A NOTE ON RINGS WITH NO NIL RIGHT IDEALS

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A result of Gupta states that if D is a division ring such that $xy^2x = yx^2y$ for all x, $y \in D$, then D must be commutative [1]. The purpose of this note is to give a generalization of this theorem. We will prove the following theorem. Throughout this paper, R, denotes a ring with identity.

Theorem

Let R be a ring with no nonzero nil right ideals. Suppose that there exist nonnegative integers n, m, k with $m > 0$ and $n = 2m + k$ such that $x^m y x^k y x^m = y x^n y$ for all x, y in R. Then R is commutative.

The proof of the above theorem consists of five lemmas. The first three lemmas are well known and easy, thus we omit their proofs here. We let the familiar notation $[x, y]$ stand for $xy - yx$.

Lemma 1

If R is a ring, x, y are elements of R and n is a positive integer, then

$$[x^n, y] = \sum_{k=0}^{n-1} x^k [x, y] x^{n-k-1}$$

Lemma 2

If $[x, [x, y]] = 0$, then $[x^k, y] = k x^{k-1} [x, y]$ for all positive integers k.

Lemma 3

If R is a ring and x, $y \in R$ satisfy $[x, [x, y]] = 0$ then $[x, [x, y^2]] = 2[x, y]^2$.

Lemma 4

Let R be a ring with no nonzero nilpotent elements. If there exist nonnegative integers n, m, k with $m > 0$ and $n = 2m + k$ such that $x^m y x^k y x^m = y x^n y$ for all x, y in R, then R is commutative.

Proof

Since R has no nilpotent elements, R is a subdirect product of domains R_α [2, 3]. Clearly, each R_α satisfies the hypotheses of the lemma. So, we may assume that R is a domain. Let x, $y \in R$. There exist integers n, m, k as in the statement of the lemma such that $x^m (x + y) x^k (x + y) x^m = (x + y) x^n (x + y)$. If we let $p = m + k + 1$, this reduces to $x^p y x^m + x^m y x^p = x^{n+1} y + y x^{n+1}$. Since $n = 2m + k$, this can be written as, $[x^p, [x^m, y]] = 0$ which by Lemma 1 easily leads to

$$[x^{pm}, [x^m, y]] = 0. \tag{1}$$

By Lemma 1, we have, $[x^{pm}, y] = \sum_{i=0}^{p-1} x^{mi} [x^m, y] x^{m(p-i-1)}$, and since x^{pm} commutes with $[x^m, y]$ by (1),

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it follows that x^{pm} commutes with $[x^{pm}, y]$, i.e.

$$[x^{pm}, [x^{pm}, y]]=0. \tag{2}$$

Let $s=pm$ so that (2) becomes

$$[x^s, [x^s, y]]=0. \tag{3}$$

Since (3) holds for all $x, y \in R$, replacing y by y^2, we can find a positive integer t such that

$$[x^t, [x^t, y^2]]=0. \tag{4}$$

From Equations (3) and (4) it is easily seen that $[x^{st}, [x^s, y]]=0$ and $[x^{st}, [x^t, y^2]]=0$. Letting $q=st$ and using Lemma 1 again in the same way it was used to derive Equation (2), we obtain

$$[x^q, [x^q, y]]=0 \tag{5}$$

and

$$[x^q, [x^q, y^2]]=0. \tag{6}$$

Since R is a domain, we have two cases:

Case 1

Char $R=r\neq 0$. Then by (5) and Lemma 2 $[(x^q)^r, y]=r(x^q)^{r-1}[x^q, y]=0$, and hence, R is commutative by Herstein's theorem [4].

Case 2

Char $R=0$. Then, using Equation (5), we have, by Lemma 3, $2[x^q,y]^2=[x^q,[x^q,y^2]]$, and hence by Equation (6) $2[x^q,y]^2=0$ which implies that $x^qy=yx^q$. Therefore, it follows by Herstein's theorem [4] that R is commutative.

Lemma 5

Let R be a ring such that there exist integers n, m, k, with $m>0$, $n=2m+k$ and $x^m yx^k yx^m=yx^n y$ for all x, y in R. If x is a nonzero element of R such that $x^2=0$, then the right ideal xR is nil.

Proof

Applying the given identity to $x+xy$ and xy we have $(x+xy)^m(xy)(x+xy)^k(xy)(x+xy)^m=xy(x+xy)^n xy$. Using the fact that $x^2=0$, this reduces to $(xy)^{m+1}(xy)^{k+1}[(xy)^m+(xy)^{m-1}x]=(xy)^{n+2}$, or $(xy)^{n+2}+(xy)^{n+1}x=(xy)^{n+2}$. Therefore $(xy)^{n+1}x=0$ or $(xy)^{n+2}=0$. Thus, every element of xR is nilpotent, i.e. xR is nil.

Proof of the Theorem

By Lemma 5, we may assume that R contains no nilpotent elements. Therefore, the conclusion of the theorem follows from Lemma 4.

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