PRÜFER RINGS IN *-DIVISION RINGS

Ismail M. Idris†

Department of Mathematics, Faculty of Science
Ain Shams University
Cairo, Egypt

الخلاصة :

لقد درسنا مفهوم المثال الكسري لحلقة تركيبية (إلتفافية) لإعطاء شروط متكافئة لحلقة جزئية لتكون حلقة بروفر (Prüfer) في حلقة قسمة D ذات ترقية إنڤوليوتية . كما نعرض لتكوين حلقة بروفر(Prüfer) معينة مرتبطة بترتيب مسبق لـ D مُعطىً .

ABSTRACT

The notion of a fractional ideal of a ring with involution is studied to give equivalent conditions for a subring to be a Prüfer ring in a division ring D with involution. Also, a construction of a certain Prüfer ring associated with a given preordering of D is given.

†Address for correspondence:
Mathematics Department, Faculty of Science
UAE University
P.O. Box 17551, Al-Ain, United Arab Emirates
E-mail: Ismail.Idris@uaeu.ac.ae

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1. INTRODUCTION

As in the commutative case, many of the rings which arise in connection with preorderings in division rings are Prüfer rings, *i.e.*, the localizations at all maximal ideals are valuation rings. Prüfer rings also help in studying Approximation Theorems for Valuations. In this work, these subrings are studied in the case of a division ring D with involution, to give more properties of orderings and valuations of division rings with involution. In general, rings with involution have been studied intensively in some applications to Lie algebras, Jordan algebras, and rings of operators. More recently, the category of rings with involution has been investigated (see [1]). The ideals of an object in this category must be closed under the involution *, called *-ideals.

In Section 3, the notion of a fractional *-ideal is studied in the case of a *-ring (a ring with involution); for the commutative case one can refer to [2]. It is shown that a symmetric subring R of D is a Prüfer ring if and only if each finitely generated fractional *-ideal of R is invertible. Furthermore, any total *-subring that contains a Prüfer ring B is the localization of B at some *-prime *-ideal of B.

For a preordering $T\subset D$, one can construct a subring consisting of elements of D, which are bounded by some rational number with respect to the preordering; this subring will turn out to be a Prüfer ring. This ring is a useful tool in studying preorderings in division rings with involution. In fact, it is shown that this subring is the intersection of all *-valuation subrings which are compatible with the preordering.

2. DIVISION RINGS WITH INVOLUTION

In this section, we state the basic definitions and some facts that will be needed in this work. Hereafter D will be a not necessarily commutative division ring with involution (an anti-automorphism of period 2), and $D^{\bullet}$ will denote its multiplicative group of non-zero elements. We simply say D is a **-division ring*. If R is a *-closed subring ($r\in R$ implies $r^*\in R$), then we say that R is a **-subring* of D. If R is a *-subring of D, a non-empty subset $S\subset R$ is called a *denominator set* if $0\notin S$, S is *-closed, S is multiplicatively closed, and S satisfies the Ore-condition (for $r\in R$ and $s\in S$ there is $b\in R$ and $t\in S$ such that $rt = sb$). In fact, for any arbitrary ring, one can define a right and left denominator set, and right and left ring of fractions of R, but from [3], any right ring of fractions of a ring with involution R is also a left ring of fractions of R. So we can speak only of rings of fractions of rings with involution. Also, from [3], if X is a ring of fractions of R, then there is a uniquely determined involution on X, which is the extension of the involution defined on R. If R is a *-subring of D, and $S\subset R$ is a denominator set, then the ring of fractions $RS^{-1} = \{rs^{-1} : r\in R,\ s\in S\}$ is a *-subring of D. Also, $R\subset RS^{-1}$, because S is non-empty and for each $r\in R$, $r = (rs)s^{-1} \in RS^{-1}$, for some $s\in S$.

Lemma 1. If $R\subset D$ is a *-subring which is closed under conjugation, then

(*i*) all ideals are two-sided ideals and left submodules of D_R are right,

(*ii*) any subset $S\subset R$ with $0\notin S$ satisfies the Ore condition.

Proof.

(*i*) If I is a left ideal in R, then for $r\in R$ and $x\in I$, $xr = xr\,x^{-1}\,x\in I$ and so I is a two-sided ideal. Similarly for R-submodules of D_R.

(*ii*) Let $r\in R$, $s\in S$. By assumption $s\neq 0$ and so $s^{-1}rs = b\in R$. Hence $rs = sb$. Take $t = s$ and the Ore-condition is satisfied.

Following [4], a *-ideal P of a *-ring R is called **-prime* if $IK\subseteq P$ implies $I\subseteq P$ or $K\subseteq P$ for *-ideals I and K. Obviously any prime ideal is *-prime. From [4], P is a *-prime *-ideal of R if and only if for a, $b\in R$ such that $aRb\subseteq P$ and $aRb^*\subseteq P$, $a\in P$ or $b\in P$.

Definition. Let R be a subring of D.

(1) R is called *total* if, for every $x\in D^{\cdot}$, x or $x^{-1}\in R$.

(2) R is called *symmetric* if it contains x^*x^{-1} for every $x\in D^{\cdot}$.

(3) R is called a **-valuation ring* if it is total and symmetric.

From [5], if R is a symmetric subring of D, then R is *-closed, closed under conjugation and each ideal I in R is *-closed, two-sided, and has the property that $ab\in I$ implies $ba\in I$. Moreover, R contains the commutator subgroup of $D^{\cdot}$, *i.e.*, $aba^{-1}b^{-1}\in R$ for every a, $b\in D^{\cdot}$. R is a symmetric subring of D if and only if R contains $x^{-1}x^*$ for every $x\in D^{\cdot}$. If R is a *-closed total subring which is closed under conjugation then R is symmetric and so it is a *-valuation subring.

Lemma 2. Let D be a *-division ring, R is a symmetric subring of D, and $P\subset R$ is a *-prime *-ideal of R, then $R-P$ is a denominator set.

Proof. Clearly $R-P$ is *-closed and does not contain zero. By Lemma 1, $R-P$ satisfies the Ore-condition. It remains to show that $R-P$ is multiplicative. Let a, $b\in R-P$. Since $a\notin P$, $b\notin P$, there is an element $x\in R$ such that $axb\notin P$ or $axb^*\notin P$. Assume $axb\notin P$, then $abx'= ab\,(b^{-1}xb) = axb\notin P$, for $x' = b^{-1}xb\in R$. If $ab\in P$, then $abx'\in P$, a contradiction. Then $ab\in R-P$. Now, assume $axb^*\notin P$. By using similar arguments, we get $ab^*\in R-P$. So $ab\, b^{-1}b^* = ab^*\notin P$. If $ab\in P$, then $ab\, b^{-1}b^*\in P$ (where $b^{-1}b^*\in R$), which is a contradiction. Hence $ab\notin P$ and $R-P$ is a multiplicative set.

Definition

(1) Let $R\subset D$ be a symmetric subring, and $P\subset R$ be a *-prime *-ideal, then from lemma 2, $R-P$ is a denominator set. Define the *localization of R at P*, denoted R_p to be $\{rs^{-1} : r\in R,\ s\in R-P\}$. This is a *-subring of D containing R and hence also symmetric.

(2) A symmetric subring $R\subset D$ is *a Prüfer ring* if the localization at each maximal *-ideal of R is a *-valuation ring.

Since any localization of R is symmetric, then a symmetric subring $R\subset D$ is a Prüfer ring if and only if the localization at each maximal *-ideal of R is a total subring.

3. FRACTIONAL *-IDEALS

We discuss in this section the notion of fractional *-ideal relative to a *-subring of D.

Definition. Let R be a *-subring of D. A *-closed R-submodule $A\subset D$ is *a fractional *-ideal* of R if there is a non-zero element $r\in R$ such that $rA\subseteq R$ and $r^*A\subseteq R$.

Since A and R are *-closed, then we also have $Ar\subseteq R$ and $Ar^*\subseteq R$; and hence we do not need to define right and left fractional *-ideals.

Clearly, every fractional *-ideal is a fractional ideal. Also, each *-ideal of R is a fractional *-ideal of R, and the intersection of two fractional *-ideals is a fractional *-ideal. We say A is a principal fractional *-ideal if $A=Ra$, $a=a^*\neq 0$, $a\notin R$, $a^{-1}\in R$. We define finitely generated fractional *-ideals in the obvious way. For a fractional *-ideal A, let $[A:R] = \{x\in D : xA\subseteq R,\ x^*A\subseteq R\} = \{x\in D : Ax\subseteq R,\ Ax^*\subseteq R\}$.

If R is a *-subring which is closed under conjugation, then $rA\subseteq R$ is equivalent to $Ar\subseteq R$, since $ra\in R$ implies $ar = r^{-1}\,(ra)\,r\in R$. The following lemma gives some more properties of such subrings.

Lemma 3. Let R be a *-subring, then

(*i*) The sum of two fractional *-ideals is a fractional *-ideal.

(*ii*) If A is a fractional *-ideal, then $[A:R]$ is a fractional *-ideal.

Proof.

(*i*) Assume A and B are fractional *-ideals of R. Clearly, $A+B = \{a+b : a\in A,\ b\in B\}$ is a *-closed R-submodule. Let $r_1, r_2\in R$ be such that $r_1A\subseteq R$, $r_1{}^*A\subseteq R$ and $r_2B\subseteq R$, $r_2{}^*B\subseteq R$. Now, $r_1r_2A\subseteq r_1A$ because $RA\subseteq A$ as A is an R-submodule. So, $r_1r_2A\subseteq r_1A\subseteq R$. Then $(r_1r_2)(A+B) = r_1r_2A+r_1(r_2B)\subseteq R$. Similarly, $(r_1r_2)^*(A+B)\subseteq R$.

(*ii*) By definition $[A:R]$ is *-closed. Since A is a fractional *-ideal, then there is $0\neq r\in R$ such that $rA\subseteq R$, $r^*A\subseteq R$. Then also both $r[A:R]\subseteq R$, $r^*[A:R]\subseteq R$. Because, if $x\in [A:R]$, so that $xA\subseteq R$ and $x^*A\subseteq R$, then $rxA\subseteq rR\subseteq R$. Hence $r[A:R]\subseteq R$. Similarly for r^*. The proof that $[A:R]$ is an R-submodule goes through in the same way.

Lemma 4. If R is a symmetric subring, and A and B are fractional *-ideals of R, then the product $AB =:$ {finite sums of elements of the form $\Sigma a_i b_i$ or $\Sigma b_i a_i$, where $a_i\in A$, $b_i\in B$}, is a fractional *-ideal.

Proof. Clearly AB is *-closed R-submodule. Let $r_1, r_2\in R$ be such that $r_1A\subseteq R$, $r_1{}^*A\subseteq R$ and $r_2B\subseteq R$, $r_2{}^*B\subseteq R$. Then for $a\in A$, $b\in B$, we have

$$r_2r_1(ab) = r_2\,(r_1a)\,r_2{}^{-1}(r_2b) \in R, \text{ and}$$

$$r_2r_1(ba) = (r_2b)\,(b^{-1}r_1br_1{}^{-1})\,(r_1a) \in R,$$

where R contains the commutator $b^{-1}r_1br_1{}^{-1}$. Thus $r_2r_1\,(AB)\subseteq R$. Similarly, $(r_2r_1)^*\,(AB)\subseteq R$ and AB is a fractional *-ideal.

Definition. A fractional *-ideal A is *invertible* if there is some fractional *-ideal A' such that $A'A=R$.

Since A, A' and R are *-closed, then $A'A=R$ is equivalent to $AA'=R$. In the commutative case, if A is invertible and $AA'=R$, then $A'=[A:R]$ (see [2, Proposition 6.4]). For a *-ring R, we have

Lemma 5. Let R be a symmetric subring, then a fractional *-ideal A is invertible if and only if $A[A:R] = R$.

Proof. If $A[A:R] = R$, then A contains an element of R and $[A:R]$ is a fractional *-ideal by Lemma 3. Hence A is invertible. Conversely, if A is invertible, then there is a fractional *-ideal A' such that $AA'=R$. Then $A'\subseteq[A:R]$, and so $R=AA'\subseteq A[A:R]\subseteq R$ Thus $A[A:R]=R$.

Lemma 6. If A is a fractional *-ideal of a symmetric subring R of D and S is a denominator set of R, then AS^{-1} is a fractional *-ideal of RS^{-1}containing A.

Proof. First we show that AS^{-1} is closed under addition. Let $as^{-1}, bt^{-1}\in AS^{-1}$ .Now

$$as^{-1} = att^{-1}s^{-1} = at(st)^{-1}, \text{ and } bt^{-1} = b(t^{-1}st)(st)^{-1}.$$

Since $t^{-1}st\in R$, as R is closed under conjugation; then $a\,t$, $b\,(t^{-1}st)\in A$, as A is an R-submodule. So $as^{-1} + bt^{-1} = (at+b(t^{-1}st))(st)^{-1}\in AS^{-1}$. Now, assume $as^{-1}\in AS^{-1}$, $rt^{-1}\in RS^{-1}$ for $a\in A$, $r\in R$, and $s,t\in S$. So,

$$
\begin{aligned}
(as^{-1})(rt^{-1}) &= a(s^{-1}r)t^{-1} \\
&= a(r_1 s_1{}^{-1})t^{-1}, && \text{for some } r_1\in R,\ s_1\in S \text{ (from the Ore condition)} \\
&= (a\,r_1)(ts_1)^{-1}\in AS^{-1} && (\text{as } A \text{ an } R\text{-submodule}),
\end{aligned}
$$

and AS^{-1} is a right RS^{-1}-submodule. Since RS^{-1} is symmetric (as it contains the symmetric subring R), then it is closed under conjugation so that:

$$(rt^{-1})(as^{-1}) = (as^{-1})\,[(as^{-1})^{-1}\,(rt^{-1})(as^{-1})\,]\in AS^{-1},$$

and AS^{-1} is also a left RS^{-1}-submodule.

Let $r \in R$ be such that $rA \subseteq R$, then r can be considered in RS^{-1} and $r(AS^{-1}) \subseteq RS^{-1}$. Finally, to show that AS^{-1} is *-closed, assume $as^{-1} \in AS^{-1}$ for some $a \in A$ and $s \in S$. Since A and S are *-closed, then $a^* \in A \subset AS^{-1}$ and $s^{*-1} \in RS^{-1}$. So, $(as^{-1})^* = s^{*-1} a^* \in AS^{-1}$ as AS^{-1} is an RS^{-1}-submodule.

The following theorem shows that any fractional *-ideal A of a symmetric subring R of D, is not only a subset of its quotient fractional *-ideal AS^{-1} for any denominator set S; but actually it is the intersection of certain fractional *-ideals in the ring of fractions of R. Also, this theorem is a key result in proving that any Prüfer ring is the intersection of all its *-valuation overrings. Now let R be a symmetric subring of D and P a *-prime *-ideal of R. Then from Lemma 2, $S = R - P$ is a denominator set, we write A_p for the fractional *-ideal AS^{-1}.

Theorem 7. Let Λ be the set of all maximal *-ideals of a symmetric subring $R \subset D$. Let A be a fractional *-ideal of R, then $A = \bigcap_{M \in \Lambda} A_M$.

Proof. Clearly $A \subseteq \bigcap_{M \in \Lambda} A_M$. Conversely, let $x \in \bigcap_{M \in \Lambda} A_M$. Then for every $M \in \Lambda$ there is $s_M \in R - M$ and $a_M \in A_M$ with $x = s_M^{-1} a_M$. So, $s_M x = a_M \in A$. Let B be the *-ideal of R generated by s_M. Then there is a maximal *-ideal M of R with $B \cap R \subseteq M$, and so $s_M \in B \cap R \subseteq M$, a contradiction. Hence $1 = r_{M1} s_{M1} + \ldots + r_{Mn} s_{Mn}$ for a finite number of maximal *-ideals $M_i \in \Lambda$, and $r_{Mi} \in R$. Multiplying on the right by x, we get $x = r_{M1} s_{M1} x + \ldots + r_{Mn} s_{Mn} x$. Since $s_{Mi} x \in A$, for every $i = 1, 2, \ldots, n$, and A is an R-submodule, then $x \in A$ as desired.

Lemma 8. Let A be a finitely generated fractional *-ideal of a symmetric subring R of D. Let M be a maximal *-ideal in R. Then $[A_M : R_M] = [A:R]_M$.

Proof. Since $[A:R]_M$ is both right and left ring of fractions, we can write any element $x \in [A:R]_M$ as $s^{-1}y$ or ys^{-1} for $y \in [A:R]$ and $s \in R - M$. For $x = s^{-1}y$, $s \in R - M$, $yA \subseteq R$, we have $sxA_M = yA_M = yAS^{-1} \subseteq RS^{-1} = R_M$. So, $xA_M = s^{-1}sxA_M \subseteq R_M$ and hence $[A:R]_M \subseteq [A_M : R_M]$.

Now, to prove the converse, we assume $A = R[a_1, a_2, \ldots, a_n]$. Let $x \in [A_M : R_M]$, then $xA_M \subseteq R_M$ and $xa_i = s_i^{-1} r_i$, for some $r_i \in R$ and $s_i \in R - M$, $i = 1, \ldots, n$. So, $s_i x a_i \in R$ for $i = 1, \ldots, n$. From $(s_1 \ldots s_n) x a_i = s_1 \ldots s_{i-1} [s_i (s_{i+1} \ldots s_n) s_i^{-1}] s_i x a_i$, and since R is closed under conjugation, we have $(s_1 \ldots s_n) x a_i \in R$ for every $i = 1, \ldots, n$, so that $(s_1 \ldots s_n) x A \subseteq R$. Hence $(s_1 \ldots s_n) x \in [A:R]$. Thus $x = (s_1 \ldots s_n)^{-1} (s_1 \ldots s_n) x \in S^{-1} [A:R] = [A:R]_M$.

Corollary 9. Let A be a finitely generated fractional *-ideal such that A_M is invertible for each maximal *-ideal M. Then A is invertible in R.

Proof. Since A_M is invertible, then by Lemma 5,

$$A_M [A_M : R_M] = R_M.$$

So,
$$A_M [A : R]_M = R_M.$$

Now, taking the intersection over the set Λ of all maximal *-ideals, and noting that $A = \bigcap_{M \in \Lambda} A_M$, $[A:R] = \bigcap_{M \in \Lambda} [A:R]_M$, and $R = \bigcap_{M \in \Lambda} R_M$ (from Theorem 7); we get $A[A:R] = R$, and hence A is invertible.

Lemma 10. Let R be a total *-subring, then every finitely generated fractional *-ideal is principal, and hence invertible.

Proof. We first claim that the set of all fractional *-ideals is totally ordered. Suppose that M, N are two fractional *-ideals such that $M \not\subset N$ and $N \not\subset M$. Let $x \in M - N$ and $y \in N - M$. Since $(xy^{-1})y = x \notin N$ and $(yx^{-1})x = y \notin M$, we have $xy^{-1} \notin R$ and $yx^{-1} \notin R$ and so R is not total, a contradiction. Now, let $A = Ra_1 + \ldots + Ra_n$ be a non-zero finitely generated fractional *-ideal of R. By the above, $\{Ra_1, \ldots, Ra_n\}$ has a largest element, say $Ra_1 \supseteq Ra_i$ $(i = 1, \ldots, n)$. Then $A \subseteq Ra_1 \subset A$, and A is principal.

We now come to the main result of this section. This result gives equivalent conditions for a symmetric subring to be a Prüfer ring. These conditions generalize those in the commutative case.

Theorem 11. The following are equivalent for a symmetric subring R of D.

(*i*) For each *-prime *-ideal P, R_p is a *-valuation ring.

(*ii*) For each maximal *-ideal M, R_M is a *-valuation ring.

(*iii*) Each finitely generated fractional *-ideal of R is invertible.

Proof.

(*i*) $\Rightarrow$ (*ii*) is clear.

(*ii*) $\Rightarrow$ (*iii*) Let A be a finitely generated fractional *-ideal. By (*ii*), R_M is total for every maximal *-ideal M. Hence, by Lemma 10, the finitely generated fractional *-ideals A_M in R_M are invertible, for every maximal *-ideal M. Thus A is invertible by Corollary 9.

(*iii*) $\Rightarrow$ (*ii*) Let P be a *-prime *-ideal in R. As in the commutative case [2], using (*iii*) one can show that the set of principal ideals of R_p is well ordered by inclusion. Now, let $x = \frac{a}{b} \in D$, where a, b are non-zero elements in R. If $x \notin R_p$, then $Rp.a \not\subset Rp.b$. So, by the above, $Rp.b \subseteq Rp.a$. Hence $x^{-1} = \frac{b}{a} \in Rp$ and Rp is a total subring. Since Rp is also symmetric, then Rp is a *-valuation subring.

Lemma 12. Suppose $B \subset D$ is a Prüfer ring and A is a total *-subring containing B. Then $A = Bp$ for some *-prime *-ideal P of B.

Proof. Since A contains the symmetric subring B, then A is symmetric, and so it is a *-valuation subring. Let I be the maximal *-ideal of non-units of A, and let $P = I \cap B$. Then, clearly P is a *-prime *-ideal in B. But B is a Prüfer ring, so Bp is a *-valuation ring, and by construction $Bp \subseteq A$. We show now that $A \subseteq Bp$. Assume $a \in A$ and $a \notin Bp$, then $a^{-1} = rs^{-1} \in P\ Bp$ for some $r \in P$ and $s \in B-P$. Now, $r \in P \subseteq I$ implies that $a^{-1} \in I$, which contradicts $a \in A$. Hence $A \subseteq Bp$, as required.

Corollary 13. Suppose $B \subset D$ is a Prüfer ring. Then B is the intersection of all its *-valuation overrings.

Proof. Let C denote the intersection of all *-valuation overrings of B. Clearly $B \subseteq C$, and we will show that $C \subseteq B$. By Lemma 12, for a *-valuation subring A containing B, we have $A = Bp$ for some *-prime *-ideal P. Since B is a Prüfer ring, then Bp is a *-valuation ring. Hence $C = \bigcap Bp$, where the intersection over all *-prime *-ideal of B. By Theorem 7, $B = \bigcap B_M$, where the intersection over all maximal *-ideals of B. Then $C = \bigcap Bp \subset \bigcap B_M = B$, as desired.

4. ORDERINGS AND PRÜFER RINGS

In this section, it will be shown that the bounded subring associated to a given preordering of D is a Prüfer ring in D. First, we give the basic facts about orderings and valuations of a division ring with involution D. Let S denote the subgroup of $D^{\cdot}$ generated by the symmetric elements ($s = s^*$, $0 \neq s \in D$). Let $S(D)$ denote the subgroup generated by the norms xx^*, $x \in D^{\cdot}$, and the commutators $xyx^{-1}y^{-1}$ and $yxy^{-1}x^{-1}$ for $x \in D^{\cdot}$ and $y \in S$. One checks easily that the subgroup $S(D)$ is closed under * and normal in $D^{\cdot}$. Let $\Sigma(D)$ denote the set of sums of elements from $S(D)$. The set $\Sigma(D)$ is a *-closed normal subgroup when it does not contain 0.

By a *preordering* of a division ring D with involution, we mean a *-closed normal subgroup T of $D^{\cdot}$ that is closed under sums and contains every xx^*, $x \neq 0$. By an *ordering* of D, we mean a preordering T such that for each non-zero symmetric element s, T contains either s or $-s$. We note that $\Sigma(D) \subset T$ for every ordering T of D, and D possesses an ordering if and only if $0 \notin \Sigma(D)$ (see [6]).

A valuation can be associated with an ordering, the following construction of such a valuation can be checked as in [7]. Call x bounded if $xx^* < r$ for some $r \in Q^+$ (the positive rationals), and call x infinitesimal if $xx^* < r$ for every $r \in Q^+$. The set V of bounded elements is a *-valuation ring [7], which we call it the bounded subring. The set J of infinitesimal elements, which equals the set of noninvertible elements in V, is a *-closed two-sided ideal in V that contains every proper ideal. By standard construction (see[5]), any *-valuation subring V gives rise to a ****-valuation*** on D (a valuation ω onto a totally ordered abelian group with the additional property that $\omega(x^*) = \omega(x)$ for all non-zero $x \in D$). We call the valuation associated to the bounded subring, the ***order valuation***. In fact, for every ordered division ring D with involution, the order valuation ω is compatible with the ordering, in the sense that $0 < x \leq y \Rightarrow \omega(x) \geq \omega(y)$ (see [7]).

Exactly as for an ordering, one can define the bounded elements and the infinitesimals at a given preordering of the *-division ring. For a preordering T_0, let V_0, J_0 denote the sets of all bounded elements and infinitesimals respectively. One can adapt the proof of Theorem 17 in [8], to get:

Theorem 14. Let $\{T_i\}_{i\in I}$ be the family of orderings containing a given preordering T_0 of the *-division ring D. Let V_i, J_i be the subring of bounded elements and the ideal of infinitesimals, respectively, attached to the ordering T_i. Then $V_0 = \bigcap V_i, J_0 = \bigcap J_i$.

Corollary 15. For any preordering T_0 of D, the bounded subring V_0 is a *-valuation subring.

Proposition 16. For any preordering T_0 of D, the bounded subring V_0 is a Prüfer ring.

Proof. Let M be a maximal *-ideal of V_0. Let $0 \neq s^* = s \in D$, and $K = Q(s)$, a *-closed commutative subfield of D. Then $T' = T_0 \cap K$ is a preordering of K, and its bounded subring is $V' = V_0 \cap K$. Let $M' = M \cap K$, then M' is a *-prime *-ideal of V'. Now V' is a Prüfer subring of the commutative field K, and so by [2, Theorem 6.6], $(V')_{M'}$ is a valuation subring. If $x \notin (V_0)_M$, then $x \in (V')_{M'}$, so that $x^{-1} \in (V')_{M'} \subseteq (V_0)_M$. Hence, $(V_0)_M$ is a *-valuation subring.

Corollary 17. V_0 is the intersection of all its *-valuation overrings. In fact, V_0 is the intersection of all *-valuation subrings which are compatible with T_0.

Corollary 17, is an immediate consequence of Proposition 16 and Corollary 13.

If D is an ordered division ring with involution, *i.e.*, $0 \notin \Sigma(D)$, then $\Sigma(D)$ is a preordering of D. Then, the bounded subring V_0 associated to $\Sigma(D)$ is the intersection of all *-valuation rings compatible with $\Sigma(D)$, which is equal to the intersection of all real *-valuation rings (see [9]). Thus, we have

Corollary 18. If D is an ordered division ring with involution, then the intersection of all real *-valuation rings is a Prüfer ring.

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