GENERALIZED LEXICOGRAPHIC PRODUCT OF GRAPHS AND STRONG ENDOMORPHISM MONOID

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الخلاصـة :

يتناول هذا البحث إيجاد علاقة رياضيّة عند صناعة المعجم بين المفردات (الألفاظ) وتوليداتها الاشتقاقية الداخلية وما ينتج عنها بواسطة العلاقة للرياضية التالية :

. SEnd $X[Y_x|x\in V(X)]$ و wr (Aut X, $\bigcup_{x,x'\in V(X)}$ SHom $(Y_x, Y_{x'})$)

ABSTRACT

The strong endomorphism monoid of generalized lexicographic product of graphs is discussed in this paper. The relation between wr (Aut X, $\bigcup_{x,x'\in V(X)}$ SHom $(Y_x, Y_{x'})$) and SEnd $X[Y_x|x\in V(X)]$ is characterized.

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The graphs that we consider here are finite, undirected, and simple. About the knowledge of graph theory and semigroup theory, we may consult [1] and [2]. About the homomorphism, strong-homomorphism, isomorphism, endomorphism, strong-endomorphism, automorphism of two graphs, we may consult [3]. About various unretractivity of graphs, we may consult [4].

Let X be a graph. For every $x\in V(X)$, let Y_x be a graph. Define a new graph $Z=X[Y_x|x\in V(X)]$, the set of vertices of Z is $\{(x, y)| y\in V(Y_x), x\in V(X)\}$. The set of edges of Z is $\{\{(x_1, y_1), (x_2, y_2)\}|\{x_1, x_2\}\in E(x)$, or $x_1=x_2$ and $\{y_1, y_2\}\in E(Y_{x_1})\}$. This new graph is called the generalized lexicographic product of X and $\{Y_x|x\in V(X)\}$. If all the graphs Y_x are the same, then this Z will become the $X[Y]$, which is called the lexicographic product and is discussed in [5]. Let SHom $(Y_x, Y_{x'})$ be the set of all strong homomorphism from Y_x to $Y_{x'}$.

Let $F = \bigcup_{x, x'\in V(X)} \text{SHom}(Y_x, Y_{x'})$.

Define wr (Aut X, F) = $\{(f, \varphi)| f\in \text{Aut } X, \varphi\in F^{V(X)}, \forall x\in V(X), \varphi(x)\in \text{SHom}(Y_x, Y_{f(x)})\}$, where $F^{V(X)}$ denotes the set of all mappings from $V(X)$ to F. Define a multiplication for the set wr (Aut X, F):

$$(f, \varphi)(g, \psi) = (fg, \varphi_g\psi)$$

Where $[\varphi_g\psi](x) = \varphi(g(x))\psi(x)\in \text{SHom}(Y_x, Y_{gf(x)})$. Then wr (Aut X, F) is a monoid under the above multiplication; wr (Aut X, F) is called the wreath product of Aut X and F by $V(X)$.

There is an action of wr (Aut X, F) on the set $V(X[Y_x|x\in V(X)])$ from the left through the following formulae:

$(f, \varphi)(x, y) = (f(x), \varphi(x)(y))$, for all $(x, y)\in V(X[Y_x|x\in V(X)])$, and all $(f, \varphi)\in$ wr (Aut X, F).

Lemma 1. wr(Aut X, F) $\subseteq$ SEnd $X[Y_x|x\in V(X)]$.

Proof. For each $(f, \varphi)\in$ wr (Aut, F), give two vertices (x_1, y_1) and (x_2, y_2) in $X[Y_x|x\in V(X)]$. Let (x_1, y_1) be adjacent to (x_2, y_2). Then $\{x_1, x_2\}\in E(X)$ or $x_1=x_2$, $\{y_1, y_2\}\in E(Y_{x_1})$. Then $\{f(x_1), f(x_2)\}\in E(X)$ or $f(x_1)=f(x_2)$, $\{\varphi(x_1)(y_1), \varphi(x_2)(y_2)\}\in$
$E(Y_{f(x_1)})$. We obtain that $(f(x_1), \varphi(x_1)(y_1))$ and $(f(x_2), \varphi(x_2)(y_2))$ are adjacent, that is, $(f, \varphi)(x_1, y_1)$ is adjacent to $(f, \varphi)(x_2, y_2)$. Conversely, if $(f, \varphi)(x_1, y_1)$ is adjacent to $(f, \varphi)(x_2, y_2)$, then $(f(x_1), \varphi(x_1)(y_1))$ is adjacent to $(f(x_2), \varphi(x_2)(y_2))$. Then one of the following two conditions holds:

(1) $f(x_1)$ is adjacent to $f(x_2)$, thus $\{x_1, x_2\}\in E(X)$.

(2) $f(x_1) = f(x_2)$, $\varphi(x_1)(y_1)$ is adjacent to $\varphi(x_2)(y_2)$. Thus we get $x_1= x_2$, $\varphi(x_1)(y_1)$ is adjacent to $\varphi(x_1)(y_2)$. Furthermore we know that $x_1=x_2$ and $\{y_1, y_2\}\in E(Y_{f(x_1)})$.

From all the above we conclude that (f, φ) belongs to SEnd $X[Y_x|x\in V(X)]$.

Theorem 2. Suppose every Y_x is a completely disconnected graph. Then wr (Aut X, F)= SEnd $X[Y_x|x\in V(X)]$ if and only if X is S-unretractive.

Proof. ($\Rightarrow$) We put a fixed vertex u_x in each Y_x . Suppose $h\in$SEnd X. Define $\alpha : (x, y) \to (h(x), u_{h(x)})$.Then $\alpha\in$SEnd $X[Y_x|x\in V(X)]$. We have $\alpha = (f, \varphi)$ such that $f\in$Aut X, $\varphi\in F^{V(x)}$. Then $\alpha(x, u_x) = (f, \varphi)(x, u_x)$, for every $x\in V(X)$. We have $(h(x), u_{h(x)}) = (f(x), \varphi(x)(u_x))$. Therefore, $h(x)=f(x)$, $h = f\in$Aut X. In conclusion, X is S-unretractive.

($\Leftarrow$) For $x_1, x_2\in V(X)$, $x_1\neq x_2$ implies $N(x_1)\neq N(x_2)$.Therefore we obtain $N(x_1, y_1)\neq N(x_2, y_2)$, for all $y_1\in Y_{x_1}$, $y_2\in Y_{x_2}$. The result is that the vertices in (x_1, Y_{x_1}) and those in (x_2, Y_{x_2}) do not satisfy the relation ν, as defined in [3]. Both two vertices in (x, Y_x) satisfy ν. Therefore $X = G|_\nu$, where $G = X[Y_x|x\in V(X)]$. By theorem 3.4 in [3], it is clear that wr (Aut X, F) = SEnd $X[Y_x|x\in V(X)]$.

Note: (x, Y_x) appeared in the proof of the above theorem and denotes the set $\{(x, y)|y\in Y_x\}$. It also denotes the subgraph of $X[Y_x|x\in V(X)]$ induced by this set.

Next, we continue to discuss the conditions for wr (Aut X,F)=SEnd $X[Y_x|x\in V(X)]$. We will give two lemmas in advance.

Lemma 3. Suppose $g\in \text{End } X[Y_x|x\in V(X)]$, $\{x', x''\}\in E(X)$. Then $g(x', Y_{x'})\cap g(x'', Y_{x''}) = \phi$.

Proof. Suppose $(x, y)\in g(x', Y_{x'})\cap g(x'', Y_{x''})$, then there exist $y'\in Y_{x'}$, $y''\in Y_{x''}$,such that $(x,y) = g(x', y') = g(x'', y'')$ $\{x', x''\}\in E(X)$ imply (x', y') is adjacent to (x'', y'').Therefore $g(x', y')$ is adjacent to $g(x'', y'')$, *i.e.* (x,y) is adjacent to (x,y) in $X[Y_x|x\in V(X)]$. This is a contradiction, because the graph that we consider here is loopless.

Lemma 4. Assume that C_{2n+1} is odd cycle, and H is a graph. Let f be homomorphism from C_{2n+1} to H, then $H[f(V(C_{2n+1}))]$ contains odd cycle.

Proof. Considering that C_{2n+1} is 3-colorable, we have that $H[f(V(C_{2n+1}))]$ is 3-colorable. Finally, it is not a bipartite graph. We know that a graph G is a bipartite graph if and only if G does not contain odd cycle. Thus $H[f(V(C_{2n+1}))]$ contains odd cycle.

Theorem 5. Suppose every Y_x has a unique odd cycle. Then wr (Aut X, F)=SEnd $X[Y_x|x\in V(X)]$ if and only if $\forall g\in \text{SEnd } X[Y_x|x\in V(X)]$, $\forall x\in V(X)$, there exists $x'\in V(X)$, such that $g(x, Y_x)\subseteq (x', Y_{x'})$.

Proof. ($\Leftarrow$) For all $g\in \text{SEnd } X$, $[Y_x|x\in V(X)]$, define f: $V(X)\to V(X)$, $x\to x'$, if $g(x, Y_x)\subseteq (x', Y_{x'})$. Then f is a mapping from $V(X)$ to $V(X)$.

First, we prove $f\in \text{End } X$. Let $\{x_1, x_2\}\in E(X)$. ① For every $y_1\in Y_{x_1}$, $y_2\in Y_{x_2}$, we have $p_1g(x_1, y_1) = p_1g(x_2, y_2)\triangleq z$, where $p_1g(x, y)$ denotes the first coordinate of $g(x, y)$. Suppose that C_1 and C_2 are the unique odd cycles of Y_{x_1} and Y_{x_2} respectively. Then $\{(x_1, y)|y\in C_1\}$ is the odd cycle of (x_1, Y_{x_1}), $\{(x_2, y)|y\in C_2\}$is the odd cycle of (x_2, Y_{x_2}). Let W_1 be a subgraph induced by $\{g(x_1, y)|y\in C_1\}$; let W_2 be a subgraph induced by $\{g(x_2, y)|y\in C_2\}$. By Lemma 3, we know $W_1\cap W_2 = \phi$. By Lemma 4, we get that both W_1 and W_2 contain odd cycle. Also, $W_1\cup W_2 \subseteq(z, Y_z)$, therefore (z, Y_z) contain two disjoint odd cycles. Furthermore, its copy Y_z contains two disjoint odd cycles also. This conclusion is contradictory to the premise. Thus condition ① does not appear. We only consider ②. There exists $y_1\in Y_{x_1}$, $y_2\in Y_{x_2}$, such that $p_1g(x_1, y_1)\neq p_1g(x_2, y_2)$. Let $x_1'=p_1g(x_1, y_1)$, $x_2'=p_1g(x_2, y_2)$, then $f(x_1)=x_1'$, $f(x_2)=x_2'$, and $x_1'\neq x_2'$. $\{x_1, x_2\}\in E(X)\Rightarrow(x_1, y_1)$ and (x_2, y_2) are adjacent $\Rightarrow g(x_1, y_1)$ and $g(x_2, y_2)$ are adjacent $\Rightarrow\{x_1', x_2'\}\in E(X)\Rightarrow\{f(x_1), f(x_2)\}\in E(X)$. Namely, $f\in \text{End } X$.

Secondly, we show $f\in \text{SEnd } X$. For each x_1, $x_2\in V(X)$, let $\{f(x_1), f(x_2)\}\in E(X)$. Then $g(x_1, y_1)$ and $g(x_2, y_2)$ are adjacent for all $y_1\in Y_{x_1}$, $y_2\in Y_{x_2}$. And g is strong – endomorphism. We conclude that (x_1, y_1) is adjacent to (x_2, y_2). Thus $\{x_1, y_1\}\in E(X)$.

Lastly, we prove $f\in \text{Aut } X$. For every x_1, $x_2\in V(X)$, $x_1\neq x_2$. We consider the following two conditions:

(1) suppose $\{x_1, x_2\}\in E(X)$. We know that $f(x_1)\neq f(x_2)$ from the process of ②.

(2) suppose $\{x_1, x_2\}\notin E(X)$. Then $f(x_1)\neq f(x_2)$. If not, suppose $f(x_1) = f(x_2)\triangleq z$. In a similar manner to ①, we choose C_1 and C_2. Then we obtain W_1, W_2. $W_1\cup W_2\subseteq(z, Y_z)$. By Lemma 4, we see that W_1 and W_2 contain two disjoint odd cycles O_1 and O_2 respectively. If $O_1\cap O_2=\phi$, then (z, Y_z) contain two disjoint odd cycles. Further, we know that Y_z contains two disjoint odd cycles. This is a contradiction. If $O_1\cap O_2\neq\phi$, let $(z, a)\in O_1\cap O_2$. Choose $(z, b)\in O_1$, such that (z, b) is adjacent to (z, a). (z, b) is image of some (x_1, y') under g, where $y'\in C_1$. Also, (z, a) is image of some (x_2, y'') under g, where $y''\in C_2$. By use of the condition that $g(x_1, y')$ is adjacent to $g(x_2, y'')$ and g is a strong-endomorphism, we know that (x_1, y') is adjacent to (x_2, y'').

But $\{x_1, x_2\}\notin E(X)$, only $x_1= x_2$, a contradiction. It is proved that f is injective. Because $V(X)$ is finite set, therefore we have that f is bijection. Therefore $f\in \text{Aut } X$.

Define $\varphi : V(X) \to F$, $x \to \varphi_x = \varphi(x)$, such that $\varphi_x : Y_x \to Y_{f(x)}$, if $g(x, y) = (f(x), y')$. It is easy to verify that $\varphi_x \in \mathrm{SHom}\,(Y_x, Y_{f(x)})$. For all $x \in V(X)$, every $y \in V(Y_x)$, we have:

$g(x, y) = (x', y') = (f(x), \varphi(x)(y)) = (f, \varphi)(x, y)$. Thus $g = (f, \varphi)$.

($\Rightarrow$): By conditions, for all $g \in \mathrm{SEnd}\, X\,[Y_x | x \in X]$, we have $g \in \mathrm{wr}\,(\mathrm{Aut}\, X, F)$. Thus $g = (f, \varphi)$. For all $x \in V(X)$, $y \in Y_x$, we have $g(x, y) = (f, \varphi)(x, y) = (f(x), \varphi(x)(y))$. Let $x' = f(x)$, then we have $g(x, y) \in (x', Y_{x'})$.

Lemma 6. Assume that X and each $Y_x (x \in V(X))$are not empty graphs, which do not contain K_3. For $x_0 \in V(X)$, Y_{x_0} contains odd cycle. Then for each $g \in \mathrm{End}\, X\,[Y_x | x \in V(X)]$, there exists $x' \in V(X)$, such that $g(x_0, Y_{x_0}) \subseteq (x', Y_{x'})$.

Proof. Suppose that there exists $f \in \mathrm{End}\, X\,[Y_x | x \in V(X)]$ and $x_0 \in V(X)$ such that:

$$f(x_0, Y_{x_0}) \subseteq (u_1, Y_{u_1}) \cup (u_2, Y_{u_2}) \cup \ldots \cup (u_k, Y_{u_k}). \tag{1}$$

We see that the relation of inclusion in (1) does not hold if some (u_i, Y_{u_i}) is deleted, where $k \geq 2$.

$$\text{Let } x_1 \in V(X),\ \{x_1, x_0\} \in E(X), \text{ and } f(x_1, Y_{x_1}) \subseteq \bigcup_{j=1}^{m} (v_j, Y_{v_j})\quad m \geq 1. \tag{2}$$

The relation of inclusion in (2) does not hold if some (v_j, Y_{v_j}) is deleted.

(*i*) We conclude that $\{v_1, v_2, \ldots, v_m\} \subseteq \{u_1, u_2, \ldots, u_k\}$.

If not, there exists some v_j, and we may assume $v_j = v_1$, without loss of generality, such that $v_1 \notin \{u_1, u_2, \ldots, u_k\}$. For each u_i $(i = 1, 2, \ldots, k)$, there exists $y_i \in V(Y_{x_0})$, such that $f(x_0, y_i) \in (u_i, Y_{u_i})$. There exists $y' \in V(Y_{x_1})$ such that $f(x_1, y') \in (v, Y_v)$. Since (x_1, y') and (x_0, y_i) are adjacent, $g(x, y')$ and $g(x_0, y_i)$ are adjacent too, $i = 1, 2, \ldots, k$. But $v_1 = p_1 g(x_1, y') = p_1 g(x_0, y_i) = u_i$. We infer that $\{v, u_i\} \in E(X)$ $i = 1, 2, \ldots, k$.

Because Y_{x_0} is connected, so $X\,[u_1, u_2, \ldots, u_k]$ is the connected subgraph of X by (1), $k \geq 2$, and $X[u_1, u_2, \ldots, u_k]$ is the subgraph of X induced by $\{u_1, u_2, \ldots, u_k\}$.The subgraph is not a completely disconnected graph.

Let $\{u_1, u_2\} \in E(X)$, then $X\,[v_1, u_1, u_2]$ is a K_3-subgraph of X. This is a contradiction.

(*ii*) $m \geq 2$. if not, $m=1$, then $f(x_1, Y_{x_1}) \subseteq (v, Y_v)$. By (*i*), without loss of generality, we suppose $v = u_1$. Suppose $\{y_1, y_2\} \in E(Y_{x_1})$, then $g(x_1, y_1) = (u_1, y_1')$, $g(x_1, y_2) = (u_1, y_2')$. Select $y_3 \in Y_{x_0}$; let $g(x_0, y_3) = (u_1, y_3')$. It is clear that (u_1, y_1'), (u_1, y_2'), (u_1, y_3') make up a K_3-subgraph of (u_1, Y_{u_1}). Go a step further, $\{y_1', y_2', y_3'\}$ will induce a K_3subgraph of Y_{u_1}, in contradiction to the conditions of this Lemma.

(*iii*) In a similar manner to (1), we conclude $\{u_1, u_2, \ldots, u_k\} \subseteq \{v_1, v_2, \ldots, v_m\}$.Thus $\{u_1, u_2, \ldots, u_k\} = \{v_1, v_2, \ldots, v_m\}$, $m = k \geq 2$. So $g(x_0, Y_{x_0}) \cup g(x_1, Y_{x_1}) \subseteq \bigcup_{i=1}^{k} (u_i, Y_{u_i})$.

(*iv*) $k=2$. if not, then $k \geq 3$. $X[u_1, u_2, \ldots, u_k]$ is a connected subgraph of X. There exists a 2-path, suppose $u_1\, u_2\, u_3$ is a path, such that $\{u_1, u_2\}$, $\{u_2, u_3\} \in E(X)$. There exists $y_0 \in Y_{x_0}$, $y_1 \in Y_{x_1}$, such that $g(x_0, y_0) \in (u_1, Y_{u_1})$, $g(x_1, y_1) \in (u_3, Y_{u_3})$. That (x_0, y_0) and (x_1, y_1) are adjacent implies that $g(x_0, y_0)$ and $g(x_1, y_1)$ are adjacent, so $\{u_1, u_3\} \in E(X)$. X contains a K_3-subgraph. A contradiction appears.

(*v*) Because X is connected, therefore for each $x \in V(X)$, we have $g(x, Y_x) \subseteq (u_1, Y_{u_1}) \cup (u_2, Y_{u_2})$. In fact, $g(x, y_x) \subsetneq (u_1, Y_{u_1})$ and $g(x, y_x) \subsetneq (u_2, Y_{u_2})$.

(*vi*) Choose $x_0 \in V(X)$ such that Y_{x_0} has odd cycle C. There exists two vertices c_1 and c_2 in C, such that $\{c_1, c_2\} \in E(C)$, and $g(x_0, c_1)$ and $g(x_0, c_2)$ belong to the same (u_i, Y_{u_i}), $i=1, 2$. Without loss of generality, we suppose that they belong to (u_1, Y_{u_1}). Let $g(x_0, c_1) = (u_1, c_1')$, $g(x_0, c_2') = (u_1, c_2')$. Choose $x_1 \in N(x_0)$. There exists $d \in V(Y_{x_1})$ such that $g(x_1, d) = (u_1, d') \in (u_1, Y_{u_1})$. $\{c_1, c_2\} \in E(Y_{x_0}) \Rightarrow \{c_1', c_2'\} \in E(Y_{u_1})$. (x_0, c_i) and (x_1, d) are adjacent $\Rightarrow (u_1, c_i')$ and (u_1, d') are adjacent $\Rightarrow c_i'$ and d' are adjacent, $i=1, 2$. $\{c_1', c_2', d'\}$ will induce K_3-subgraph. A contradiction appears.

All the above reasoning indicates that the assumption at the beginning of this proof is wrong. This completes the proof of Lemma 6.

Notice that S End $G \subseteq$ End G. We can conclude the following theorem by Theorem 5 and Lemma 6.

Theorem 7: Let X and Y_x ($x \in V(X)$) be connected graphs which do not have K_3-subgraph. For each $x \in V(X), Y_x$ has a unique odd cycle. Then:

SEnd $X[Y_x | \, x \in V(X)]$ = wr (Aut X, F).

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