notation for Global Nodal Displacements

2. FINITE ELEMENT FORMULATION

In this paper, the problem of analysis of large elastic–plastic deformations will be formulated in terms of the finite element method. Since the basic procedures of the finite element method are described in numerous papers and monographs (see, for example Reference [2]), only the aspects related to the present problem will be emphasized in this paper.

The fundamental equation in the following arguments is the *principle of virtual work* (Equation (2.26), Reference [1]:

$$\int_V \delta\mathbf{e}^{\mathrm{T}}\mathbf{s}\,\mathrm{d}V - \int_V \delta\mathbf{u}^{\mathrm{T}}\mathbf{f}\,\mathrm{d}V - \int_S \delta\mathbf{u}^{\mathrm{T}}\mathbf{p}\,\mathrm{d}S = 0 \quad (1)$$

which is the condition of equilibrium in terms of the displacement field $\mathbf{u}(x, y)$. In the finite element method, the displacement field $\mathbf{u}$ is approximated by a discrete model which contains a finite number of independent nodal displacements. For this purpose, the regions of integration in Equation (1) are divided into a finite number of subregions, or elements. Within each element, the displacement field is approximated by known functions which are continuous across the element boundaries. For a nonlinear elastic body, for which the stress vector is a *function* of the displacement gradients, Equation (1) results in a system of *nonlinear algebraic equations.* In the present problem of an elastic–plastic body, the stress $\mathbf{s}$ is a *functional* of the displacement gradients, Equation (1) leads to a system of *functional* equations for the nodal displacements.

Figure 1 shows a typical element mesh in an arbitrary body with m elements and n nodal points. The components of displacement at a node i are u_i and v_i. The nodal displacement vector for the body, or the *system*, is the $2n \times 1$ matrix $\mathbf{q}$ whose elements are

$$q_{2i-1} = u_i,\ q_{2i} = v_i,\ i = 1, 2, \ldots, n \quad (2)$$

i.e.,

$$\mathbf{q}^{\mathrm{T}} = (u_1, v_1, u_2, v_2, \ldots, u_n, v_n) \quad (3)$$

A typical element with its nodal displacements is shown in Figure 2. The nodal displacement vector $\mathbf{q}_k$ of the element k is

$$\mathbf{q}_k^T = (u_1, v_1, u_2, v_2, u_3, v_3) \quad (4)$$

Since, at the corresponding nodes, the nodal displacements of an element are identical with the nodal displacements of the body, it is evident that

$$\mathbf{q}_k = \mathbf{T}_k\,\mathbf{q} \quad (5)$$

where the form of $\mathbf{T}_k$ is implied by the definitions of $\mathbf{q}$ and $\mathbf{q}_k^T$ stated by Equations (3) and (4), respectively.

The displacement field within the element k is approximated by linear functions of x and y,

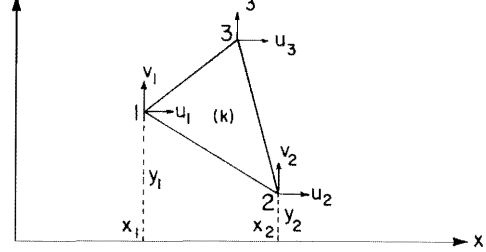

Figure 2. Typical Triangular Element; Notation for Element Displacements

$$\left.\begin{aligned} u &= \frac{1}{2\Delta}[(a_1 + b_1 x + c_1 y)u_1 + (a_2 + b_2 x + c_2 y)u_2 \\ &\quad + (a_3 + b_3 x + c_3 y)u_3] \\ v &= \frac{1}{2\Delta}[(a_1 + b_1 x + c_1 y)v_1 + (a_2 + b_2 x + c_2 y)v_2 \\ &\quad + (a_3 + b_3 x + c_3 y)v_3] \end{aligned}\right\} \quad (6)$$

where

$$2\Delta = 2 \times (\text{area of triangle}) = \det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

and

$$a_1 = x_2 y_3 - x_3 y_2,\ b_1 = y_2 - y_3,\ c_1 = x_3 - x_2$$

with a_2, b_2, c_2, a_3, b_3, and c_3 obtained by cyclic permutation of indices.

In matrix notation, with

$$\mathbf{u}_k = \begin{Bmatrix} u \\ v \end{Bmatrix}_k,$$

$$N_1 = (a_1 + b_1 x + c_1 y)/2\Delta,\ N_2 = \ldots \text{etc.}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{N}_k = (\mathbf{I}\,N_1, \mathbf{I}\,N_2, \mathbf{I}\,N_3).$$

Equation (6) becomes

$$\mathbf{u}_k = \mathbf{N}_k \mathbf{q}_k. \quad (7)$$

In view of Equation (5),

$$\mathbf{u}_k = \mathbf{N}_k \mathbf{T}_k \mathbf{q} \quad (8)$$

The components of strain are computed from the components of displacement according to equation (8). [1]. The result is

$$\mathbf{e}_k = (\mathbf{B}'_k + \tfrac{1}{2}\mathbf{B}''_k)\mathbf{q}_k \quad (9)$$

where the matrix $\mathbf{B}'_k$ corresponds to the linear terms in the strain–displacement relations, while $\mathbf{B}''_k$ takes into account the nonlinear terms. It can be easily verified that the matrices $\mathbf{B}'_k$ and $\mathbf{B}''_k$ are shown by Equations (10) and (11), respectively.

$$\mathbf{B}'_k = \frac{1}{2\Delta}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \quad (10)$$

$$\mathbf{B}''_k = \mathbf{F}_k \mathbf{G}_k \mathbf{H}_k \quad (11)$$

where

$$\mathbf{F}_k = \frac{1}{4\Delta^2}\begin{bmatrix} b_1 & 0 & b_1 & 0 & b_2 & 0 & b_2 & 0 & b_3 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_1 & 0 & c_2 & 0 & c_2 & 0 & c_3 & 0 & c_3 \\ c_1 & b_1 & c_1 & b_1 & c_2 & b_2 & c_2 & b_2 & c_3 & b_3 & c_3 & b_3 \end{bmatrix}$$

$$\mathbf{G}_k = \begin{bmatrix} u_1 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 \\ 0 & v_1 & 0 & 0 \\ 0 & 0 & 0 & v_1 \\ u_2 & 0 & 0 & 0 \\ 0 & 0 & u_2 & 0 \\ 0 & v_2 & 0 & 0 \\ 0 & 0 & 0 & v_2 \\ u_3 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 \\ 0 & v_3 & 0 & 0 \\ 0 & 0 & 0 & v_3 \end{bmatrix} \quad \mathbf{H}_k = \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & b_1 & 0 & b_2 & 0 & b_3 \\ c_1 & 0 & c_2 & 0 & c_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \end{bmatrix}$$

In the following, the differential $d\mathbf{e}_k$ will be needed rather than the strain $\mathbf{e}_k$ itself. Differentiation of (9) yields

$$d\mathbf{e}_k = \mathbf{B}'_k d\mathbf{q}_k + \tfrac{1}{2}\,d\mathbf{B}''_k\,\mathbf{q}_k + \tfrac{1}{2}\,\mathbf{B}''_k\,d\mathbf{q}_k.$$

It can be shown, however, that [2]:

$$d\mathbf{B}''_k\,\mathbf{q}_k = \mathbf{B}''_k\,d\mathbf{q}_k;$$

consequently,

$$d\mathbf{e}_k = (\mathbf{B}'_k + \mathbf{B}''_k)\,d\mathbf{q}_k = \overline{\mathbf{B}}_k\,d\mathbf{q}_k \quad (12)$$

and

$$d\mathbf{e}_k = \overline{\mathbf{B}}_k\,\mathbf{T}_k\,d\mathbf{q}. \quad (13)$$

From the stress–strain relation Equation (3.23), Reference [1];

$$d\mathbf{s}_k = \overline{\mathbf{D}}_k d\mathbf{e}_k = \overline{\mathbf{D}}_k \overline{\mathbf{B}}_k d\mathbf{q}_k \quad (14)$$

and

$$\mathbf{s}_k = \int_0^t \overline{\mathbf{D}}_k \overline{\mathbf{B}}_k d\mathbf{q}_k \quad (15)$$

Substitution of (8) and (13) into (1) results in

$$\delta\mathbf{q}^{\mathrm{T}} \sum_{k=1}^{k=m} \mathbf{T}_k^{\mathrm{T}} \int_{V_k} \bar{\mathbf{B}}_k^{\mathrm{T}} \mathbf{s}_k\,dV - \delta\mathbf{q}^{\mathrm{T}} \sum_{k=1}^{k=m} \mathbf{T}_k^{\mathrm{T}} \int_{V_k} \mathbf{N}_k^{\mathrm{T}} \mathbf{f}\,dV - \delta\mathbf{q}^{\mathrm{T}} \sum_k \mathbf{T}_k^{\mathrm{T}} \int_{S_k} \mathbf{N}_k^{\mathrm{T}} \mathbf{p}\,dS = 0 \quad (16)$$

The above equation is valid for any virtual $\delta\mathbf{q}^{\mathrm{T}}$; therefore

$$\sum_{k=1}^{k=m} \mathbf{T}_k^{\mathrm{T}} \int_{V_k} \overline{\mathbf{B}}_k^{\mathrm{T}} \mathbf{s}_k\,dV - \sum_{k=1}^{k=m} \mathbf{T}_k^{\mathrm{T}} \mathbf{F}_k - \sum_{k=1}^{k=m} \mathbf{T}_k^{\mathrm{T}} \mathbf{P}_k = 0 \quad (17)$$

where

$$\mathbf{F}_k = \int_{V_k} \mathbf{N}_k^{\mathrm{T}} \mathbf{f}_{dV}$$

$$\mathbf{P}_k = \int_{s_k} \mathbf{N}_k^{\mathrm{T}} \mathbf{p}\, \mathrm{d}S$$

and the last summation in Equations (16) and (17) extends only over the boundary elements.

Equation (17) is a nonlinear functional equation in $\mathbf{q}$, because $\overline{\mathbf{B}}_k$ is a function of $\mathbf{q}_k$ and, thus, of $\mathbf{q}$, and $\mathbf{s}$ is a functional of $\mathbf{q}$ as shown by (15).

The first term on the left-hand side of Equation (17) will be denoted by $\mathbf{Q}(\mathbf{q})$; the second and the third terms combined, by $\mathbf{R}$. Thus, Equation (17) can be written as

$$\mathbf{Q}(\mathbf{q}) = \mathbf{R} \tag{18a}$$

or

$$Q_i(\mathbf{q}) = R_i,\ i = 1, 2, \ldots, 2n \tag{18b}$$

The numerical solution of (18) or (19) will require the derivatives

$$\frac{\delta Q_i}{\delta q_j} \equiv K_{ij} \quad \begin{array}{l} i = 1, 2, \ldots, 2n \\ j = 1, 2, \ldots, 2n \end{array} \tag{19}$$

Differentiation of Equation (17) yields

$$\mathrm{d}\mathbf{Q} = \sum_{k=1}^{k=m} \mathbf{T}_k^{\mathrm{T}} \int_{V_k} (\mathrm{d}\overline{\mathbf{B}}_k^{\mathrm{T}} \mathbf{s}_k + \overline{\mathbf{B}}_k^{\mathrm{T}} \mathrm{d}\mathbf{s}_k)\, \mathrm{d}V \tag{20}$$

From (12), (10) and (11),

$$\mathrm{d}\overline{\mathbf{B}}_k = \mathrm{d}\mathbf{B}_k'' = \frac{1}{4\Delta^2} \begin{bmatrix} \mathrm{d}\left(\frac{\partial u}{\partial x}\right) & \mathrm{d}\left(\frac{\partial v}{\partial x}\right) & 0 & 0 \\ 0 & 0 & \mathrm{d}\left(\frac{\partial u}{\partial y}\right) & \mathrm{d}\left(\frac{\partial v}{\partial y}\right) \\ \mathrm{d}\left(\frac{\partial u}{\partial y}\right) & \mathrm{d}\left(\frac{\partial v}{\partial y}\right) & \mathrm{d}\left(\frac{\partial u}{\partial x}\right) & \mathrm{d}\left(\frac{\partial v}{\partial x}\right) \end{bmatrix}$$

$$\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & b_1 & 0 & b_2 & 0 & b_3 \\ c_1 & 0 & c_2 & 0 & c_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \end{bmatrix} \tag{21}$$

and

$$\mathrm{d}\overline{\mathbf{B}}^{\mathrm{T}} \mathbf{s}_k = \mathbf{C}_k \,\mathrm{d}\mathbf{q}_k \tag{22}$$

where

$$\mathbf{C}_k \equiv \begin{bmatrix} b_1 & 0 & c_1 & 0 \\ 0 & b_1 & 0 & c_1 \\ b_2 & 0 & c_2 & 0 \\ 0 & b_2 & 0 & c_2 \\ b_3 & 0 & c_3 & 0 \\ 0 & b_3 & 0 & c_3 \end{bmatrix} \begin{bmatrix} s_{11} & 0 & s_{12} & 0 \\ 0 & s_{11} & 0 & s_{12} \\ s_{12} & 0 & s_{22} & 0 \\ 0 & s_{12} & 0 & s_{22} \end{bmatrix}$$

$$\cdot \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & b_1 & 0 & b_2 & 0 & b_3 \\ c_1 & 0 & c_2 & 0 & c_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \end{bmatrix}$$

The first term in Equation (20) is thus,

$$\int_{V_k} \mathrm{d}B_k^T s_k \,\mathrm{d}V = \mathbf{K}_k^s \,\mathrm{d}\mathbf{q} \tag{23}$$

where

$$\mathbf{K}_k^{\mathrm{S}} = \int_{V_k} \mathbf{C}_k \,\mathrm{d}V \tag{24}$$

is the so-called *initial stress*, or *geometric*, *stiffness matrix* of the element k. The second term in Equation (20) can be written, with (14) as

$$\int_{V_k} \overline{\mathbf{B}}_k^{\mathrm{T}} \overline{\mathbf{D}}_k \overline{\mathbf{B}}_k \,\mathrm{d}\mathbf{q}_k \,\mathrm{d}V = \overline{\mathbf{K}}_k \,\mathrm{d}\mathbf{q}_k \tag{25}$$

where

$$\overline{\mathbf{K}}_k = \int_{V_k} \mathbf{B}_k'^{\mathrm{T}} \overline{\mathbf{D}}_k \mathbf{B}_k' \,\mathrm{d}V + \int_{V_k} (\mathbf{B}_k'^{\mathrm{T}} \overline{\mathbf{D}}_k \mathbf{B}_k'' + \mathbf{B}_k''^{\mathrm{T}} \overline{\mathbf{D}}_k \mathbf{B}_k' + \mathbf{B}_k''^{\mathrm{T}} \overline{\mathbf{D}}_k \mathbf{B}_k''^{\mathrm{T}})\,\mathrm{d}V \tag{26}$$

The matrix $\overline{\mathbf{K}}_k$ will be called the *material stiffness matrix* of the element k. (In the problems of nonlinear elasticity, the matrix $\overline{\mathbf{K}}_k$ is known as the *elastic* stiffness matrix.)

With (23), (25), and (5), (20) becomes

$$\mathrm{d}\mathbf{Q} = \sum_{k=1}^{k=m} \mathbf{T}_k^{\mathrm{T}} (\mathbf{K}_k^s + \overline{\mathbf{K}}_k)\, \mathbf{T}_k \,\mathrm{d}\mathbf{q} \tag{27}$$

and, hence, the matrix $\mathbf{K}$, whose elements are $K_{ij} = \partial Q_i / \partial q_j$, is

$$\mathbf{K} = \sum_{k=1}^{k=m} \mathbf{T}_k^{\mathrm{T}} (\mathbf{K}_k^s + \overline{\mathbf{K}}_k)\, \mathbf{T}_k \tag{28}$$

The matrix

$$\mathbf{K}_k = \mathbf{K}_k^s + \overline{\mathbf{K}}_k$$

is known as the *tangent stiffness matrix* of the element k. The matrix $\mathbf{K}$ defined by Equation (28) is the *global tangent stiffness matrix*.

3. METHOD OF SOLUTION

The functional equations (18) are solved by applying incremental steps of loading and performing iterations within each increment. The computational procedure is then as follows:

Let $\mathbf{q}^{(n)}$ and $\mathbf{R}^{(n)}$ be the nodal displacements and the right-hand side of (18), respectively, at the end of the

nth load increment, for which

$$\mathbf{Q}(\mathbf{q}^{(n)}) - \mathbf{R}^{(n)} = 0. \quad (29)$$

Let $\Delta\mathbf{R}$ be the next load increment, and let $\Delta\mathbf{q}$ be the corresponding nodal displacement increment; they satisfy

$$\mathbf{Q}(\mathbf{q}^{(n)} + \Delta\mathbf{q}) - (\mathbf{R}^{(n)} + \Delta\mathbf{R}) = 0. \quad (30)$$

The first approximation for $\Delta\mathbf{q}$ will be obtained by taking the first term of the Taylor expansion of $\mathbf{Q}$ at $\mathbf{q}^{(n)}$, i.e., by replacing Equation (30) with

$$\mathbf{Q}(\mathbf{q}^{(n)}) + \mathbf{K}_o\Delta\mathbf{q} - (\mathbf{R}^{(n)} + \Delta\mathbf{R}) = 0$$

or, in view of (29),

$$\mathbf{K}_o\Delta\mathbf{q} - \Delta\mathbf{R} = 0 \quad (31)$$

where

$$\mathbf{K}_o = \left(\frac{\partial\mathbf{Q}}{\partial\mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}^{(n)}}$$

is the tangent stiffness matrix at $\mathbf{q} = \mathbf{q}^{(n)}$ (the tangent stiffness matrices have been discussed in Section 2; see Equations (19), (27), and (28)).

From (31), the first approximation of $\Delta\mathbf{q}$ follows as

$$\Delta\mathbf{q}_{(1)} = \mathbf{K}_o^{-1}\Delta\mathbf{R}.$$

If the above approximate value is substituted into Equation (30), we have

$$\mathbf{Q}(\mathbf{q}^{(n)} + \Delta\mathbf{q}_{(1)}) - (\mathbf{R}^{(n)} + \Delta\mathbf{R}) = -\Psi. \quad (32)$$

The next correction to $\Delta\mathbf{q}$ is obtained by expanding $\mathbf{Q}$ at $\mathbf{q} = \mathbf{q}^{(n)} + \Delta\mathbf{q}_{(1)}$, which results in

$$\mathbf{Q}(\mathbf{q}^{(n)} + \Delta\mathbf{q}_{(1)}) + \mathbf{K}_{(1)}\Delta\mathbf{q} - (\mathbf{R}^{(n)} + \Delta\mathbf{R}) = 0$$

or, with (32) in

$$\mathbf{K}_{(1)}\Delta\mathbf{q} = \Psi. \quad (33)$$

where $\mathbf{K}_{(1)}$ is the tangent stiffness at $\mathbf{q} = \mathbf{q}^{(n)} + \Delta\mathbf{q}_{(1)}$. From (33),

$$\Delta\mathbf{q}_{(2)} = \mathbf{K}_{(1)}^{-1}\Psi, \quad (34)$$

and, now,

$$\Delta\mathbf{q} = \Delta\mathbf{q}_{(1)} + \Delta\mathbf{q}_{(2)}. \quad (35)$$

The ith correction to $\Delta\mathbf{q}$ follows, thus, from

$$\Delta\mathbf{q}_{(i)} = \mathbf{K}_{(i-1)}^{-1}\Psi_{(i-1)} \quad (36)$$

where

$$\mathbf{K}_{(i-1)} = \mathbf{K}(\mathbf{q}^{(n)} + \Delta\mathbf{q}_{(1)} + \ldots + \Delta\mathbf{q}_{(i-1)}) \quad (37)$$

$$\Psi_{(i-1)} = \mathbf{Q}(\mathbf{q}^{(n)} + \Delta\mathbf{q}_{(1)} + \ldots + \Delta\mathbf{q}_{(i-1)}) - (\mathbf{R}^{(n)} + \Delta\mathbf{R}) \quad (38)$$

The value of $\Delta\mathbf{q}$ after i corrections is

$$\Delta\mathbf{q} = \Delta\mathbf{q}_{(1)} + \Delta\mathbf{q}_{(2)} + \ldots + \Delta\mathbf{q}_{(i)} \quad (39)$$

The above procedure is formally identical with the known *Newton–Raphson* method for the solution of a system of nonlinear algebraic equations. The convergence of this method for a system of nonlinear algebraic equations has been widely discussed (see, for example, [31]).

It should be kept in mind, however, that the equations of the present problem (Equations (18)) are not algebraic equations in $\mathbf{q}$. Consequently, many properties of the Newton–Raphson method, especially those related to the nature of the approximation and to the convergence of the process, are not transferable to the case of Equations (18). The differences between the Newton–Raphson method applied to a system of nonlinear algebraic equations (e.g. of a nonlinear elastic problem) and the system of functional equations (as in the present problem) is illustrated in Figures 3a, b.

In the case of algebraic equations (Figure 3a), after the first value $\Delta\mathbf{q}_{(1)}$ has been found, the corresponding

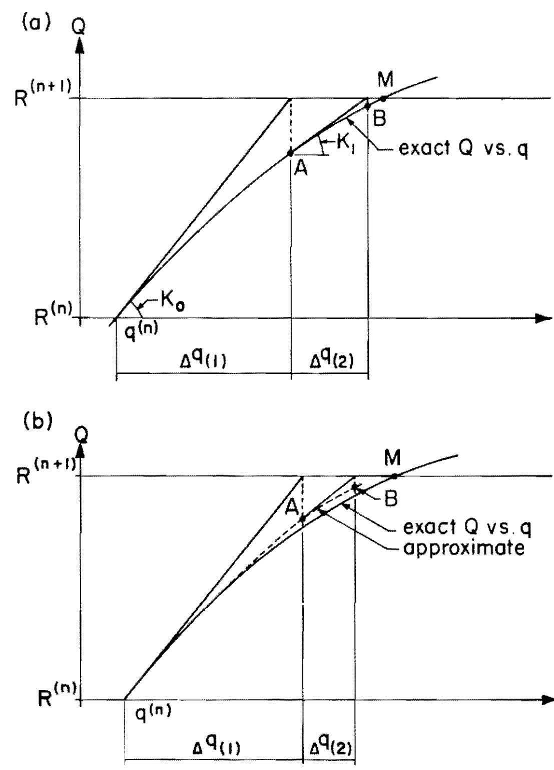

Figure 3. Iteration Process for (a) A Nonlinear Elastic Solid and for (b) An Elasto–Plastic solid

value of $\mathbf{Q}(\mathbf{q}^{(n)} + \Delta\mathbf{q}_{(1)})$ (point A) and the slope $\mathbf{K}_1$ for the next step can be computed exactly. The process is known to converge to the exact solution [3].

In the case of functional equations (Figure 3b), from the value $\Delta\mathbf{q}_{(1)}$, the value of $\mathbf{Q}(\mathbf{q}^{(n)} + \Delta\mathbf{q}_{(1)})$ can be determined only approximately, i.e., the point A cannot be located exactly on the true curve $\mathbf{Q}(\mathbf{q})$, and the slope $\mathbf{K}_1$ for the next step is also only approximate. Consequently, the convergence theorems of the Newton–Raphson method, developed for systems of algebraic equations, are not directly applicable to the present problem. The same objections apply to the modified Newton–Raphson method (in which $\mathbf{K}_i$ is replaced by the matrix $\mathbf{K}_0$ in every step of iteration) as well as to other iterative procedures developed for the finite element analysis of elasto–plastic solids, such as the 'initial strain' method [4] and the 'initial stress' method [5]. The treatment of the incremental Equations (31) as a system of ordinary differential equations (proposed in Reference [9]) and the application of higher-order numerical integration methods does not remove the difficulties outlined above.

In spite of the lack of a formal proof of convergence, the method described in this section has been shown to yield results of remarkable accuracy. References [6–8] contain a detailed discussion and numerical results which prove, at least heuristically, the validity of the procedure.

4. SUMMARY AND CONCLUSIONS

The problem of analysis of displacements, stresses, and strains in elements made of elasto–plastic metals, subjected to arbitrarily large deformations under the conditions of plane strain has been formulated in terms of the finite element method. The resulting system of integral equations for the nodal displacement can be solved by using a combination of Euler's forward integration with the Newton–Raphson iteration at each step.

The work described in this paper seems to demonstrate the feasibility of the theory of plasticity of large deformations and the finite element technique in solving complex problems (i.e., any shape and any deformation) of the mechanics of elasto–plastic metals. At the same time, certain topics have been exposed as requiring further extensive investigations. They are

1. Efficient methods of solution of the integral equations resulting from the application of the finite element technique to the problems of stress and strain analysis. Specially, reduction of computer time and clarification of the nature of convergence appear to be of utmost urgency.
2. Criteria for selection of optimal types of finite elements. Clearly, the experience accumulated in linearly elastic problems is not directly transferable to the present problems.

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