FITTING IDEALS AND CYCLIC DECOMPOSITION OF FINITELY GENERATED MODULES

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الخلاصـــة :

لتكن R حلقة تقييمية و M معياراً ذا طراز منته على R بحيث تكون كل مثالياته المتلائمة مثاليات رئيسة ؛ وهذا يتطلب : $M \cong \oplus_{i=1}^{i=n} R/(a_i)$ ؛

حيث n هو العدد الدنيوي لمولدات M و a_i تقسم a_{i+1}. وإذا كانت كل مثاليات M المتلائمة غير منعدمة فإن ؛ $M \cong \oplus_{i=1}^{i=n} R/(F_k(M) : F_{k+1}(M))$ ؛

حيث $F_k(M)$ هي مثالية متلائمة من الصف k. وقد خلصنا إلى أن مثاليات التلائم تميز فصيلة M التقابلية وتمكننا من معرفة ما إذا كان M حرا أو لا.

ABSTRACT

Let R be a valuation ring and let M be a finitely generated R-module whose Fitting ideals are principal ideals. Then $M \cong \oplus_{i=1}^{i=n} R/(a_i)$ where n is the minimal number of generators of M, the a_i are in R, and a_i divides a_{i+1}. If furthermore the Fitting ideals of M are nonzero, then $M \cong \oplus_{i=1}^{i=n} R/(F_k(M) : F_{k+1}(M))$ where $F_k(M)$ is the k^{th} Fitting ideal of M. It is also shown that the Fitting ideals characterize the isomorphism class of M, and allow us to decide whether M is finite free or not.

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1 INTRODUCTION

Let R be a commutative ring with unit, and M a finitely generated R-module. In this paper we are concerned with some conditions which force M to be direct sum of cyclically presented modules (*i.e.*, modules of the form $R/(a)$, a in R). For our purpose, we use the Fitting ideals which embody a great deal of information about M and serve as complete and independent set of invariants of M.

This paper is organized in the following way: In Section 2, we record some known properties about the Fitting ideals of a finitely generated R-module. We also cite Proposition 1 and Proposition 2 which are crucial tools in this work, and which concern the Fitting ideals of direct sum of finitely generated modules.

We begin Section 3 by proving Proposition 4 in which we give a condition under which a finitely generated module have a cyclically presented direct summand. This proposition is a basic step in the proof of our main result, Theorem 1, in which we give a condition under which any finitely generated module over a valuation ring is direct sum of cyclically presented modules. As first corollary of this theorem we obtain a result of R.B. Warfield (see [1, Theorem 1]). We also show Corollary 2 related to the investigation of the extent to which a finitely generated module deviate from being free (the problem which motivated H. Fitting in introducing these invariants). Our last corollary expresses a very interesting feature of the Fitting ideals, namely their characterization of isomorphism classes for some modules.

2 PRELIMINARIES AND NOTATION

Throughout this paper R will denote a commutative ring and M a finitely generated R-module. Let $\underline{x} = (x_1, ..., x_n)$ be a set of generators of M. A relation of M is a vector $(a_1, \ldots, a_n)$ in R^n such that $\sum_{i=1}^{i=n} a_i x_i = 0$. For a positive integer $k = 0, ..., n-1$, the k^{th} Fitting ideal of M is defined to be the ideal $F_k(M)$ generated by the determinants of all $(n-k) \times (n-k)$-submatrices of the matrix

$$K(M) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{i1} & \cdots & a_{in} \\ & \vdots & \end{pmatrix}$$

where the vectors $(a_{i1}, \ldots, a_{in})$ are the relations of M. If $k \geq n$, we define $F_k(M) = R$. These ideals are invariants for M independently of the choice of $\underline{x}$ (see [2] or [3]).

Remark 1. *The matrix $K(M)$ is generally infinite, but when M is a finitely presented module this matrix can be assumed to be finite. In this case the Fitting ideals of M are finitely generated ideals.*

The Fitting ideals of a finitely generated module M form an ascending sequence of invariant ideals for M, and satisfy the following properties (see [2], [3], or [4]):

(1.1) If n is the size of a generating set for M then

$$(Ann_R(M))^n \subseteq F_0(M) \subseteq Ann_R(M).$$

In particular, $F_0(\sum_{i=1}^{i=n} R/I_i) = I_1 ... I_n$ where the I_i are ideals from R.

(1.2) If S is a multiplicatively closed subset of R then

$$F_k(S^{-1}.M) = S^{-1}.F_k(M) \quad \text{for any positive integer } k.$$

(1.3) Let f : $R \longrightarrow R'$ be a homomorphism of rings. Then $F_k(M \otimes_R R') = F_k(M).R'$ for any positive integer k.

(1.4) Let M_1 and M_2 be two finitely generated R-modules . Then

$$F_k(M_1 \oplus M_2) = \sum_{p+q=k} F_p(M_1)F_q(M_2) \quad \text{for any positive integer } k.$$

We shall need the following propositions whose proofs are in [5, Proposition 1.11 and Proposition 1.12] (see also [6]).

Proposition 1. *Let $M = M_1 \oplus M_2$ be a direct sum of two finitely generated R-modules. Then*

(i) for any positive integer k, $F_k(M_1) \subseteq F_{n_2+k}(M)$ and $F_k(M_2) \subseteq F_{n_1+k}(M)$ where n_1 and n_2 are sizes of sets of generators of M_1 and M_2 respectively.

(ii) $F_k(M_i) \subseteq F_{k+1}(M_i).F_{n-1}(M)$, where $k \leq n_i - 1$, $i = 1, 2$ and $n = n_1 + n_2$.

Proposition 2. *Let R be a commutative ring. If $I_1, ..., I_n$ are ideals from R such that $I_i \subseteq I_{i+1}$, then*

$$F_k(\oplus_{i=1}^{i=n} R/I_i) = I_{k+1} ... I_n, \quad \text{for } k = 0, ..., n-1.$$

In particular, $F_k(\sum_{i=1}^{i=n} R/(a_i)) = (a_1 ... a_{n-k})$ where the a_i are in R such that a_i divides a_{i+1} for i=1,...,n-1.

3 DECOMPOSITION OF FINITELY GENERATED MODULES

In this section we use the Fitting ideals to give some conditions under which a module will be direct sum of cyclically presented modules (*i.e.*, direct sum of modules of the form $R/(a)$, a in R). We will denote by $\mu(M)$ the minimal number of generators of a finitely generated R-module M. The sequence of Fitting ideals of M allows us to define two other numerical invariants for M namely, $\nu(M) := min\{k \mid F_k(M) = R\}$ and $\omega(M) := min\{k \mid F_k(M) \neq 0\}$. We have the following proposition:

Proposition 3. *Suppose R is a local ring. Let M be a finitely generated R-module. Then $\mu(M) = \nu(M)$.*

Proof. Let $r = \nu(M)$, and $\mu(M) = n$. Since $F_n(M) = R$, we have $r \leq n$. We must show that $r \geq n$. To do this let $\boldsymbol{m}$ be the maximal ideal of R. Then $F_r(M/\boldsymbol{m}M) = F_r(M \otimes R/\boldsymbol{m}) = F_r(M).R/\boldsymbol{m}$ (by (1.3)). Hence $F_r(M/\boldsymbol{m}M) = R/\boldsymbol{m}$ ($F_r(M) = R$), and then $r \geq dim_{R/\boldsymbol{m}}(M/\boldsymbol{m}M) = \mu(M) = n$ ($dim_{R/\boldsymbol{m}}(M/\boldsymbol{m}M)$ is the dimension of the $R/\boldsymbol{m}$-vector space $M/\boldsymbol{m}M$). The result is then proved. □

Proposition 4. *Let R be a commutative local ring. Let M be a finitely generated R-module, and $\mu(M) = n$. Suppose that $F_{n-1}(M)$ is a principal ideal. Then $R/F_{n-1}(M)$ is a direct summand of M.*

Proof. Let $K(M)$ be the matrix of relations of M (see in the introduction). Let $F_{n-1}(M) = (\alpha)$. Since $F_{n-1}(M)$ is generated by the entries a_{ij} of $K(M)$, and since any generating set for a module over a local ring contains a minimal generating set, we may assume that $\alpha = a_{11}$. Let $(\varepsilon_1, ..., \varepsilon_n)$ be a basis of R^n, and K the submodule of R^n generated by the vector rows $\lambda_1, \lambda_2...$ of the matrix $A = K(M)$. Setting $a_{ij} = \alpha a'_{ij}$, with $a'_{11} = 1$, $\varepsilon'_1 = \varepsilon_1 + \sum_{j=2}^{j=n} a'_{j1}\varepsilon_j$. Then $(\varepsilon'_1, \varepsilon_2, ..., \varepsilon_n)$ is a basis of R^n, and $\lambda_1 = \alpha\varepsilon'_1$. Let K' be the submodule of R^n generated by $\varepsilon_2, ..., \varepsilon_n$, and $K'' = K \cap K'$. It is easy to see that $K = (\lambda_1) \oplus K''$, So that $M \cong R^n/K \cong R/(\alpha) \oplus K'/K''$. □

In what follows, we will concern ourselves with finitely generated modules over a valuation ring. We recall that a valuation ring is a ring whose ideals are totally ordered by inclusion.

Proposition 5. *Suppose R is a valuation ring. Let M be a finitely generated R-module whose Fitting ideals are principal ideals, and $n = \mu(M)$. If N is direct summand of M such that $M = N \oplus R/F_{n-1}(M)$, then the Fitting ideals of N are also principal ideals.*

Proof. Let M be a finitely generated module over a valuation ring R. Suppose that $F_k(M)$ is a principal ideal for any positive integer k, and $M = N \oplus R/F_{n-1}(M)$ where $n = \mu(M)$. Then by (1.4) $F_k(M) = F_k(N)F_{n-1}(M) + F_{k-1}(N)$ for any positive integer k $(F_i(R/F_{n-1}(M)) = R$ for $i \geq 1)$. So, Proposition 1 (ii) implies $F_k(M) = F_k(N)F_{n-1}(M)$ $(*)$. In the following we take $F_k(M) = \alpha_{k+1}$ for $k = 0, ..., n-1$ (α_{k+1} could be zero). If $F_{n-1}(M) = 0$, then $F_k(N) = 0$ for $k = 0, ..., n-2$ (see Proposition 1 (i)). Thereby we suppose that $F_{n-1}(M) \neq 0$. So, $(*)$ implies that $(\alpha_{k+1}) = F_k(N)(\alpha_n)$. Since $F_k(M) \subseteq F_{n-1}(M)$, there exists β_k in R such that $\alpha_{k+1} = \beta_k\alpha_n$. Hence $\alpha_n(\beta_k) = \alpha_n F_k(N)$ for $k = 0, ..., n-2$. Then, by [7, Corollary 4.2], $F_k(N) = (\beta_k)$ for $k = 0, ..., n-2$ $(\alpha_n \neq 0)$. On the other hand, $n-1$ is the minimal number of generators of N (R is a local ring). This completes the proof. □

Remark 2. *Through the previous proof, one sees that it is not sufficient to assume that the ring is local. Indeed, the cancellation of nonzero principal ideals does not hold for local rings.*

Theorem 1. *Suppose R is a valuation ring. Let M be a finitely generated R-module whose Fitting ideals are principal ideals. Let $n = \mu(M)$. Then there exist elements $a_1, ..., a_n$ in R such that $M \cong \oplus_{i=1}^{i=n} R/(a_i)$, and a_i divides a_{i+1} for i=1,...,n-1.*

Proof. Let $n = \mu(M)$. We use induction on n. When $n = 1$, we have $M = R/F_0(M)$ $(Ann_R(M) = F_0(M)$, by (1.1)). So the assertion holds for $n = 1$. Suppose $n > 1$. Then Proposition 4 implies $M \cong R/F_{n-1}(M) \oplus N$, where N is a finitely generated R-module and $\mu(N) = n-1$ (R is a local ring). Hence, by Proposition 5 the Fitting ideals of N are principal ideals. So, there exist $b_1, ..., b_{n-1}$ in R such that $N = \oplus_{i=1}^{i=n-1} R/(b_i)$ (by the induction hypothesis). We take $(a_1) = F_{n-1}(M)$, and $a_{i+1} = b_i$ for $i = 1, ..., n-1$. Then $M = \oplus_{i=1}^{i=n} R/(a_i)$. Since b_i divides b_{i+1} for $i = 1, ..., n-1$, it remains us to show that a_1 divides a_2. By Proposition 2 $F_{n-2}(N) = (b_1) = (a_2)$, and by Proposition 1 (ii) $F_{n-2}(N) \subseteq F_{n-1}(M)$. So $(a_2) \subseteq (a_1)$, which completes the proof. □

Remark 3. *The converse is true for an arbitrary commutative ring: that is, if $M \cong \oplus_{i=1}^{i=n} R/(a_i)$, and a_i divides a_{i+1} for i=1,...,n-1, then $F_k(M)$ is a principal ideal (see Proposition 2).*

The following result was proved by R. B. Warfield (see [1, Theorem 1]).

Corollary 1. *Every finitely presented module over a valuation ring is direct sum of cyclically presented modules.*

Proof. Let M be a finitely presented module over a valuation ring R. Since the Fitting ideals of M are finitely generated, they are also principal ideals. So, we can apply the previous theorem and we have the result. □

Corollary 2. *Suppose R is a valuation ring. Let M be a finitely generated R-module whose Fitting ideals are principal ideals. M is finite free if and only if there exists a positive integer r such that $F_{r-1}(M) = 0$ and $F_r(M) = R$.*

Proof. Suppose that M is R-finite free of rank r. Then $K(M) = (0)$. So, by definition, $F_r(M) = R$ and $F_k(M)=0$ for $k = 1, ..., r-1$. Conversely, suppose that $F_{r-1}(M)=0$ and $F_r(M) = R$ for some integer r. By Theorem 1 we have $M \cong \oplus_{i=1}^{i=n} R/(a_i)$, where a_i divides a_{i+1} for $i = 1, ..., n$-1, and $n = \mu(M)$. Then $F_{r-1}(M) = (a_1 ... a_{n-r+1})$ and $F_r(M) = (a_1 ... a_{n-r})$ (Proposition 2). Therefore, by hypothesis, $a_1 ... a_{n-r+1} = 0$ and $a_1 ... a_{n-r}$ is a unit in R. Hence $a_{n-r+1} = ... = a_n = 0$. However, $r = \nu(M) = \mu(M) = n$ (by Proposition 3). Then $a_1 = ... = a_n = 0$ and $M \cong R^n$. This completes the proof. □

Remark 4. *In virtue of this corollary, the computation of the Fitting ideals of a finitely generated module M will allows us to decide whether M is free or not.*

Let I and J be two ideals from R. We denote by $(I : J)$ the quotient ideal of I and J, *i.e.*, $(I : J) = \{a \in R \mid aI \subseteq J\}$.

Lemma 1. *Suppose R a valuation ring. Let $a_1, ..., a_n$ be elements in R such that a_i divides a_{i+1} for $i = 1, ..., n-1$. Let $M = \oplus_{i=1}^{i=n} R/(a_i)$, $p = \omega(M)$, and let $q = \nu(M)$. Then*

$$(F_k(M) : F_{k+1}(M)) = \begin{cases} R & \text{if } k < p-1. \\ Ann_R((a_1 ... a_{n-p})) & \text{if } k = p-1. \\ (a_{n-k}) & \text{if } p \leq k \leq q-1. \\ (a_1 ... a_{n-q+1}) & \text{if } k = q-1. \\ R & \text{if } k \geq q. \end{cases}$$

Proof. Let k be a positive integer. We have the following cases :

Case 1: $k < p-1$. In this case both $F_k(M)$ and $F_{k+1}(M)$ are zero. So, their quotient ideal is R.

Case 2: $k = p-1$. In this case $(F_k(M) : F_{k+1}(M)) = (0 : (a_1 ... a_{n-p})) = Ann_R((a_1 ... a_{n-p}))$.

Case 3: $p \leq k \leq q-1$. We have $(F_k(M) : F_{k+1}(M)) = ((a_1 ... a_{n-k}) : (a_1 ... a_{n-k-1}))$. Then $(a_{n-k}) \subseteq (F_k(M) : F_{k+1}(M))$. Conversely, let $x \in (F_k(M) : F_{k+1}(M))$. Then $x a_1 ... a_{n-k-1} = y a_1 ... a_{n-k}$ for a y in R. So, $x - y a_{n-k} \in Ann_R((a_1 ... a_{n-k-1}))$. Since (a_{n-k}) is not included in $Ann_R((a_1 ... a_{n-k-1}))$, we have $Ann_R((a_1 ... a_{n-k-1})) \subseteq (a_{n-k})$ (R is a valuation ring). Then $x - y a_{n-k} \in (a_{n-k})$, so that $x \in (a_{n-k})$. Then $(F_k(M) : F_{k+1}(M)) \subseteq (a_{n-k})$. Finally, $(F_k(M) : F_{k+1}(M)) = (a_{n-k})$.

Case 4: $k = q-1$. We have $(F_{q-1}(M) : F_q(M)) = ((a_1 ... a_{n-q+1}) : R) = (a_1 ... a_{n-q+1})$.

Case 5: $k \geq q$. In this case we have $F_k(M) = F_{k+1}(M) = R$. Then the ideal quotient is the ring R.□

Corollary 3. *Suppose R is a valuation ring. Let M be a finitely generated R-module whose Fitting ideals are nonzero principal ideals. Then $M \cong \oplus_{k=0}^{k=n-1} R/(F_k(M) : F_{k+1}(M))$, where $n = \mu(M)$.*

Proof. By Theorem 1 we have $M \cong \oplus_{i=1}^{i=n} R/(a_i)$, where a_i divides a_{i+1} for i=1,...,n-1, and $n = \mu(M)$. Since R is a local ring and the Fitting ideals of M are nonzero, $\omega(M) = 0$ and $\mu(M) = \nu(M)$(by Proposition 3). So, by Lemma 1 *(ii)* $(F_k(M) : F_{k+1}(M)) = (a_{n-k})$ for $k = 0, ..., n-1$. Hence $(a_i) = (F_{n-i}(M) : F_{n-i+1}(M))$ for $i = 1, ..., n$. Then $M \cong \oplus_{k=0}^{k=n-1} R/(F_k(M) : F_{k+1}(M))$. □

Corollary 4. *Suppose R is a valuation ring. Let M and M' be two finitely generated R-modules. Suppose that the Fitting ideals of M are principal ideals. If $F_k(M) = F_k(M')$ for any positive integer k, then $\mu(M) = \mu(M')$.*

Proof. Let $\mu(M) = r$ and $\mu(M') = r'$. By hypothesis the Fitting ideals of M and M' are principal ideals. Then, by Theorem 1, there exist $a_1, ..., a_r, b_1, ..., b_{r'}$ in R such that a_i divides a_{i+1}, b_i divides b_{i+1}, $M \cong \oplus_{i=1}^{i=r} R/(a_i)$, and $M' \cong \oplus_{i=1}^{i=r'} R/(b_i)$. Then $F_k(M) = (a_1...a_{r-k})$ for $k = 0, ..., r-1$ and $F_k(M') = (b_1...b_{r'-k})$ for $k = 0, ..., r'-1$. Suppose that $r < r'$. Then $F_k(M) = F_k(M')$ implies $R = (b_1...b_{r'-r})$. Hence $(b_i) = R$ for $i = 1, ..., r'-r$. This is impossible since $\mu(M') = \nu(M') = r'$ (see Proposition 3) In the same way, the case $r > r'$ is impossible. So $r = r'$. □

We could see easily that two isomorphic finitely generated modules have the same Fitting ideals (indeed, they have the same relations). The converse fails in general. We have the following corollary:

Corollary 5. *Suppose R is a valuation ring. Let M and M' be two finitely generated R-modules. Suppose that the Fitting ideals of M are nonzero principal ideals. Then, $M \cong M'$ if and only if $F_k(M) = F_k(M')$ for any positive integer k.*

Proof. Suppose that $F_k(M) = F_k(M')$ for any positive integer k. Let $r = \mu(M)$. In virtue of Theorem 1 and Corollary 4 there exist $a_1, ..., a_r$ and $b_1, ..., b_r$ in R such that $M \cong \oplus_{i=1}^{i=r} R/(a_i)$, and $M' \cong \oplus_{i=1}^{i=r} R/(b_i)$ $(r = \mu(M) = \mu(M')$. Then we show by induction on i that $(a_i) = (b_i)$ for $i = 1, ..., r$. For $i = 1$ we have $F_{r-1}(M) = F_{r-1}(M')$. Then $(a_1) = (b_1)$ (by Proposition 2). Let i be in $\{1, ..., r-1\}$. Suppose that $(a_j) = (b_j)$ for any positive integer $j \leq i$. We must show that $(a_{i+1}) = (b_{i+1})$. Since $F_{r-(i-1)}(M) = F_{r-(i-1)}(M')$, $(a_1...a_{i+1}) = (b_1...b_{i+1})$. So, $a_1...a_i(a_{i+1}) = b_1...b_i(b_{i+1}) = a_1...a_i(b_{i+1})$. Since $(a_1...a_i) \neq 0$ (by hypothesis), [7, Corollary 4.2] implies $(a_{i+1}) = (b_{i+1})$. So, by the induction hypothesis, $(a_i) = (b_i)$ for any i in $\{1, ..., r-1\}$. This completes the proof. □

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