RADICAL SEMIGROUPS WHERE $S^2=S$

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الخلاصة :

ندرس في هذا البحث نصف الزمر الجزئية تحت الجذر البسيط. كذلك نفحص خواص نصف الزمر الجذرية المغلقة. أما التركيز الأساسي في هذا البحث فإنه على بناء نصف الزمر الجذرية الجامدة ذات التطابق اليميني الأقصى .

ABSTRACT

This paper looks at radical semigroups under the simple radical. Properties under which the class of radical semigroups is closed are examined. The main focus of the paper is the structure of globally idempotent radical semigroups with maximal right congruences.

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RADICAL SEMIGROUPS WHERE $S^2 = S$

In this work we will look at one of many semigroup analogs to the Jacobson radical and use it to help understand the structure of certain semigroups [1, 2]. We will consider the semigroup radical, $\overline{rad}(S)$, introduced by Hoehnke [1] and referred to as the simple radical by Roiz and Schein [3]. We will use the representation of $\overline{rad}(S)$ as the intersection of all maximal modular right congruences. In the second section we will point out some closure properties of the class of all radical semigroups. The focus of this paper is the third section, where we will look at the structure of radical semigroups with maximal right congruences in which $S^2 = S$. In the language of Schein [4], this would be the globally idempotent radical semigroups with maximal right congruences. This then classifies the structure of all finite globally idempotent radical semigroups. Prior work by Oehmke [2] classified radical monoids and Tishchenko [5] and Hoehnke [1] have classified all radical semigroups with zero. In the final section, we take a brief look at non-globally idempotent radical semigroups.

1 ELEMENTARY DEFINITIONS AND RESULTS

Throughout this paper S is a semigroup and S^1 is the semigroup with an identity adjoined. All concepts and notation of the theory of semigroups not defined here are as in [6]. If ρ is a [right] congruence on S with $a, b \in S$, we will use the notation $(a, b) \in \rho$ to denote that a and b are related under ρ. $[a]_\rho$ will denote the equivalence class of ρ containing a. The partial order for equivalence relations can be used as a partial order for [right] congruences as well. The meet and join of two [right] congruences are defined accordingly. We will denote the trivial congruence by ι and the universal congruence by ν.

If I is a [right] ideal of S, we will denote the Rees [right] congruence associated with I by $\mathcal{R}_I$; this is defined by $(a, b) \in \mathcal{R}_I \Leftrightarrow a = b$ or $a, b \in I$.

If T is an equivalence class of a right congruence and $a \in S$, then define the set $a^{-1}T = \{s \in S^1 | \ as \in T\}$. The following is then the largest right congruence having T as an equivalence class: $(a, b) \in \mathcal{T}_T \Leftrightarrow a^{-1}T = b^{-1}T$.

2 CLOSURE PROPERTIES OF RADICAL SEMIGROUPS

The intersection of all maximal modular right congruences, $\overline{rad}(S)$, is a two sided congruence and following Hoehnke [1], we will take this as the definition of the radical of S. A semigroup S is said to be *semisimple* if $\overline{rad}(S) = \iota$. S is said to be *radical* if $\overline{rad}(S) = \nu$, in other words if S has no maximal modular right congruences. As examples, the reader can verify that a left zero semigroup is radical and the infinite cyclic semigroup on a single generator is semisimple.

We will primarily be looking at radical semigroups in this paper. Let $\mathcal{A}$ be the class of all radical semigroups. We begin this study by looking at some closure properties of $\mathcal{A}$. The first is true of radicals in general. The second shows that the class of radical semigroups is closed under taking direct products and the proof can be found in [7].

Proposition 2.1 *If S is a radical semigroup and φ is a semigroup homomorphism from S onto $\varphi(S)$, then $\varphi(S)$ is a radical semigroup. In other words, $\mathcal{A}$ is closed under homomorphisms.*

Proposition 2.2 *If S_α for $\alpha \in A$, where A is an index set, is a family of radical semigroups then $\prod_\alpha S_\alpha$ is a radical semigroup. In other words, $\mathcal{A}$ is closed under direct products.*

We now turn our attention to the ideals of semigroups. Tishchenko [8] has shown, having given a counter example, that the radical of an ideal is not the restriction of the radical of the semigroup to the ideal. However, Slover [9] showed that if the semigroup has a left zero, then $\overline{rad}(S)|_I = \overline{rad}(I)$. We show here that in the case of radical semigroups we also get the result. In other word, $\mathcal{A}$ is closed under ideals.

As Oehmke showed in [10], if ρ is a right congruence on S and $a \in S$ then the *conjugate*, ρa, of ρ by a is a right congruence on S where ρa is defined as follows;

$$(x, y) \in \rho a \;\Leftrightarrow (ax, ay) \in \rho.$$

In addition to giving us new right congruences on a semigroup, the idea of a conjugate can also be used to extend a maximal right congruence τ on an ideal T of S to a right congruence on the entire semigroup. By conjugating a maximal modular right congruence τ on T by its left identity, we are then able to create a maximal modular right congruence on S whose restriction to T is exactly τ. This then gives us our desired result.

Lemma 2.1 *Let τ be a maximal right congruence on an ideal T of S and let $e \in T$. Then $\tau e = \{(x, y) \in S \times S \mid (ex, ey) \in \tau\}$ is a right congruence on S.*

Proof: τe is clearly an equivalence relation on S, thus we only need to verify that τe is right compatible. To this end, let $s \in S$ and $(x, y) \in \tau e$. We must show that $(xs, ys) \in \tau e$ or that $(exs, eys) \in \tau$. Note that since s may not be in T, this is not trivial. First notice that $exs, eys \in T$ and $(ex, ey) \in \tau$. Since T is a two-sided ideal, for any $u \in T$ we have $(exsu, eysu) \in \tau$. Now assume that there is an $\bar{s} \in S$ such that $(ex\bar{s}, ey\bar{s}) \notin \tau$. Fix $v \in T$ and let $V = [ex\bar{s}v]_\tau$. Note that $(ex\bar{s}v, ey\bar{s}v) \in \tau$. Now construct $\mathcal{T}_V$. It is straightforward to see that $(ex\bar{s}, ey\bar{s}) \in \mathcal{T}_V$. From the definition of $\mathcal{T}_V$ and the fact that V is an equivalence class of τ we see that $\tau \leq \mathcal{T}_V$, but since τ is maximal it must be that $\tau = \mathcal{T}_V$ or $\mathcal{T}_V = \nu$. The first of these implies that $(ex\bar{s}, ey\bar{s}) \in \tau$ which is a contradiction and the second implies that $V = T$ which is also a contradiction. So it must be the case that $\forall s \in S \;\; (exs, eys) \in \tau$ or that τe is a right congruence on S. □

We now state and prove our desired theorem.

Theorem 2.1 *If S is a radical semigroup with a two-sided ideal T, then when treated as a semigroup on its own, T is a radical semigroup.*

Proof: Assume that T is not a radical semigroup. Then there exists a maximal modular right congruence τ on T. Let $e \in T$ be the left identity for τ. By Lemma 2.1 τe is a right congruence on S. It is straightforward to see that τe is modular with left identity e and that $\tau e|_T = \tau$. Furthermore, note that since for all $s \in S$ we have $(es, s) \in \tau e$, T intersects every τe-class. To see that τe is maximal assume that σ is a right congruence on S such that $\tau e < \sigma$. Then $\tau e|_T < \sigma|_T$. But $\tau e|_T = \tau$ and τ is a maximal right congruence on T, so $\sigma|_T = \nu$ and hence $\sigma = \nu$ on S. Thus τe is a maximal modular right congruence on S and hence S is not a radical semigroup. □

3 THE STRUCTURE OF RADICAL SEMIGROUPS WITH $S^2 = S$

For this section, we assume that S is globally idempotent. The simplest way to achieve this is to have an identity. The structure of radical monoids was determined by Oehmke in [2], and the theorem is included here for completeness.

Theorem 3.1 (Oehmke) *If S is a radical monoid, then S is a group with no maximal subgroups.*

From this point on, we will assume that in general S does not have an identity element. Let ρ be a right congruence on a semigroup S. By a *generator* for ρ we mean an element $m \in S$ such that for all $y \in S$ there exists $z \in S$ such that $(mz, y) \in \rho$. Notice that if ρ is modular, then any left identity is a generator for ρ. Conjugation of a right congruence by one of its generators will produce a modular right congruence.

Lemma 3.1 *Let ρ be a right congruence with generator m on a semigroup S. Then ρm is a modular right congruence.*

Proof: Let $e \in S$ be such that $(me, m) \in \rho$. Thus, for any $s \in S$ we have $(mes, ms) \in \rho$ and hence $(es, s) \in \rho m$. Thus e is a left identity for ρm. □

We now show how to find a generator for a maximal right congruence.

Lemma 3.2 *Let S be a semigroup such that $S^2 = S$. Let ρ be a maximal right congruence on S then ρ has a class that is not a right ideal if and only if ρ has a generator.*

Proof: Let x be in a class, $[x]_\rho$, that is not a right ideal. Then, since $S^2 = S$, find $m, y \in S$ such that $x = my$. Now consider the right ideal $xS \cup \{x\} = xS^1$. This right ideal intersects $[x]_\rho$ but can not be contained in it, otherwise $[x]_\rho$ is a right ideal [2, Lemma 2]. But then xS^1 must intersect every class of ρ [2, Lemma 1]. Now notice that $xS^1 \subseteq mS$ and hence mS intersects every ρ-class as well. So, for any $u \in S$ we can choose $z \in S$ such that $mz \in [u]_\rho$. Thus m is a generator for ρ. To see the converse, note that any element of a right ideal class can not be a generator. □

The fact that every element can be written as a product is a necessary hypothesis that was neglected by the author in a similar result, Lemma 4, in [2]. Consider the three element semigroup $S = \{a, b, z\}$ where any product of elements is z. Then the partition $\{a, b\}$ and $\{z\}$ forms a maximal right congruence without a generator that has a class, $\{a, b\}$, that is not a right ideal.

We now know when a maximal right congruence has a generator and we know that conjugating by a generator gives a modular right congruence. The next question that begs to be asked is what happens when a maximal right congruence is conjugated by a generator. We see that the conjugate is itself a maximal right congruence.

Lemma 3.3 *Let ρ be a maximal right congruence on S with generator m. Then ρm is a maximal right congruence on S.*

Proof: Assume that ρm is not a maximal right congruence. Then there exists a right congruence σ such that $\rho m < \sigma < \nu$. We first show that the following *inverse conjugate*, $m\sigma$, is a right congruence on S:

$$(a, b) \in m\sigma \Leftrightarrow \exists\, s, t \in S \ni (s, t) \in \sigma \text{ and } (ms, a), (mt, b) \in \rho.$$

We must first show that $m\sigma$ is an equivalence relation. Let $a \in S$, then since m is a generator for ρ we can find $s \in S$ such that $(ms, a) \in \rho$. Since σ is a right congruence $(s, s) \in \sigma$, so $(a, a) \in m\sigma$ and $m\sigma$ is reflexive. To see that it is symmetric, simply switch the order in which s and t are chosen. To see that $m\sigma$ is transitive assume that (a, b), $(b, c) \in m\sigma$. Thus there exists $s, t, u, v \in S$ such that (s, t), $(u, v) \in \sigma$ and (ms, a), (mt, b), (mu, b), $(mv, c) \in \rho$. We choose s and v as our elements of S. Clearly (ms, a), $(mv, c) \in \rho$, so all that needs to be shown is that $(s, v) \in \sigma$. Notice that (mt, b), $(mu, b) \in \rho$ so $(mt, mu) \in \rho$ and $(t, u) \in \rho m$. But this implies that $(t, u) \in \sigma$ and hence that $(s, v) \in \sigma$. Thus $m\sigma$ is an equivalence relation. Showing that $m\sigma$ is right compatible is straightforward.

We now show that if our original assumption, that $\rho m < \sigma < \nu$, is true, then $\rho < m\sigma < \nu$, which contradicts the maximality of ρ and thus ρm must be maximal.

To see that $m\sigma$ is proper, notice that if not, then for all $u, v \in S$ we have $(mu, mv) \in m\sigma$. This implies that there are $s, t \in S$ such that $(s,t) \in \sigma$ and (mu, ms), $(mv, mt) \in \rho$. So (u,s), $(v,t) \in \rho m < \sigma$. Now by transitivity of σ, we get $(u,v) \in \sigma$. Hence $\sigma = \nu$, which is a contradiction, so $m\sigma$ is proper.

To see that $\rho < m\sigma$, note that if $(a,b) \in \rho$ then there exists $s \in S$ such that (ms, a), $(ms, b) \in \rho$ and hence $(a,b) \in m\sigma$, thus $\rho \leq m\sigma$. We now consider the case that $\rho = m\sigma$. Let $(s,t) \in \sigma$. We show that $(s,t) \in \rho m$. Clearly (ms, ms), $(mt, mt) \in \rho$ and thus there exists $s, t \in S$ such that $(s,t) \in \sigma$ and (ms, ms), $(mt, mt) \in \rho$ and so by definition $(ms, mt) \in m\sigma$. But if $m\sigma = \rho$ then $(ms, mt) \in \rho$ and $(s,t) \in \rho m$. So if $\rho = m\sigma$ then $\rho m = \sigma$ which is a contradiction. Hence $\rho < m\sigma < \nu$. But as mentioned, this contradicts the maximality of ρ and thus ρm is maximal. □

We are now ready to discuss the structure of radical semigroups with maximal right congruences such that $S^2 = S$.

Theorem 3.2 *Let S be a radical semigroup with a maximal right congruence such that $S^2 = S$. Then $S = I \cup J$ where I and J are disjoint right ideals.*

Proof: Let ρ be a maximal right congruence on S. Then ρ has at least two classes. If ρ has any classes that are not right ideals then by the above lemmas we can construct a maximal modular right congruence contrary to the radicalness of S. So all of the ρ classes must be right ideals and by [2, Lemma 3] there must be exactly two right ideal classes I and J. □

We point out here that this decomposition need not be unique, consider a left zero semigroup. There are, however, semigroups in which this decomposition can be done in only one way. It is also worth pointing out that this result is not a biconditional. There are in fact semisimple semigroups which can be decomposed in this way.

The next question to consider is whether or not these semigroups can have two-sided ideals. We will see that radical semigroups with maximal right congruences such that $S^2 = S$ have no zeros and the factor semigroups of those that have ideals can have no maximal nor modular right congruences. We begin by stating a result of Hoehnke [1].

Theorem 3.3 (Hoehnke) *If S is a semigroup with a left zero, then every proper modular right congruence is contained in a maximal, and necessarily modular, right congruence.*

Theorem 3.4 *If S is a radical semigroup where $S^2 = S$ and S has a maximal right congruence, then S does not have a zero.*

Proof: Decompose S into its two disjoint right ideals I and J. If one contains a two-sided zero then the other is not a right ideal. □

Theorem 3.5 *If S is a radical semigroup where $S^2 = S$, S has a maximal right congruence and S has a proper two-sided ideal T, then S/T has no maximal nor modular right congruences.*

Proof: First note that since S/T is a homomorphic image of S it is radical by Proposition 2.1 and clearly every element in S/T can be written as a product. Since S/T has a zero, it has a left zero and hence by Hoehnke's result, S/T can not have a modular right congruence and remain radical. If S/T has a maximal right congruence the above theorem is contradicted. Thus S/T has no maximal nor modular right congruences. □

We point out here that since S/T has no maximal right congruences it can not have a maximal ideal and thus must be infinite. Furthermore, since $S = I \cup J$, the ideal T and all ideals containing it must intersect both I and J.

Since finite semigroups have maximal right congruences we get the following corollary.

Corollary 3.1 *A finite radical semigroup such that $S^2 = S$ is simple.*

So finite globally idempotent radical semigroups are simple semigroups that can be decomposed into a disjoint union of two right ideals.

4 WHEN $S^2 \neq S$

We now give a brief discussion to the case in which $S^2 \neq S$ and a few conjectures are made as to the structure of radical semigroups of this type. We begin by noting that the free semigroup on a set of elements is not radical, in fact it is semisimple.

Proposition 4.1 *The free semigroup, S, on a set of elements A is semisimple.*

Proof: Fix a prime p. Create the following equivalence relation ρ_p,

$$(x, y) \in \rho_p \Leftrightarrow |x| \equiv |y| \bmod p$$

where $|x|$ is length of the word x in terms of the number of elements of A. Since there are no relations on the generators this is well defined and is clearly right compatible. It is also modular, where the left identities are those words whose length is a multiple of p. To see that it is also maximal, let $\sigma > \rho_p$. Then there are elements $x, y \in S$ such that $(x, y) \in \sigma$ but $(x, y) \notin \rho_p$. Since $\sigma > \rho_p$ we can assume with out loss of generality that $|x| < |y| \leq p$. Let $n = |y| - |x|$. Then by multiplying on the right by an element in A it can be shown that if $|s| \equiv |t| \bmod n$ then $(s, t) \in \sigma$. Since p is prime and $n < p$ we see that $(n, p) = 1$ and hence for any $s, t \in S$, where $|s| > |t|$, there exists positive integers k and m such that $kn - mp = |s| - |t|$ or $kn + |t| = mp + |s|$. Now find $z \in S$ such that $|z| = kn + |t| = mp + |s|$. Then $(s, z), (z, t) \in \sigma$ and thus $(s, t) \in \sigma$. So $\sigma = \nu$, and ρ_p is maximal. It is now clear that $\overline{rad}(S) = \cap\{\rho \mid \rho \text{ is maximal and modular }\} \leq \cap\{\rho_p \mid p \text{ is prime }\} = \iota$. □

Let S be a semigroup such that $S^2 \neq S$. Decompose S in the following way. Let $A_1 = \{x \in S \mid \forall s, t \in S \; x \neq st\}$. These are the elements of S that can not be written as a product. Now create the ideal $I_1 = S - A_1$. Let $A_2 = \{x \in I_1 \mid \forall s, t \in I_1 \; x \neq st\}$ and $I_2 = I_1 - A_2$. Continue to recursively define A_n and I_n. Note that each of the I_n are ideals of S. It can be shown that any element in A_i can be written as a product of elements in A_1. From the above result, if S is a radical semigroup then there must be some relations on the generators A_1. With the additional assumption that the semigroup has the descending chain condition on ideals, we can get a clearer understanding of the structure of these radical semigroups.

Proposition 4.2 *Let S be a radical semigroup with the descending chain condition on ideals such that $S^2 \neq S$. Then there exists an ideal T of S such that $T^2 = T$ and S/T is a nilpotent semigroup.*

Proof: As in the above paragraph, create the chain of ideals $I_1, I_2, \ldots$. Due to the chain condition this chain must terminate. Let the smallest ideal in the chain be $T = I_n$. Then by its construction $T^2 = T$, else another ideal in the chain can be created that is properly contained in T. By Theorem 2.1, T is radical and, clearly, $S/T = S/I_n$ is nilpotent. □

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