GELFAND AND EXCHANGE RINGS: THEIR SPECTRA IN POINTFREE TOPOLOGY

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الخلاصـة :

يتعلق هذا البحث بحلقات جلفاند والحلقات المتبادلة A بدلالة طيفياتها من حيث نظيراتها الخالية من النقط وإطارات *RIdA* للمثاليات الجذرية و *JRIdA* للمثاليات الجذرية لجاكبسون. حيث يسمح هذا الاتجاه بدراسة مستقلة عن أيّ اختبار للأساسيات ومع هذا تؤدي إلى نفس النتائج عند استعمال الاختيار التابع. والنتائج التي حصلنا عليها هي أعم وأشمل وينتج عنها كل ماهو معروف في هذا المجال.

وعلى وجه الخصوص فإن حلقات جيلفاند تتميز بطبيعة الـ *RIdA* حيث أثبتنا مايلي:

* الحلقة تكون حلقة تبادل إذا وفقط و(إذا) هي حلقة جيلفاند ذات البعد صفر *JRIdA*؛
* الحلقة تكون حلقة تبادل إذا وفقط إذا هي حلقة ذات عناصر global التي نسميها حلقة محلية مدعمة جيداً في الـ Topos والـ sheaves في إطار منتظم مضغوط؛
* الحلقة تكون حلقة تبادل إذا وفقط إذا هي حلقة ذات عناصر global لحلقة محلية في الـ Topos والـ sheaves في إطار مضغوط ذي البعد صفر.

هذا وقد قمنا بتقديم تحليل لاختبار الأساسيات المطلوبة في بعض الأعمال السابقة في هذا المجال.

ABSTRACT

This paper deals with Gelfand and exchange rings A in terms of their spectra, the latter understood not in the usual sense as the *spaces* of prime, or maximal, ideals, but as their pointfree counterparts, the *frames RIdA* of radical ideals and *JRIdA* of Jacobson radical ideals. This approach permits a treatment which is independent of any choice principles but which still leads to the same kind of results as those obtained in the classical choice-dependent setting. Further, the present results are more general; they imply the classical ones.

In particular, Gelfand rings A are characterized by the normality of *RIdA*, and we show that:

- a ring A is an exchange ring iff it is Gelfand with zerodimensional *JRIdA*;
- a ring is Gelfand iff it is the ring of global elements of what we call a well-supported local ring in the topos of sheaves on a compact regular frame; and
- a ring is an exchange ring iff it is the ring of global elements of a local ring in the topos of sheaves on a compact zerodimensional frame.

In addition, we provide an analysis of the choice principles required in some previous work in this area.

GELFAND AND EXCHANGE RINGS: THEIR SPECTRA IN POINTFREE TOPOLOGY

Pointfree topology deals with particular complete lattices, called frames, which may be regarded as abstractly defined lattices of open sets of topological spaces. Apart from the fact that a considerable number of results in classical topology turn out to be consequences of results in this pointfree setting, frames have the important feature that certain kinds of spaces often have their useful properties only by virtue of some choice principle whereas the corresponding frames of open sets, which may serve various purposes just as well as the spaces, already exist independent of such assumptions.

For the following, the important case in point is that of the *prime spectrum* $PrimA$ of a commutative ring A with unit, familiarly described as the space of prime ideals of A with the Zariski topology given by the basic open sets $\{P \in PrimA \mid a \notin P\}$ for each $a \in A$. Now, these spaces will only be useful if one assumes the *Prime Ideal Theorem* that any non-trivial Boolean algebra has a prime ideal: otherwise $PrimA$ may well be empty and thus utterly fail to carry any information concerning A. On the other hand, the radical ideals of A, that is, the ring ideals J such that $a^n \in J$ implies $a \in J$, form a frame $RIdA$ which constitutes the pointfree antecedent of $PrimA$ regardless of any choice assumption. Indeed, if the Prime Ideal Theorem is assumed $RIdA$ is isomorphic to the lattice of open sets of $PrimA$ but in many situations the frame $RIdA$ can perform the same function as the latter, independent of that isomorphism. The main purpose here is to establish several new results of this nature.

In this vein, this paper presents a strengthened version of the result by Johnstone that a commutative ring A with unit is an exchange ring iff it is a Gelfand ring whose maximal ideal space is zero-dimensional [1] in which the latter *space* is replaced by a naturally corresponding *frame* derived from $RIdA$ by a general construction which turns out to be the frame of Jacobson radical ideals of A. In addition, it gives a new characterization of exchange rings in terms of a property of their radical ideals and derives some known results of Nicholson [2] on exchange rings by way of particularly suggestive proofs made possible by the present approach. Further, it provides characterizations of Gelfand and exchange rings as the rings of global elements of local rings in the topos of sheaves on certain types of frames which are pointfree analogs of classical results by Mulvey [3] and Monk [4]. Finally, it carries out an analysis of the precise rôle of the choice principles required in some previous work in this area.

The arguments involved here make use of various results in pointfree topology, some specifically proved here, others part of the general background, including the relevant refinement of the classical result that the spectral spaces are exactly the prime spectra of commutative rings with unit (Hochster [5], Banaschewski [6]). Regarding foundations, we are working in Zermelo–Fraenkel set theory (as usually understood: without the Axiom of Choice), on the basis of classical logic, but it will be clear that several of the proofs presented here are in fact constructively valid in the sense of topos theory.

1 DEFINITIONS AND EXAMPLES

All rings in this paper are taken to be commutative and with unit 1. The particular rings to be considered here are defined as follows.

A ring A is called a *Gelfand ring* if $a + b = 1$ in A implies that $(1 + ar)(1 + bs) = 0$ for some $r, s \in A$; on the other hand, A is called an *exchange ring* provided that, for each $a \in A$, there exist idempotent $u \in A$ such that $a + u$ is invertible.

We illustrate the scope of these notions by a number of examples, mostly familiar but some perhaps new. Regarding Gelfand rings, we have:

(G1) Exchange rings. If $a+b=1$ and u is an idempotent such that $u-a$ is invertible then:

$$\left(1+\frac{a}{u-a}\right)\left(1-\frac{b}{u-a}\right)=\frac{1}{(u-a)^2}\,u(u-a-b)=\frac{1}{(u-a)^2}\,u(u-1)=0.$$

(G2) f-Rings with bounded inversion. Recall that an f-ring is a lattice-ordered ring in which $(a\wedge b)c=(ac)\wedge(bc)$ for all a,b and all $c\geq 0$, and bounded inversion means that any $a\geq 1$ has an inverse. Now, if $a+b=1$ then $a\vee b$ has an inverse because

$$a\vee b=a\vee(1-a)=\left((a-\tfrac{1}{2})\vee(\tfrac{1}{2}-a)\right)+\tfrac{1}{2}=\left|a-\tfrac{1}{2}\right|+\tfrac{1}{2}\geq\tfrac{1}{2}\,,$$

so that we can consider:

$$\left(1-\frac{a}{a\vee b}\right)\left(1-\frac{b}{a\vee b}\right)=\frac{1}{(a\vee b)^2}\,(0\vee(b-a))(0\vee(a-b)).$$

Here the last product is of the form c^+c^- for $c=a-b$, and for f-rings this is zero, as a consequence of the fact that $c^+\wedge c^-=0$ in any lattice-ordered ring.

In particular it follows that the rings $C(X)$ of real-valued continuous functions on topological spaces X are Gelfand rings, and it is easily seen that such rings need not be exchange rings: take $X=\mathbb{R}$ and note the absence of any idempotents other than 0 and 1.

(G3) As an example of function rings which are Gelfand rings but not lattice-ordered rings we have the rings $C^\infty(X)$ of all real-valued smooth functions on smooth manifolds X. Let $w\in C^\infty(\mathbb{R})$ be such that:

$$w(t)\geq 0 \text{ for all } t,\ w(t)=0 \text{ for } t\leq\tfrac{1}{4},\ w(t)=1 \text{ for } t\geq\tfrac{1}{2}\,.$$

Then, for any $a\in C^\infty(X)$, put $u=w\circ a$ so that:

$$u(x)=0 \text{ if } a(x)\leq\tfrac{1}{4} \text{ and } u(x)=1 \text{ if } a(x)\geq\tfrac{1}{2}\,,$$

and define the function r on X by:

$$r(x)=\begin{cases}-\frac{u(x)}{a(x)} & \text{if } \ a(x)\neq 0\\[1ex] 0 & \text{if } \ a(x)=0.\end{cases}$$

It follows that $r(x)=0$ whenever $a(x)<\frac{1}{4}$ and hence $r\in C^\infty(X)$; further, $1+ar$ is zero on the set $\{x\in X\mid a(x)\geq\frac{1}{2}\}$.

Now let $a+b=1$ in $C^\infty(X)$, r as given above, and s defined analogously for b. Then $(1+ar)(1+bs)=0$ because $a(x),b(x)<\frac{1}{2}$ is excluded since $a+b=1$.

(G4) An integral domain A is a Gelfand ring iff, for any $a\in A$, a or $1-a$ is invertible which is exactly the condition that the non-invertible elements of A form an ideal, or equivalently that A has a largest proper ideal, saying that A is a *local* ring.

The following are examples of exchange rings, supplying further examples of Gelfand rings in view of (G1).

(E1) Boolean rings. The fact that a Boolean ring has only one invertible element, namely 1, is conveniently counteracted here by the fact that every element is idempotent so that the trivial identity $a+(1-a)=1$ proves the point.

(E2) Regular rings. For any element a, if $a = a^2b$ by regularity then $1 - ab$ is idempotent and

$$(a + 1 - ab)(1 + ab(b - 1)) = a + 1 - ab + a(b - 1) = 1$$

since $a^2b = a$ and $(1 - ab)ab = 0$.

(E3) The rings $C(X)$ for zero-dimensional compact Hausdorff (that is: Boolean) spaces X. For any $a \in C(X)$, compactness and zero-dimensionality provide open-closed $U \subseteq X$ such that:

$$\{x \in X \mid a(x) = 0\} \subseteq U \subseteq \{x \in X \mid |a(x)| < \tfrac{1}{2}\},$$

and for the characteristic function $u \in C(X)$ of U, $a + u$ is never zero and consequently invertible in $C(X)$. Note that $C(X)$ is regular iff all cozero sets are open-closed (Gillman and Jerison [7], 14.29); hence any Boolean space not of this type, such as the classical Cantor set, determines an exchange ring which is not regular. Furthermore, such rings are also Jacobson semisimple (in the sense that $a = 0$ whenever all $1 + ab$ are invertible). The existence of exchange rings of this kind played a rôle in the early history of this subject; the examples provided at the time were obtained by ad hoc constructions (Monk [4], Nicholson [2]) while here we see that such rings actually occur quite naturally.

(E4) An indecomposable ring is an exchange ring iff it is local: any local ring is obviously an exchange ring, and any indecomposable ring of that kind must be local, 0 and 1 being its only idempotents.

By way of general construction principles we add that, for either type of ring, any homomorphic image or directed union of rings of this kind is again of this kind. The same holds for any finite products in general, and for arbitrary products provided the Axiom of Choice is assumed.

Regarding the history of these notions, Gelfand rings were introduced by Mulvey [3], albeit with a somewhat different definition and dealing with general, not necessarily commutative, rings. In the commutative case, the same kind of ring, but again with a different definition, was already studied earlier by De Marco and Orsatti [8]. In either case, the motivation for singling out these particular rings was that they were perceived as a class of rings which in many ways resemble the function rings $C(X)$. The precise relation between the definition adopted here and those used by these earlier authors will be discussed in Section 5.

While the Gelfand rings have a very straightforward origin, the history of exchange rings is considerably more involved: it is deeply rooted in module theory, connected with the problem of finding isomorphic refinements of direct decompositions of a given module. Thus, a module M over a ring R (not necessarily commutative) is called *exchangeable* if, for any R-module K, $K = A_1 \oplus \cdots \oplus A_n$ and $K = N \oplus P$ where $N \cong M$ implies that $K = N \oplus B_1 \oplus \cdots \oplus B_n$ for suitable submodules B_i of A_i, and R is called an *exchange ring* if it is exchangeable as a module over itself (Warfield [9]). In due course, it turned out that the latter property could be characterized by various first order conditions on the ring (Monk [4], Nicholson [2]), and specifically in the commutative case this led to the condition used here.

Finally these two notions of such disparate origins were placed in interesting juxtaposition by Johnstone [1] who established the following remarkable result:

A ring A is an exchange ring iff it is Gelfand and its maximal ideal space $MaxA$ is zero-dimensional.

Given that the properties of the space involved here depend on choice principles, as indicated by the fact that $MaxA \neq \emptyset$ for every Gelfand ring A iff the Prime Ideal Theorem holds, the question arises whether there is a choice-independent antecedent of this, and one of the present purposes, as already indicated, is to show this is indeed the case. As an additional feature of the proof presented below in Section 4 we note that it is entirely direct and does not depend on the Grothendieck sheaf representation of rings used as the main tool in [1]. Moreover, it shows that some very general facts in pointfree topology form the ultimate basis of this particular subject.

2 THE FRAME OF RADICAL IDEALS

Recall that a *frame* is a complete lattice L in which

$$a \wedge \bigvee S = \bigvee\{a \wedge t \mid t \in S\}$$

for all $a \in L$ and $S \subseteq L$, and a *frame homomorphism* is a map $h : L \to M$ between frames preserving all finitary meets, including the unit (= top) e, and arbitrary joins, including the zero (= bottom) 0.

As basic examples we note the frames $\mathfrak{O}X$ of open subsets of topological spaces X on the one hand and the complete Boolean algebras on the other. A frame isomorphic to some $\mathfrak{O}X$ is called *spatial*; natural non-spatial examples of frames are the non-atomic complete Boolean algebras. General references to frames are Johnstone [1] and Vickers [10].

Of particular interest in the present context will be the following frame properties which extend familiar topological notions. A frame L is called:

compact if $e = \bigvee S$ implies $e = \bigvee T$ for some finite $T \subseteq S$;

regular if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, where $x \prec a$ means that $a \vee x^* = e$ for the pseudocomplement $x^* = \bigvee\{y \in L \mid y \wedge x = 0\}$ of x;

zero-dimensional if every element of L is a join of complemented elements, that is, of elements $c \in L$ for which $c \vee c^* = e$;

normal if $a \vee b = e$ in L implies there exist $c, d \in L$ such that $a \vee c = e = b \vee d$ and $c \wedge d = 0$.

Further, an element $c \in L$ is called *compact* if $c \leq \bigvee S$ implies $c \leq \bigvee T$ for some finite $T \subseteq S$, and a frame L is called *coherent* if every element of L is a join of compact elements and the meet of any finitely many compact elements is compact, where the latter is equivalent to saying that L is compact and $c \wedge d$ is compact for any compact c and d.

Concerning regularity, we note that any subframe of a frame which is generated by some regular subframes is itself regular, and consequently any frame L has a largest regular subframe $RegL$.

Regarding rings, recall that a *radical ideal* of a ring A is a ring ideal J such that $a^n \in J$ implies $a \in J$ for any $a \in A$ and exponent n. In particular, for each $a \in A$,

$$[a] = \{x \in A \mid x^n \in Aa \text{ for some } n\}$$

is a radical ideal, the principal radical ideal generated by a, and $[a] \cap [b] = [ab]$ for any $a, b \in A$. Note that $a \in [0]$ iff $a^n = 0$ for some n and hence $[0] = \{0\}$ means A is semiprime; further, $A/[0]$ is the semiprime reflection of A, that is, the quotient map $A \to A/[0]$ is the universal homomorphism from A to semiprime rings.

The radical ideals of a ring A, partially ordered by inclusion, obviously form a complete lattice $RIdA$: meet is intersection. Further, $RIdA$ is distributive and directed join is given by union – which makes it a frame. Finally, the compact elements of $RIdA$ are exactly the finitely generated radical ideals, that is, the:

$$J = [a_1] \vee \cdots \vee [a_n],$$

for some $a_1, \ldots, a_n \in A$, and since $[a] \cap [b] = [ab]$ it follows that $RIdA$ is coherent.

For any ring A, $IdpA$ will be the Boolean algebra of its *idempotents*, with the familiar operations:

$$u \wedge v = uv, \;\; u \vee v = u + v - uv, \;\; 1 - u = \text{ complement of } u\,,$$

for any $u, v \in IdpA$. On the other hand, for any frame L, its *Boolean part* BL will be the Boolean algebra of its complemented elements, with its lattice operations induced from L. Then we have:

Lemma 1. *For any ring A, the map $IdpA \to B(RIdA)$ taking u to $[u]$ is an isomorphism.*

Proof. These $[u]$ are obviously complemented in $RIdA$, with complement $[1-u]$, and the map $u \mapsto [u]$ is clearly a homomorphism, one-one since $[u] = [0]$ evidently implies $u = 0$. To see that it is onto consider any $J \in B(RIdA)$ so that $J \vee H = [1]$ and $J \cap H = [0]$ for some $H \in RIdA$. Then also $J + H = [1]$ and hence $a + b = 1$ for some $a \in J$ and $b \in H$ where $ab \in [0]$ so that $(ab)^n = 0$ for some n. It follows that:

$$1 = (a+b)^{2n} = a^n c + b^n d\,,$$

for some c and d, and then $u = a^n c$ and $v = b^n d$ are complementary idempotents. Now, for any $x \in J$, $x = xu + xv$ and since $xv \in J \cap H$ there exist m such that $(xv)^m = 0$. As a result, $x^m = x^m u \in [u]$ so that $x \in [u]$, showing that $J \subseteq [u]$ and therefore $J = [u]$. □

As a first, familiar instance of the way in which $RIdA$ encodes important ring properties we note the following:

Corollary. *A semiprime ring A is regular iff $RIdA$ is zero-dimensional.*

Proof. ($\Rightarrow$) If $a = a^2 b$ then $[a] = [u]$ for the idempotent $u = ab$, and $[u]$ is complemented by the lemma; since $RIdA$ is obviously generated by the $[a]$ this proves the claim.

($\Leftarrow$) For any $a \in A$, $[a]$ is compact in $RIdA$ and hence complemented by zero-dimensionality so that $[a] = [u]$ for some idempotent u by the lemma. Consequently, $u = ab$ and $a^n = uc$ for some $b, c \in A$ and exponent n. Now, the latter implies $(a(1-u))^n = 0$, and since A is semiprime this shows $a = au = a^2 b$, as desired. □

In a similar vein, we now have the following chacterization of Gelfand rings.

Proposition 1. *A ring A is Gelfand iff $RIdA$ is normal.*

Proof. ($\Rightarrow$) If $J \vee H = [1]$ in $RIdA$ then also $J + H = [1]$ so that $a + b = 1$ for some $a \in J$ and $b \in H$. Now, if $r, s \in A$ are such that $(1 + ar)(1 + bs) = 0$ then:

$$J \vee [1 + ar] = [1] = H \vee [1 + bs],$$

since $ar \in J$ and $bs \in H$, and $[1 + ar] \cap [1 + bs] = [(1 + ar)(1 + bs)] = [0]$.

($\Leftarrow$) If $a + b = 1$ in A then $[a] \vee [b] = [1]$ in $RIdA$ and by normality there exist $J, H \in RIdA$ such that:

$$[a] \vee J = [1] = [b] \vee H \text{ and } J \cap H = [0].$$

It then follows from the first part that:

$$(*) \qquad a_0 + c = 1 = b_0 + d\,,$$

for some $a_0 \in [a]$, $c \in J$, $b_0 \in [b]$, and $d \in H$. In particular, $a_0^n = ax$ and $b_0^m = by$ for suitable exponents n, m and $x, y \in A$, and hence $(*)$ implies:

$$ax + c_0 = 1 = by + d_0 \ ,$$

with new $c_0 \in J$ and $d_0 \in H$. Further $c_0 d_0 \in [0]$ so that:

$$(1 - ax)^k (1 - by)^k = 0,$$

for some k, and multiplying out these powers then shows there exist r and s for which $(1 + ar)(1 + bs) = 0$, as desired. □

3 COMPACT NORMAL FRAMES

In view of Proposition 1, it is clear that these frames will play a central rôle in our context.

To begin with, for *any* compact frame L, one considers the map $s_L : L \to L$ such that:

$$s_L(a) = \bigvee\{x \in L \mid x \vee y = e \text{ implies } a \vee y = e\}.$$

The $x \in L$ occurring here are called a-small; they clearly form an ideal in L, and by compactness $s_L(a)$ itself is a-small and hence the largest a-small element. Furthermore, s_L is a closure operator such that $s_L(a \wedge b) = s_L(a) \wedge s_L(b)$ which makes it a *nucleus*, with the effect that $SL = Fix(s_L)$ is a frame and the map $s_L : L \to SL$ a frame homomorphism. Finally, s_L is *codense*, that is, $s_L(a) = e$ implies $a = e$, as a result of which SL is compact, and $s_L : L \to SL$ is the unique smallest codense quotient of L (Banaschewski and Harting [11]).

For any ring A, the frame $S(RIdA)$ has a very concrete significance: the radical ideals J of A such that $s_{RIdA}(J) = J$ are exactly the *Jacobson radical ideals* of A, understood in the sense that any $a \in A$ for which all $1 + ab$ are invertible modulo J belongs to J. This, incidentally, hinges on commutativity: in general, the condition $s_{RIdA}(J) = J$ characterizes the Brown–McCoy radical ideals (Banaschewski and Harting [11]). In the following, we refer to $S(RIdA)$ as the frame $JRIdA$ of Jacobson radical ideals of A.

We now specifically turn to compact normal frames L.

The first result is from Banaschewski [12]; we include the proof for the sake of convenience.

Lemma 2. *SL is compact regular.*

Proof. For any $x < a$ in SL there exist $b \in L$ such that $x \vee b < e = a \vee b$ by the definition of s_L, and the normality of L then supplies $c, d \in L$ such that $a \vee c = e = b \vee d$ and $c \wedge d = 0$. It follows that $d \prec a$ in L but $d \not\leq x$ and for $y = s_L(d)$ this implies that $y \prec a$ in SL but $y \not\leq x$. As a consequence, a is the join of all $z \prec a$ in SL, showing that SL is regular. □

There is another special map on L, $r_L : L \to L$ defined by:

$$r_L(a) = \bigvee\{x \in L \mid x \prec a\}.$$

For arbitrary frames not much can be said about this but in the present situation we have:

Lemma 3. *r_L is a homomorphism providing a retraction of L to $RegL$; further, $s_L r_L = s_L$ and $r_L s_L = r_L$.*

Proof. It is obvious by the definition of $\prec$ that r_L preserves $0, \wedge$, and e. Further, $r_L(\bigvee D) = \bigvee r_L[D]$ for updirected D follows immediately by compactness, and $r_L(a \vee b) = r_L(a) \vee r_L(b)$ is an easy consequence of normality: if $x \prec a \vee b$ so that $a \vee b \vee x^* = e$ then there exist disjoint u and v such that $a \vee u = b \vee x^* \vee v = e$, and we have further:

$$b \vee w = e = x^* \vee v \vee z \ ,$$

with disjoint w and z. As a result, $v \prec a$, $z \prec b$, and:

$$x = x \wedge (x^* \vee v \vee z) = (x \wedge v) \vee (x \wedge z) \leq r_L(a) \vee r_L(b),$$

which proves the non-trivial part of the desired identity. In all this shows r_L is a homomorphism.

Next, $r_L^2 = r_L$. If $x \prec a$ then also $x \prec y \prec a$ for some y by normality and hence $x \prec r_L(a)$ so that $x \leq r_L(r_L(a))$; this shows that $r_L(a) \leq r_L(r_L(a))$ and hence equality, the reverse inequality being automatic.

As a result, $Im(r_L) = Fix(r_L)$ is regular:

$$r_L(a) = r_L(r_L(a)) = \bigvee\{r_L(x) \mid x \prec a\}$$

and $x \prec a$ implies $r_L(x) \prec r_L(a)$. It follows that $Im(r_L) \subseteq RegL$, but the reverse inclusion is trivial: for any $a \in RegL$,

$$a = \bigvee\{x \in RegL \mid x \prec a \text{ in } RegL\}$$

implies $r_L(a) = a$ since $x \prec a$ in $RegL$ implies $x \prec a$ in L.

Regarding the relations between r_L and s_L, $x \prec s_L(a)$ implies $x \prec a$ by the definition of s_L and consequently $r_L(s_L(a)) \leq r_L(a)$, the non-trivial part of the desired equality. On the other hand, a is $r_L(a)$-small because $a \vee y = e$ implies $r_L(a) \vee r_L(y) = e$ and hence also $r_L(a) \vee y = e$, showing that $s_L(a) \leq s_L(r_L(a))$ while the reverse inequality is again trivial. □

Corollary. *s_L induces an isomorphism $RegL \to SL$ with inverse effected by r_L.*

Proof. s_L is codense and any codense homomorphism h on a regular frame is one-one because $h(a) = h(b)$ and $x \prec a$ implies $h(b \vee x^*) = h(a \vee x^*) = e$ so that $b \vee x^* = e$ and hence $x \leq b$. On the other hand, s_L maps $RegL$ onto SL because $s_L r_L = s_L$. Finally, for $a \in RegL$, $r_L s_L(a) = r_L(a) = a$, showing that r_L induces the inverse of the isomorphism $RegL \to SL$ given by s_L. □

Remark. The above map $r_L : L \to L$ is obviously defined for any frame L but need not be a homomorphism. In fact, *for compact L, r_L is a homomorphism iff L is normal*: if $a \vee b = e$ in L implies $r_L(a) \vee r_L(b) = e$ then also $x \vee y = e$ where $a \vee x^* = e = b \vee y^*$ by compactness, and here:

$$x^* \wedge y^* = (x \vee y)^* = e^* = 0 \ ,$$

showing normality.

For the following, recall that a *cover* of a frame L is any subset C of L such that $\bigvee C = e$, and a cover B *refines* a cover C if each element of B is below some element of C; further, a *partition* of L is a cover by pairwise disjoint elements, meaning: any two distinct elements have zero meet.

A frame L will be called *weakly zero-dimensional* if every finite cover of L is refined by a finite partition. Note this is equivalent to the weaker condition that whenever $a \vee b = e$ in L there exist c and d in L such that $c \leq a$, $d \leq b$, $c \vee d = e$, and $c \wedge d = 0$. Given this, the general case follows by induction: if $a_0 \vee a_1 \vee \cdots \vee a_n = e$, take a partition $\{b_1, \ldots, b_n\}$ such that $b_1 \leq a_0 \vee a_1$ and $b_i \leq a_i$ for $i = 2, \ldots, n$ by induction hypothesis and then $d_i \leq a_i \vee b_2 \vee \cdots \vee b_n$ $(i = 0, 1)$ such that $d_0 \vee d_1 = e$ and $d_0 \wedge d_1 = 0$ to obtain the desired partition $\{d_0 \wedge b_1, d_1 \wedge b_1, b_2, \ldots, b_n\}$.

It is clear that any compact zero-dimensional frame is weakly zero-dimensional but not conversely as the product $\mathbf{2} \times \mathbf{3}$ of the two-element with the three-element chain shows.

Lemma 4. *For any compact frame L, the following are equivalent:*

(1) *L is normal and SL is zero-dimensional.*

(2) *L is weakly zero-dimensional.*

(3) *For each $a \in L$, the homomorphism $L \to \uparrow a$, $x \mapsto x \vee a$, maps BL onto $B(\uparrow a)$.*

Proof. (1) $\Rightarrow$ (2). If $a \vee b = e$ in L then also $r_L(a) \vee r_L(b) = e$ in $RegL \cong SL$, and hence there exist $c \leq r_L(a)$ and $d \leq r_L(b)$ such that $c \vee d = e$ and $c \wedge d = 0$; since $RegL$ is a subframe of L and $r_L(x) \leq x$ for all $x \in L$ this proves the claim.

(2) $\Rightarrow$ (3). For any $b \in B(\uparrow a)$, if $c \in \uparrow a$ such that $b \vee c = e$ and $b \wedge c = a$ accordingly, and further $x \leq b$ and $y \leq c$ in L for which $x \vee y = e$ and $x \wedge y = 0$ by hypothesis, it follows that:

$$x \vee a \leq b = (b \wedge x) \vee (b \wedge y) \leq x \vee a\ ,$$

and hence $b = x \vee a$, as required.

(3) $\Rightarrow$ (1). If $a \vee b = e$ in L then $a \in B(\uparrow (a \wedge b))$ trivially and hence $a = x \vee (a \wedge b)$ for some $x \in BL$. It follows that $a \vee x^* = e = b \vee x$, showing that L is normal. Similarly, if $x \prec a$ in L and hence $a \vee x^* = e$ there exist $z \in BL$ such that $a = z \vee (a \wedge x^*)$, and consequently:

$$x = x \wedge a = x \wedge (z \vee (a \wedge x^*)) \leq z \leq a.$$

Thus, for any $a \in L$, $r_L(a) = \bigvee\{x \in L \mid x \prec a\}$ is a join of complemented elements of L, and since these themselves belong to $RegL$ it follows that $SL \cong RegL$ is zero-dimensional. $\square$

Remark. A compact frame L with zero-dimensional SL need not be normal. Thus, for any finite L, $SL \cong \uparrow m = \{a \in L \mid a \geq m\}$ for the meet m of all maximal elements of L and hence SL is Boolean, but clearly not all finite distributive lattices are normal.

We close with an observation concerning the spectrum of SL for a compact normal frame L.

Recall that the spectrum of any frame L is the space ΣL of prime elements $p \in L$ (meaning: $p < e$ and if $a \wedge b \leq p$ then $a \leq p$ or $b \leq p$) with the open sets $\Sigma_a = \{p \in \Sigma L \mid a \not\leq p\}$, $a \in L$. As is familiar, this may also be described as the space of all frame homomorphisms $\xi : L \to \mathbf{2}$, with the sets $\{\xi \mid \xi(a) = 1\}$, $a \in L$, as the open sets, but in the present context, the first description seems preferable.

Note that any maximal (= maximal less than e) element of a frame is prime; on the other hand, in a *regular* frame, any prime element p is maximal: if $p < a$ then $x \not\leq p$ for some $x \prec a$, and since $x \wedge x^* = 0 \leq p$ it follows that $x^* \leq p$, hence also $x^* \leq a$, and therefore $a = a \vee x^* = e$.

Now we have:

Lemma 5. *For any compact normal frame L, $\Sigma(SL)$ is the space of maximal elements of L.*

Proof. By Lemma 2, the points of $\Sigma(SL)$ are the maximal elements of SL. Now, any such element u is also maximal in L since $u < a$ in L implies $u < s_L(a)$, hence $s_L(a) = e$, and therefore $a = e$. On the other hand, any maximal element v of L belongs to SL because $v \leq s_L(v) < e$ implies $v = s_L(v)$. Regarding the topology of $\Sigma(SL)$, it is clear that this is the same as the subspace topology induced from ΣL: for any maximal $u \in L$, $a \not\leq u$ iff $s_L(a) \not\leq u$ since $a \vee u = e$ iff $s_L(a) \vee u = e$, for any $a \in L$. □

Corollary. *For any Gelfand ring A, $\Sigma(JRIdA)$ is the space $MaxA$ of maximal ideals of A.*

Proof. As noted earlier, $S(RIdA)$ is the frame $JRIdA$ of Jacobson radical ideals of A, and the maximal ideals of A are clearly the maximal elements of $RIdA$. □

Note that, for Gelfand rings, this identifies $JRIdA$ as the pointfree version of $MaxA$, as indicated earlier.

4 EXCHANGE RINGS

We are now able to establish the desired pointfree form of the characterization of exchange rings referred to in Section 1 as well as a new characterization directly in terms of radical ideals.

Proposition 2. *The following are equivalent for any ring A.*

(1) *A is an exchange ring.*

(2) *$RIdA$ is weakly zero-dimensional.*

(3) *A is Gelfand and $JRIdA$ is zero-dimensional.*

Proof. (1) ⇒ (2). If $I \vee J = [1]$ in $RIdA$, take $a \in I$ and $b \in J$ such that $a + b = 1$ and further an idempotent u for which $u - b$ is invertible with inverse c. Then

$$auc = (1 - b)uc = (u - b)uc = u$$

and

$$b(u - 1)c = (b - u)(u - 1)c = 1 - u,$$

showing that $u \in I$ and $1 - u \in J$ and therefore $[u] \subseteq I$ and $[1 - u] \subseteq J$ which proves the claim by Lemma 1.

(2) ⇒ (3). Immediate by Lemma 4 and Proposition 1.

(3) ⇒ (1). For any $a \in A$, apply the map $r = r_{RIdA}$ of Lemma 3 to the relation $[a] \vee [1 + a] = [1]$ in $RIdA$ and use the given zero-dimensionality of $Reg(RIdA) \cong JRIdA$ together with Lemma 1 to obtain idempotents $u \in r([a])$ and $v \in r([1 + a])$ such that $[u] \vee [v] = [1]$. Then also $u + v - uv = 1$ and we may assume that $u + v = 1$ and $uv = 0$. Now, since $r([x]) \subseteq [x]$ for any $x \in A$, $u = ab$ and $v = (1 + a)c$ for suitable $b, c \in A$, and then

$$\begin{aligned}(a + v)(ab^2 + (1 + a)c^2) &= u + a(1 + a)c^2 + v(1 + a)c^2 \\ &= u + a(1 + a)c^2 + (1 + a)c^2 = u + v = 1,\end{aligned}$$

showing that $a + v$ is invertible. □

Remark 1. Specifically, the above equivalence (1) ≡ (3) is the choice-free characterization of exchange rings corresponding to the result of Johnstone [1] mentioned in Section 1. Indeed, by (1) ⇒ (3), any exchange ring is a

Gelfand ring with zero-dimensional $MaxA$ because the spectrum of a zero-dimensional frame is zero-dimensional and $MaxA = \Sigma(JRIdA)$ for any Gelfand ring A by the corollary of Lemma 5. On the other hand, though, a Gelfand ring A with zero-dimensional $MaxA$ need not be an exchange ring: in the absence of the appropriate choice principle the latter property will not be strong enough to ensure this. We refer to Section 5 for the details.

Remark 2. It is implicit in the proof of Proposition 2 that a ring A is an exchange ring iff $a+b=1$ in A implies there exist $c, d \in A$ such that $ac + bd = 1$ and $acbd = 0$. That exchange rings have this property is also shown by Nicholson [2].

Recall the familiar terminology by which idempotents are said to *lift modulo an ideal* J of a ring A whenever the natural homomorphism $A \to A/J$ maps $IdpA$ *onto* $Idp(A/J)$. With this, we have the following variant of a result of Nicholson [2] as an easy consequence of Lemma 4 and the present proposition.

Corollary 1. *A ring A is an exchange ring iff idempotents lift modulo every radical ideal of A.*

Proof. By the results referred to, A is an exchange ring iff, for each $H \in RIdA$, the frame homomorphism $RIdA \to {\uparrow}H$ taking J to $J \vee H$ maps $B(RIdA)$ onto $B({\uparrow}H)$. On the other hand, the isomorphism of Lemma 1 provides a commuting square

$$\begin{array}{ccc} IdpA & \longrightarrow & Idp(A/H) \\ \cong\downarrow & & \downarrow\cong \\ B(RIdA) & \longrightarrow & B(RId(A/H)) \end{array}$$

for each $H \in RIdA$ where the horizontal maps are induced by the natural homomorphism $\nu : A \to A/H$. It follows that the top map is onto iff this holds for the bottom map, and in view of the obvious isomorphism $RId(A/H) \cong {\uparrow}H$ also induced by ν this proves the claim. $\square$

Remark. Formally this corollary is partly stronger and partly weaker than the original result of Nicholson [2] that A is an exchange ring iff idempotents lift modulo *every ideal* of A but, actually, the two conditions are a priori equivalent, as an easy consequence of the fact that, in any ring, two idempotents are equal whenever they are equal modulo [0].

The following is a further result of Nicholson [2] which has a very suggestive proof in the present context. Regarding notation we let $\langle 0\rangle = s_{RIdA}([0])$, the Jacobson radical of A.

Corollary 2. *A ring A is an exchange ring iff* $A/\langle 0\rangle$ *is an exchange ring and idempotents lift modulo* $\langle 0\rangle$.

Proof. ($\Rightarrow$) Any homomorphic image of an exchange ring is an exchange ring, and idempotents lift modulo every ideal of A.

($\Leftarrow$) Note first that the natural homomorphism $\nu : A \to A/\langle 0\rangle$ reflects invertibility: if $ab - 1 \in \langle 0\rangle$ then $[ab-1] + [ab] = 1$ implies $[ab] = 1$ since $\langle 0\rangle$ is [0]-small, and hence $abc = 1$ for some c. Now, for any $a \in A$, if $\nu(a) + u$ is invertible for some idempotent u of $A/\langle 0\rangle$ then $u = \nu(w)$ for some idempotent $w \in A$ so that $\nu(a+w)$ is invertible, and as noted this makes $a + w$ invertible. $\square$

In a very different vein, we further have:

Corollary 3. *Any Gelfand ring A for which $RIdA$ is finite is an exchange ring.*

Proof. For any finite normal frame L, SL is a finite regular frame, hence Boolean and consequently zero-dimensional. □

5 THE RÔLE OF CHOICE PRINCIPLES

Formulated for the rings under consideration here, the condition Mulvey [3] introduced was that, for any distinct maximal ideals P and Q of the ring A, there exist $r \notin P$ and $s \notin Q$ such that $rs = 0$. As a property of $RIdA$, expressed for a general frame L, this says that for any distinct maximal elements u and v of L there exist $a, b \in L$ for which $a \not\leq u$, $b \not\leq v$, and $a \wedge b = 0$. We shall call a frame of this kind *weakly normal* and a ring A with weakly normal $RIdA$ *weakly Gelfand.*

On the other hand, De Marco and Orsatti [8] were dealing with rings in which every prime ideal is contained in a unique maximal ideal and called these *pm*-rings. Accordingly, a frame L will be called a *pm-frame* whenever each prime element of L is below a unique maximal element.

Note that any compact normal frame L is both weakly normal and a *pm*-frame. For the first part, $u \vee v = e$ for any distinct maximal $u, v \in L$, and normality then provides the required $a, b \in L$. The second assertion is an immediate consequence of the fact that $s_L(p)$ is maximal for any prime element $p \in L$ (Banaschewski [12], Remark 1.4) which is seen as follows: if $s_L(p) < a$ then $s_L(p) \vee b < e = a \vee b$ for some b, and if further $a \vee c = e = b \vee d$ where $c \wedge d = 0$ then $d \not\leq p$ because $s_L(p) \vee b < e$ so that $c \leq p$ and hence $a = a \vee c = e$.

Now, one of the main features of either of the above ring notions was that, given the Axiom of Choice, they imply $RIdA$ is normal, that is, A is Gelfand in the sense of the definition adopted here. In the following we shall determine the exact extent to which this conclusion is choice dependent.

For this purpose, we first derive appropriate results in the context of frames and then transfer them to rings by the general principle that,

for any coherent frame L there exist rings A such that $RIdA \cong L$.

This was originally proved by Hochster [5] in terms of prime spectra and spectral spaces, with corresponding dependence on choice principles, but a subsequent refinement of the arguments involved showed that the result holds in this pointfree form independent of the latter (Banaschewski [6]).

Regarding weak normality we now have

Lemma 6. (1) *The Axiom of Choice implies that every weakly normal compact frame is normal.*

(2) *If the Axiom of Choice fails there exists a coherent weakly normal frame which is not normal.*

Proof. (1) is proved in 2.7 of Banaschewski [13].

(2) We show there exists a non-normal coherent frame *without maximal elements* which is then weakly normal vacuously.

First, a general observation. For any complete partially ordered set S with compact unit e, let $\mathfrak{A}$ be the bounded sublattice of the lattice of all downsets U of $S - \{e\}$ ($x \leq y \in U$ implies $x \in U$) generated by the downsets $S - \uparrow a = \{x \in S \mid a \not\leq x\}$, $a \in S$, and $\lambda : S \to \mathfrak{A}$ the map taking a to $S - \uparrow a$. Then $\mathfrak{A}$ consists of all

$\lambda(a_1) \cap \ldots \cap \lambda(a_n)$ for $a_1, \ldots, a_n$ in S, and λ is a partial order embedding preserving all joins, the latter because $a = \bigvee X$ in S implies:

$$\uparrow a = \bigcap \{\uparrow s \mid s \in X\}.$$

Further, for any non-void $U, V \in \mathfrak{A}$, $0 \in U \cap V$ and hence the zero $\emptyset$ of $\mathfrak{A}$ is prime. In particular, this implies that $\mathfrak{A}$ *is not normal* whenever there exist $a, b \in S - \{e\}$ such that $a \vee b = e$: normality would imply that:

$$\lambda(a) \cup U = \lambda(e) = \lambda(b) \cup V \text{ and } U \cap V = \emptyset,$$

for some $U, V \in \mathfrak{A}$, but then $U = \emptyset$ or $V = \emptyset$ so that $a = e$ or $b = e$, a contradiction.

Next, let $\mathfrak{M}$ be any maximal ideal of the lattice $\mathfrak{A}$. Then $u = \bigvee \lambda^{-1}[\mathfrak{M}]$ is a maximal element of $S - \{e\}$. First, $u < e$ by the compactness of e since $u = e$ implies $a_1 \vee \cdots \vee a_n = e$ for some $a_i \in S$ such that $\lambda(a_i) \in \mathfrak{M}$, but then also $\lambda(e) \in \mathfrak{M}$, a contradiction. Secondly, $\mathfrak{M} \subseteq \downarrow \lambda(u)$ since $\mathfrak{M}$, being prime, is generated by the $\lambda(a) \in \mathfrak{M}$; hence $\mathfrak{M} = \downarrow \lambda(u)$, showing that $u = v$ whenever $u \leq v < e$ in S.

We apply these considerations as follows. For any onto map $\varphi : X \to E$ of sets, let S be the set consisting of all subsets of X which φ maps one-one, partially ordered by inclusion, together with an added top element e. Then S is complete because any subset of $S - \{e\}$ bounded above in $S - \{e\}$ has a least upper bound in $S - \{e\}$, namely its union. Further, e is compact: for any updirected subset of $S - \{e\}$ the union belongs to $S - \{e\}$.

Now assume that $\varphi : X \to E$ has no section, exhibiting a violation of the Axiom of Choice. Then, in particular, φ is not one-one and for distinct $x, y \in X$ such that $\varphi(x) = \varphi(y)$, $\{x\}, \{y\} \in S - \{e\}$ and $\{x\} \vee \{y\} = e$ in S. As a result, the bounded distributive lattice $\mathfrak{A}$ associated with S as above is *not normal.* On the other hand, the maximal elements of $S - \{e\}$ are obviously the $A \subseteq X$ which φ maps one-one onto E and which therefore provide a section. Hence, by our earlier observation, $\mathfrak{A}$ *has no maximal ideals.* As a result, the ideal lattice of $\mathfrak{A}$ is a coherent frame without maximal elements which is not normal. □

Remark. We note that the above arguments regarding S and $\mathfrak{A}$ in connection with $\varphi : X \to E$ also provide a proof of the familiar result of Klimovsky [14] that the Axiom of Choice is equivalent to the Maximal Ideal Theorem for bounded distributive lattices.

Turning now to *pm*-frames we have

Lemma 7. *The Prime Ideal Theorem holds iff every coherent pm-frame is normal.*

Proof. ($\Rightarrow$) We show first that, in a coherent frame L, the join of any prime ideal P is a prime element. Indeed, let $a \wedge b \leq s$ for $s = \bigvee P$ and suppose $a \not\leq s$ so that there also exist compact $c \leq a$ such that $c \not\leq s$. Now, for any compact $d \leq b$, $c \wedge d$ is compact by coherence, and since $c \wedge d \leq s$ this implies $c \wedge d \in P$ so that $d \in P$ because $c \notin P$; it follows that $d \leq s$ for any compact $d \leq b$, showing that $b \leq s$.

Now, let $a \vee b = e$ in L and assume, by way of contradiction, that the filter $F = \{x \wedge y : a \vee x = e = b \vee y \text{ in } L\}$ is proper. Then, as a familiar consequence of the Prime Ideal Theorem, there is a prime ideal P in L disjoint from F, and we let $s = \bigvee P$. Now, $a \vee s < e$ for otherwise $a \vee x = e$ for some $x \in P$ and then $x \in P \cap F$, a contradiction. Similarly, $b \vee s < e$, and we then have prime elements p and q such that $a \vee s \leq p$ and $b \vee s \leq q$, again by the Prime Ideal Theorem (Banaschewski [15]). Further, since L is a *pm*-frame there exist maximal $u \geq p$ and $v \geq q$ in L, and since s is prime by the first part of this proof it follows that $u = v$. Finally, this implies $a, b \leq u$ and hence $e = a \vee b \leq u$, a contradiction.

($\Leftarrow$) We show that any failure of the Prime Ideal Theorem determines a coherent *pm*-frame which is not normal. Given any non-trivial Boolean algebra B without prime ideals, let A be the sublattice of $B \times B \times B$ consisting of all $a = (a_1, a_2, a_3)$ such that $a_1, a_2 \leq a_3$. Then A is not normal:

$$(0, e, e) \vee (e, 0, e) = (e, e, e);$$

but if:

$$(0, e, e) \vee (a_1, a_2, a_3) = (e, e, e) = (e, 0, e) \vee (b_1, b_2, b_3),$$

in A then $a_1 = e = b_2$, hence $a_3 = e = b_3$, and therefore $a \wedge b \neq 0$. Further, A has no prime ideals because B has none and B is embedded in A by $b \mapsto (b, b, b)$. It follows that the ideal lattice of A is now the desired frame: it is coherent, (vacuously) a *pm*-frame, and not normal. □

Remark. The ($\Rightarrow$) part of the above proof is a slightly modified version of an argument given by Johnstone [1].

As an immediate consequence of these two lemmas and the relationship between coherent frames and rings referred to earlier we now have the following.

Proposition 3. (1) *Every weakly Gelfand ring is Gelfand iff the Axiom of Choice holds.*

(2) *Every pm-ring is Gelfand iff the Prime Ideal Theorem holds.*

As an analogous result concerning exchange rings we have:

Proposition 4. *Every Gelfand ring with zero-dimensional maximal ideal space is an exchange ring iff the Prime Ideal Theorem holds.*

Proof. ($\Leftarrow$) It is a familiar consequence of the Prime Ideal Theorem that every compact regular frame is spatial, and given the latter the zero-dimensionality of $MaxA = \Sigma(JRIdA)$ trivially implies that of $JRIdA$.

($\Rightarrow$) We show that any failure of the Prime Ideal Theorem provides a Gelfand ring A for which $MaxA$ is empty and hence trivially zero-dimensional but $JRIdA$ is not zero-dimensional. Again, the approach is to construct an appropriate coherent frame. For this, let M be the ideal lattice of a non-trivial Boolean algebra without prime ideals, making it a non-trivial compact zero-dimensional frame with empty spectrum, and N any compact regular frame which is not zero-dimensional such as the frame of open sets of the real unit interval (or its pointfree counterpart, as in IV, 1.2 of Johnstone [1]). Then the compact regular frame $K = M \oplus N$ clearly also has empty spectrum. Further, the ideal lattice $\mathfrak{J}K$ of K is a *normal* coherent frame, by the normality of compact regular frames, and since the homomorphism $\mathfrak{J}K \to K$ taking the ideals of K to their joins in K is onto trivially and codense by compactness it follows that $S(\mathfrak{J}K) \cong K$. Finally, by (A3) in the Appendix, K is not zero-dimensional because N is not zero-dimensional. Altogether then this provides a normal coherent frame L such that SL has empty spectrum but is not zero-dimensional, as required. □

6 SHEAF REPRESENTATIONS

In this section it will be convenient to change notation and use $\mathfrak{L}, \mathfrak{M}, \ldots$ for frames, $U, V, \ldots$ for their elements, with 0 and E for zero and unit, respectively. A *sheaf* (of sets) on a frame $\mathfrak{L}$ is then defined in verbatim the same way as it is for a topological space X (the case $\mathfrak{L} = \mathfrak{O}X$), as a presheaf satisfying the familiar separation and patching conditions. If S is any sheaf on a frame $\mathfrak{L}$, SU will be the set it assigns to $U \in \mathfrak{L}$, and the notation for the restriction map $SU \to SV$, $V \leq U$, will be $x \mapsto x|V$.

A ring $\mathcal{A}$ in the category $Sh\mathfrak{L}$ of sheaves on a frame $\mathfrak{L}$ will be called a *ring on* $\mathfrak{L}$, and its *ring of global elements* will be the ring $\mathcal{A}E$ (in the category of sets). For a given ring A, a *sheaf representation* of A is any ring $\mathcal{A}$ on a frame $\mathfrak{L}$ such that $A \cong \mathcal{A}E$. Naturally, of particular interest are the $\mathcal{A}$ with some special property which improves upon the properties of A. The following general result is of this kind.

For any ring A, the assignments:

$$[a] \mapsto A[a^{-1}] = a[X]/(1 - aX)$$

$$[a] \subseteq [b] \mapsto A[b^{-1}] \to A[a^{-1}],$$

the homomorphism resulting from the fact that
b becomes invertible in $A[a^{-1}]$ if some $a^n = bc$,

define a sheaf on the frame $RIdA$, providing a ring $\mathcal{A}$ on $RIdA$ whose ring of global elements is A: $\mathcal{A}[1] = A[1^{-1}] = A$. Moreover, $\mathcal{A}$ is local in the sense that:

$$\mathcal{A} \models (\forall x)((\exists y)(xy = 1)) \vee (\exists z)((1 - x)z = 1)), \quad \text{and} \quad \mathcal{A} \models 7(0 = 1)$$

in the internal logic of the topos of sheaves on $RIdA$, meaning that, for any $J \in RIdA$ and $a \in \mathcal{A}J$, $J = I \vee H$ for some I and H in $RIdA$ such that $a|I$ and $(1 - a)|H$ are invertible and $0|v = 1/v$ only if $U = 0$. For the details, see Chapter V of Johnstone [1] except for the fact that it presents $\mathcal{A}$ as a sheaf on $PrimA$ but the arguments involved can equally well be understood as applying to the pointfree version $RIdA$ of the latter.

Now, if A is a *Gelfand* ring let $\mathfrak{M}A = Reg(RIdA)$ and $r = r_{RIdA}$ the retraction $RIdA \to \mathfrak{M}A$ considered earlier for arbitrary compact normal frames. Then, the restriction to $\mathfrak{M}A$ of the ring $\mathcal{A}$ on $RIdA$ determined by A as above is again local: for any $J \in \mathfrak{M}A$ and $a \in \mathcal{A}J$, if $J = I \vee H$ in $RIdA$ such that $a|I$ and $(1 - a)|H$ are invertible then also $J = r(I) \vee r(H)$ in $\mathfrak{M}A$, and $a|r(I)$ and $(1 - a)|r(H)$ are invertible since $r(G) \subseteq G$ for all $G \in RIdA$. As a result, any Gelfand ring has a sheaf representation by *a local ring on a compact regular frame.* It is our aim to make this more precise in order to obtain an actual characterization of Gelfand rings.

The following notion will be crucial for this purpose. For any local ring $\mathcal{A}$ on a frame $\mathfrak{L}$, we define the *support map* $spt : \mathcal{A}E \to \mathfrak{L}$ by

$$spt(a) = \bigvee\{U \in \mathfrak{L} \mid a|U \text{ is invertible in } \mathcal{A}U\}.$$

This does indeed have the properties suggested by the name. Trivially $spt(0) = 0$ and $spt(1) = E$, the identity $spt(ab) = spt(a) \wedge spt(b)$ is similarly obvious, and the inequality

$$spt(a + b) \leq spt(a) \vee spt(b)$$

holds exactly because $\mathcal{A}$ is local: if $(a + b)|U$ is invertible and consequently $(a|U + b|U)c = 1$ for some $c \in \mathcal{A}U$ then $U = V \vee W$ where $(a|V)(c|V)$ and $(b|W)(c|W)$ are invertible since $\mathcal{A}$ is local, and as this makes $a|V$ and $b|W$ invertible it follows that $V \leq spt(a)$ and $W \leq spt(b)$, proving $U \leq spt(a) \vee spt(b)$. Incidentally, as a familiar consequence of this, there is a frame homomorphism σ: $RId(\mathcal{A}E) \to \mathfrak{L}$ such that $\sigma(J) = \bigvee\{spt(a) \mid a \in J\}$.

A local ring $\mathcal{A}$ on a compact regular frame $\mathfrak{L}$ will be called *well-supported* whenever

(WS1) Each $U \in \mathfrak{L}$ is the join of all $spt(a)$, $a \in \mathcal{A}E$, for which there exist $b \in \mathcal{A}E$ such that $ab = 0$ and $U \vee spt(b) = E$.

(WS2) If $spt(a_1) \vee \cdots \vee spt(a_n) = E$ for some $a_1, \ldots, a_n$ in $\mathcal{A}E$ then there exist $r_1, \ldots, r_n$ in $\mathcal{A}E$ such that $a_1r_1 + \cdots + a_nr_n = 1$.

Note here that $ab = 0$ implies $spt(a) \wedge spt(b) = 0$, and if also $U \vee spt(b) = E$ it follows that $spt(a) \prec U$. Hence (WS1) should be viewed as a strengthened form of regularity of $\mathfrak{L}$, appropriately involving $\mathcal{A}$.

Now the desired characterization is as follows.

Proposition 5. *The Gelfand rings are exactly the rings of global elements of well-supported local rings on compact regular frames.*

Proof. ($\Rightarrow$) Given our earlier observation concerning any Gelfand ring A, it only has to be shown that the restriction to $\mathfrak{M}A$ of the associated ring $\mathcal{A}$ on $RIdA$ is well-supported.

To begin with, note that the support in $\mathfrak{M}A$ of any $a \in A$ is $r([a])$ where r, as earlier, is the retraction $RIdA \to \mathfrak{M}A$. For any $c \in A$, $a|[c]$ is invertible in $\mathcal{A}[c] = A[c^{-1}]$ iff $[c] \subseteq [a]$: $a|[c]$ is invertible iff there exist polynomials $p(X)$ and $q(X)$ in $A[X]$ such that:

$$1 - ap(X) = q(X)(1 - cX),$$

and a simple calculation based on comparison of coefficients proves the claim. As a result $a|J$ is invertible, for any $J \in RIdA$, iff $[c] \subseteq [a]$ for all $c \in J$, that is, iff $J \subseteq [a]$. Now, for $J \in \mathfrak{M}A$, this holds iff $J \subseteq r([a])$, and this proves the claim.

Next, by the definition of the retraction r,

$$J = \bigvee\{[a] \mid [a] \prec J\},$$

in $RIdA$ for any $J \in \mathfrak{M}A$. On the other hand, $[a] \prec J$ means $J \vee [a]^* = [1]$ and this holds iff $J \vee [b] = [1]$ for some $b \in [a]^*$. Now the latter says that $[a] \cap [b] = [0]$, hence $ab \in [0]$, and therefore $(ab)^n = 0$ for some n; finally, since $[a] = [a^n]$ and $[b] = [b^n]$ we may assume $ab = 0$. In all, this shows, for any $J \in \mathfrak{M}A$, that:

$$J = \bigvee\{[a] \mid ab = 0 \text{ and } J \vee [b] = [1]\},$$

in $RIdA$, and by acting r and noting that $J \vee [b] = [1]$ iff $J \vee r([b]) = [1]$ we obtain (WS1).

Regarding (WS2), if $r([a_1]) \vee \cdots \vee r([a_n]) = [1]$ in $\mathfrak{M}A$ then also $[a_1] \vee \cdots \vee [a_n] = [1]$ in $RIdA$, hence $b_1 + \cdots + b_n = 1$ for some $b_i \in [a_i]$, and this implies that $a_1r_1 + \cdots + a_nr_n = 1$ for suitable $r_i \in A$.

($\Leftarrow$) Let $\mathcal{A}$ be any well-supported local ring on a compact regular frame $\mathfrak{L}$ and $a + b = 1$ in $\mathcal{A}E$. Then $U \vee V = E$ where $a|U$ and $b|V$ are invertible since $\mathcal{A}$ is local, and by compactness it follows from (WS1) that there exist $a_1, \ldots, a_n, \bar{a}_1, \ldots, \bar{a}_n$ and $b_1, \ldots, b_m, \bar{b}_1, \ldots, \bar{b}_m$ in $\mathcal{A}E$ such that:

$$\begin{array}{lll} spt(a_i) \leq U, & U \vee spt(\bar{a}_i) = E, & a_i\bar{a}_i = 0 \\ spt(b_i) \leq V, & V \vee spt(\bar{b}_j) = E, & b_j\bar{b}_j = 0 \end{array}$$

$$\bigvee spt(a_i) \vee \bigvee spt(b_j) = E.$$

Next, the first two of these conditions imply that:

$$(*) \qquad U \vee spt(\bar{a}) = E = V \vee spt(\bar{b}),$$

for $\bar{a} = \bar{a}_1 \cdots \bar{a}_n$ and $\bar{b} = \bar{b}_1 \cdots \bar{b}_m$, by the properties of supports. On the other hand, by the third condition and (WS2):

$$\Sigma a_ir_i + \Sigma b_js_j = 1$$

with suitable r_i and s_j, and since $a_i\bar{a} = 0$ and $b_j\bar{b} = 0$ it follows that $\bar{a}\bar{b} = 0$. Finally, because $U \leq spt(a)$ and $V \leq spt(b)$, $(*)$ and (WS2) imply that:

$$ar + \bar{a}\bar{r} = 1 = bs + \bar{b}\bar{s}$$

and hence $(1 - ar)(1 - bs) = \bar{a}\bar{r}\bar{b}\bar{s} = 0$, showing that $\mathcal{A}E$ is Gelfand. $\square$

Remark 1. This characterization of Gelfand rings may be viewed as a pointfree relative of the one given by Mulvey [3] in that the latter deals with sheaves of rings on compact Hausdorff spaces for which, again, the global elements are tied to the space by some separation requirements. It should be pointed out, however, that the direct transfer of the present characterization to the spatial setting does not yield that of [3], and whether there are alternatives to our conditions (WS1) and (WS2) for which this is the case remains to be investigated.

Remark 2. A natural weakening of (WS1) would be to require only that the supports of the global elements of $\mathcal{A}$ *generate* $\mathfrak{L}$. It would obviously be of interest to know whether this, together with (WS2) as given, is already sufficient to make $\mathcal{A}E$ Gelfand – but this has eluded us so far.

The following counterpart of Proposition 5 is the pointfree version of the result of Monk [4] (see also Johnstone [1], V, 2.7) that the exchange rings are exactly the rings of global sections of sheaf spaces on Boolean spaces whose stalks are local rings.

Proposition 6. *The exchange rings are exactly the rings of global elements of local rings on compact zero-dimensional frames.*

Proof. ($\Rightarrow$) Obvious by Proposition 2 and the remarks preceding Proposition 5.

($\Leftarrow$) For any local ring $\mathcal{A}$ on a compact zero-dimensional frame $\mathfrak{L}$, if $a \in \mathcal{A}E$ and $E = U \vee V$ where $a|U$ and $(1+a)|V$ are invertible we may assume that $U \wedge V = 0$ by the properties of $\mathfrak{L}$. Then take $b, c \in \mathcal{A}E$ such that:

$$b|U = (a|U)^{-1},\ b|V = 0,\ c|U = 0,\ c|V = ((1+a)|V)^{-1}.$$

It follows that $u = ab$ and $v = (1+a)c$ are complementary idempotents and

$$(a+v)(bu+cv) = u + acv + cv = u + (1+a)cv = u + v = 1,$$

showing that $\mathcal{A}E$ is an exchange ring. □

Remark. This could also be obtained, in the manner of the original proof by Monk [4], by using the Peirce sheaf representation of A on the ideal lattice of the Boolean algebra $IdpA$, but in the present context this proof seems more natural.

One might expect that the above characterizations of Gelfand and exchange rings are actually the object parts of certain category equivalences, partly because that is a frequent feature of representation theorems but specifically since this is indeed the case in the classical situations considered by Mulvey [3] and Monk [4]. The setting here would obviously be appropriate categories of *ringed frames*, that is, pairs $(\mathcal{A}, \mathfrak{L})$ where $\mathcal{A}$ is a ring on the frame $\mathfrak{L}$, with maps $(\mathcal{A}, \mathfrak{L}) \to (\mathcal{B}, \mathfrak{M})$ given by a frame homomorphism $h : \mathfrak{L} \to \mathfrak{M}$ together with a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}h$ of rings on $\mathfrak{L}$.

For Gelfand rings, the correspondence $A \mapsto (\mathcal{A}|\mathfrak{M}A, \mathfrak{M}A)$, in our earlier notation, is indeed functorial in this sense, as a consequence of the functoriality of the basic correspondence $A \mapsto (\mathcal{A}, RIdA)$ for arbitrary rings. In the special case of exchange rings, which involves the ringed frames $(\mathcal{A}, \mathfrak{L})$ where $\mathcal{A}$ is a local ring and $\mathfrak{L}$ compact zero-dimensional, the desired equivalence seems to present no problem but for Gelfand rings in general there are some difficulties which have not yet been resolved. In any event, we shall return to this topic at some later stage.

We close with a comment concerning the rôle of choice principles in the context of sheaf representations. For any local ring $\mathcal{A}$ on a *topological space* (meaning: on a spatial frame $\mathfrak{L}$) the ring $\mathcal{A}E$ has prime ideals: by the properties of the support map, $\{a \in \mathcal{A}E | spt(a) \leq P\}$ is a prime ideal of $\mathcal{A}E$ for any prime element $P \in \mathfrak{L}$ above $spt(0)$. As an immediate consequence, any of the following is equivalent to the Prime Ideal Theorem:

(1) Every ring has a sheaf representation by a local ring on a topological space.

(2) Every Gelfand ring has a sheaf representation by a well-supported local ring on a compact Hausdorff space.

(3) Every exchange ring has a sheaf representation by a local ring on a Boolean space.

APPENDIX: COMPLEMENTED ELEMENTS IN A FRAME COPRODUCT

Recall that, for any nontrivial (meaning $e \neq 0$) frames L and M, one has their coproduct $L \oplus M$ with coproduct maps $i_L : L \to L \oplus M$ and $i_M : M \to L \oplus M$ which are embeddings such that $a \oplus b = i_L(a) \wedge i_M(b) = 0$ iff $a = 0$ or $b = 0$, and any element of $L \oplus M$ is a join of elements $a \oplus b$. Further, $L \oplus M$ is compact whenever L and M are compact.

(A1) *For any $a \oplus b \neq 0$ in $L \oplus M$, $a \oplus b$ is complemented iff a and b are complemented.*

Proof. The "if" part being obvious we only have to consider the "only if" part. For any $a \oplus b$,

$$(a \oplus b)^* = \bigvee \{c \oplus d \mid (a \wedge c) \oplus (b \wedge d) = 0\}$$

and $(a \wedge c) \oplus (b \wedge d) = 0$ iff $a \wedge c = 0$ or $b \wedge d = 0$, hence iff $c \leq a^*$ or $d \leq b^*$ which implies $c \oplus d \leq a^* \oplus e$ or $c \oplus d \leq e \oplus b^*$ and consequently $c \oplus d \leq (a^* \oplus e) \vee (e \oplus b^*)$. It follows that $(a \oplus b)^* = (a^* \oplus e) \vee (e \oplus b^*)$ and hence $a \oplus b$ is complemented iff:

$$e \oplus e = (a \oplus b) \vee (a^* \oplus e) \vee (e \oplus b^*).$$

As an immediate consequence,

$$e \oplus b = (e \oplus e) \wedge (e \oplus b) = (a \oplus b) \vee (a^* \oplus b) = (a \vee a^*) \oplus b$$

and since $b \neq 0$ the quotient frame $\downarrow b = \{x \in M \mid x \leq b\}$ is non-trivial so that passing from $L \oplus M$ to $L \oplus \downarrow b$ we obtain that $a \vee a^* = e$, showing a is complemented, and by symmetry the same then follows for b. □

(A2) *If L and M are compact and M is zero-dimensional then $u \in L \oplus M$ is complemented iff $u = (a_1 \oplus b_1) \vee \cdots \vee (a_n \oplus b_n)$ with complemented a_i and b_i.*

Proof. Again, "if" is obvious. For "only if" note first that $u = (a_1 \oplus b_1) \vee \cdots \vee (a_n \oplus b_n)$ with complemented b_i by the zero-dimensionality of M, the compactness of $L \oplus M$, and the fact that any complemented element in a compact frame is compact. Further, using the atoms of the (finite!) Boolean subalgebra of M generated by the b_i, we may assume that $b_i \wedge b_k = 0$ whenever $i \neq k$. It then follows that $(a_i \oplus b_i) \wedge (a_k \oplus b_k) = 0$ for $i \neq k$ and consequently each $a_i \oplus b_i$ is complemented, which proves the claim by (A1). □

(A3) *If L and M are compact, L is regular, and M and $L \oplus M$ are zero-dimensional then L is also zero-dimensional.*

Proof. For any non-zero $c \in L$, $c = \bigvee \{x \in L \mid 0 < x \prec c\}$ and hence also $c \oplus e = \bigvee \{x \oplus e \mid 0 < x \prec c\}$. Now, $x \prec c$ implies $x \oplus e \prec c \oplus e$ and since $L \oplus M$ is compact zero-dimensional (A2) implies that there exist complemented $a_1, \ldots, a_n$ in L and $b_1, \ldots, b_n$ in M such that:

$$x \oplus e \leq (a_1 \oplus b_1) \vee \cdots \vee (a_n \oplus b_n) \leq c \oplus e.$$

Further, we may again assume that $b_i \wedge b_k = 0$ if $i \neq k$, and of course that all a_i and b_i are non-zero. Now, for $b = b_1$, taking meet with $e \oplus b$ we obtain:

$$x \oplus b \leq a_1 \oplus b \leq c \oplus b\,,$$

and since $b \neq 0$ this implies $x \leq a_1 \leq c$, showing c is a join of complemented elements, as claimed. □

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REFERENCES

[1] P.T. Johnstone, "Stone Spaces", *Cambridge Studies in Adv. Math,* vol. 3. Cambridge: Cambridge University Press, 1982.

[2] W.K. Nicholson, "Lifting Idempotents and Exchange Rings", *Trans. American Math. Soc.*, **229** (1977), pp. 269–278.

[3] C.J. Mulvey, "A Generalization of Gelfand Duality", *J. Algebra*, **56** (1979), pp. 499–505.

[4] G.S. Monk, "A Characterization of Exchange Rings", *Proc. American Math. Soc.*, **35** (1972), pp. 349–353.

[5] M. Hochster, "Prime Ideal Structure in Commutative Rings", *Trans. American Math. Soc.*, **142** (1969), pp. 749–758.

[6] B. Banaschewski, "Radical Ideals and Coherent Frames", *Comm. Math. Univ. Carolinae*, **37** (1996), pp. 349–370.

[7] L. Gillman and M. Jerison, "Rings of Continuous Functions", *Grad. Texts in Math*, vol. 43. Berlin: Springer-Verlag, 1976.

[8] G. De Marco and A. Orsatti, "Commutative Rings in Which Every Prime Ideal is Contained in a Unique Maximal Ideal", *Proc. American Math. Soc.*, **30** (1971), pp. 459–466.

[9] R. Warfield, "Exchange Rings and Decomposition of Modules", *Math. Ann.*, **199** (1972), pp. 31–36.

[10] S. Vickers, "Topology via Logic", *Cambridge Tracts in Theor. Comp. Sci.*, vol. 5. Cambridge: Cambridge University Press, 1985.

[11] B. Banaschewski and R. Harting, "Lattice Aspects of Radical Ideals and Choice Principles", *Proc. London Math. Soc.*, **50** (1985), pp. 385–404.

[12] B. Banaschewski, "Pointfree Topology and the Spectra of f-Rings", in *Ordered Algebraic Structures, Proc. Curaçao Conference 1995.* Dordrecht: Kluwer Academic Publishers, 1997, pp. 123–148.

[13] B. Banaschewski, "Functorial Maximal Spectra", *J. Pure Appl. Alg.*, to appear.

[14] G. Klimovsky, "El teorema de Zorn y la existencia de filtros y ideales maximales en los reticulados distributivos", *Rev. Un. Mat. Argentina*, **18** (1958), pp. 160–164.

[15] B. Banaschewski, "Prime Elements from Prime Ideals", *Order*, **2** (1985), pp. 211–213.

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Note Added in Proof (15 December 2000). Regarding Remark 1 following Proposition 5 (p.19), recent joint work with J.J.C. Vermeulen has produced equivalent alternatives to our condition (SW1) and (SW2) which quite evidently embody the pointfree essence of the condition given by Mulvey [3] and lead to a proof that the sheaves of rings considered there in the case of compact Hausdorff spaces X are in fact *exactly the well-supported local rings on* $\mathfrak{O}X$. The details will be published in due course.