RIGOROUS ANALYSIS OF AN IMPLICIT SPECTRAL METHOD FOR KdV EQUATION

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الخلاصة :

نعرض في هذا البحث تطبيق طريقة (فوريير) الطيفية مقروناً بتفرد (أويلر) العياري لمعادلة KdV. ثم نستقصي إمكان حلولها الشاملة وتقاربها. ونقدّم كذلك طريقة توليدية لحلّ الأنظمة الجبرية غير الخطية، وأخيراً نقوم بتقدير عواملها الإنشائية بدقة بالغة.

ABSTRACT

A Fourier spectral method in combination with the standard midpoint Euler temporal discretization for the KdV equation is considered. The global existence and convergence of the numerical solution are investigated rigorously. An iteration method for solving the nonlinear algebraic systems at each time level is also proposed, with its compression factor strictly estimated.

Keywords: KdV equation; spectral method; implicit scheme; existence; convergence; iteration method.

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1. INTRODUCTION

The KdV equation is an important mathematical model, with wide applications in quantum mechanics, nonlinear optics, *etc.* [1]. It can be described as follows:

$$\begin{cases} \partial_t U(t) + \Phi(U(t))_x + U_{xxx}(t) = 0 , & -\infty < x < \infty , \quad 0 < t \le T \\ U|_{t=0} = U_0 , & -\infty < x < \infty , \end{cases} \tag{1.1}$$

where $\partial_t U = \frac{\partial U}{\partial t}$, $U_x = \frac{\partial U}{\partial x}$, $\Phi(U) = U^{n+1}$ with n being a nonnegative integer. U_0 is a given function.

The KdV equation has been intensively investigated during the past decades. There is a lot of work in the literature concerning its numerical solutions, *e.g.* [2–5]. It is well known that a key point in constructing numerical schemes is the reasonable simulation of the conservations satisfied by the original problem. Usually the schemes possessing such a property are implicit in treating nonlinear terms. Meanwhile, implicit schemes often permit larger step sizes in time marching, which are critical in the study of long-time behaviors. Thus the analysis of implicit schemes is of practical significance. However, relatively less work has been done on its rigorous analysis.

Guo Benyu [4, 6] proposed a class of (linearly) implicit difference schemes and spectral schemes for the KdV equation. He proved that the numerical solutions satisfy the first and second conservations for some specific choices of scheme parameters. Also, he analyzed strictly the generalized stability and the convergence of these schemes. Wineberg *et al.* [7] proposed a spectral method with Crank–Nicolson time differencing (both for linear and nonlinear terms) for the KdV equation, and generalized it to 2-dimensional K–P equations. They also presented a rough estimation of the compression factor for the iterative solutions of the resulting nonlinear algebraic equations, and reported lots of numerical results. But it still leaves open as for the global existence and convergence of the numerical solutions. Frutos and Sanz-Serna [8] proposed a kind of fourth order temporal discretization for (1.1). The kernel of their algorithm is the simple midpoint Euler discretization, with positive and negative step sizes. The numerical results presented therein seem very attractive, especially when such a technique collaborates with the spatial spectral discretizations. However there is also no theoretical analysis about this nonlinearly implicit scheme. Recently, He Guoqiang [9–11] proposed some techniques for the analyses of implicit difference approximations of nonlinear evolution equations. One of his essential ideas is to study the existence and convergence of a solution to a nonlinear scheme at the same time.

In this paper, we extend some of the ideas in [9–11] to deal with a nonlinearly implicit spectral scheme for (1.1). It combines the temporal midpoint Euler discretization and the spatial Fourier spectral discretization, and possesses the first and second conservations. We analyze the global existence and convergence of the numerical solution rigorously. Also, an iteration method for solving the nonlinear algebraic equations at each time level is proposed. We show strictly the compression factor of this iteration is of order $O(r^{1/2})$, which clarifies the rough estimation, $O(r^{2/3})$, of the compression factor in [7]. Besides, since the analysis in this paper is also true for negative time step sizes, therefore all the results can be applied to the analysis of the schemes in [8].

An outline of this paper is as follows: in Section 2 the implicit spectral scheme for (1.1) is described. In Section 3 , we first show an *a priori* estimate of the error between the numerical solution and the exact solution. Then we propose an iteration for solving the nonlinear algebraic equations resulting at each time level. Based on the analysis of this iteration, we prove the global existence of the numerical solution. Finally, the convergence is readily obtained by the existence and the *a priori* estimate.

2. SCHEME

In this section, we describe the implicit spectral scheme for (1.1). First we give some notations. Let $I = (0, 2\pi)$, and $L^2(I)$ is the space consisting of all the square integrable, complex valued functions. Its inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ are defined as follows

$$(u,v) = \frac{1}{2\pi}\int_I u\bar{v}dx \ , \qquad \|u\| = \sqrt{(u,u)} \ .$$

Let $m \geq 0$ be an integer and $1 \leq q \leq \infty$. Denote as usual by $L^q(I)$,$W^{m,q}(I)$ and $H^m(I) = W^{m,2}(I)$ the classical Sobolev spaces. $\|\cdot\|_{m,q}$ is the norm of $W^{m,q}(I)$, $\|\cdot\|_m$ and $|\cdot|_m$ are the norm and semi-norm of $H^m(I)$, resp., (see [12] for details). Furthermore, let $C_p^\infty(I)$ be the set of all the 2π- periodic, infinitely differentiable functions. $W_p^{m,q}(I)$, $H_p^m(I)$ are the closures of $C_p^\infty(I)$ in $W^{m,q}(I)$ and $H^m(I)$, resp.

Now we consider the Fourier spectral approximation in spatial direction. Let N be a positive integer, define

$$V_N = \{v(x) = \sum_{|j|\leq N} v_j e^{ijx} \ \mid \ v_j = \bar{v}_{-j} \ , \ \ \forall |j| \leq N\},$$

where i is the complex unit. Actually V_N is the set of real valued trignometric polynomials of degree $\leq N$. Denote by P_N the L^2-orthogonal projection upon V_N.

Next we consider the finite difference discretization in temporal direction. Let τ be the step size in t. $S_\tau = \{t = k\tau \mid 0 \leq k \leq [\frac{T}{\tau}]\}$ and

$$v_t(t) = \frac{1}{\tau}(v(t+\tau) - v(t)),$$

$$\hat{v}(t) = \frac{1}{2}(v(t+\tau) + v(t)).$$

The implicit Fourier spectral scheme for (1.1) considered in this paper is described as follows: to find $u(t) \in V_N$ for all $t \in S_\tau$, such that

$$\begin{cases} u_t(t) + P_N\Phi(\hat{u}(t))_x + \hat{u}_{xxx}(t) = 0 \ , & \forall t \in S_\tau \ , \quad t \leq T-\tau \ , \\ u(0) = P_N U_0 \ . & \end{cases} \tag{2.1}$$

It is easy to verify that the solution to (2.1) possesses the discrete analogues of the first two conservations of (1.1). More precisely,

$$\int_I u(t)dx = \int_I u(0)dx \ , \quad \forall t \in S_\tau$$

and

$$\int_I u^2(t)dx = \int_I u^2(0)dx \ , \quad \forall t \in S_\tau \ .$$

3. THEORETICAL ANALYSIS

3.1. A Priori Error Estimation

Let $U(t) = U(x,t)$ be the exact solution to (1.1). Define $U^N(t) = P_N U(t)$. Then we have from (1.1) that:

$$\begin{cases} U_t^N(t) + P_N\Phi(\hat{U}^N(t))_x + \hat{U}_{xxx}^N(t) = \tilde{f}(t) \ , & \forall t \in S_\tau \ , \quad t \leq T-\tau \ , \\ U^N(0) = P_N U_0 \ , & \end{cases} \tag{3.1}$$

where

$$\tilde{f}(t) = U_t^N(t) - \partial_t \hat{U}^N(t) + P_N[\Phi(\hat{U}^N(t))_x - \hat{\Phi}(U(t))_x] \, .$$

Suppose for the present that solution $u(t')$ to scheme (2.1) exists for $t' = 0, \tau, \cdots, t+\tau$, and its difference $e(t')$ from $U^N(t')$, *i.e.*, $e(t') = u(t') - U^N(t')$, satisfies

$$\|e(t')\|_{L^\infty} \le M_0 \, , \qquad \forall t' \le t+\tau \, . \tag{3.2}$$

with M_0 being a positive constant. Then subtraction of (3.1) from (2.1) gives:

$$e_t(t) + P_N(\tilde{\Phi}(t))_x + \hat{e}_{xxx}(t) = -\tilde{f}(t) \, , \tag{3.3}$$

with

$$\tilde{\Phi}(t) = \Phi(\hat{U}^N(t) + \hat{e}(t)) - \Phi(\hat{U}^N(t)) \, .$$

By taking L^2 inner product with $2\hat{e}(t)$ on both sides of (3.3), we get:

$$(\|e(t)\|^2)_t - 2(\tilde{\Phi}(t), \hat{e}_x(t)) = -2(\tilde{f}(t), \hat{e}(t)) \, . \tag{3.4}$$

Now we estimate the inner products in the above equality. Clearly

$$|2(\tilde{f}(t), \hat{e}(t))| \le \|\hat{e}(t)\|^2 + \|\tilde{f}(t)\|^2 \le \frac{1}{2}(\|e(t+\tau)\|^2 + \|e(t)\|^2) + \|\tilde{f}(t)\|^2 \, .$$

By integration by parts,

$$\begin{aligned}
|2(\tilde{\Phi}(t), \hat{e}_x(t))| &= \left| 2\sum_{l=0}^{n} \binom{n+1}{l} ((\hat{U}^N(t))^l (\hat{e}(t))^{n+1-l}, \hat{e}_x(t)) \right| \\
&= \left| 2\sum_{l=0}^{n} \binom{n+1}{l} \left((\hat{U}^N(t))^l, \left(\frac{(\hat{e}(t))^{n+2-l}}{n+2-l} \right)_x \right) \right| \\
&\le 2\sum_{l=1}^{n} \frac{l}{n+2-l} \binom{n+1}{l} |((\hat{U}^N(t))^{l-1} \hat{U}_x^N(t), (\hat{e}(t))^{n+2-l})| \\
&\le c_1 \left(\sum_{l=1}^{n} \|\hat{e}(t)\|_{L^\infty}^l \right) \|\hat{e}(t)\|^2 \, ,
\end{aligned}$$

where c_1 is a positive constant depending only on $max_{0 \le t \le T} \|U^N(t)\|_{1,\infty}$. It can be bounded above by a constant independent of N , provided that $U(t) \in H_p^s(I)$ with $s > \frac{3}{2}$ (cf. [13]). Hereafter we will use c_i to denote the general constants, independent of τ and N , which can be of different values in different cases. Thus it follows from (3.2) that

$$|2(\tilde{\Phi}(t), \hat{e}_x(t))| \le c_2 \|\hat{e}(t)\|^2 \le c_2(\|e(t+\tau)\|^2 + \|e(t)\|^2) \, ,$$

with $c_2 = c_1 \sum_{l=1}^{n} (M_0)^l$. By the above estimates, we derive from (3.4) that:

$$(1 - \frac{\tau}{2} - c_2\tau)\|e(t+\tau)\|^2 \le (1 + \frac{\tau}{2} + c_2\tau)\|e(t)\|^2 + \tau\|\tilde{f}(t)\|^2 \, .$$

Assume τ is sufficiently small, then:

$$\begin{aligned}
\|e(t+\tau)\|^2 &\le \tfrac{1+\tau/2+c_2\tau}{1-\tau/2-c_2\tau}\|e(t)\|^2 + \tfrac{\tau}{1-\tau/2-c_2\tau}\|\tilde{f}(t)\|^2 \\
&\le (1 + c_3\tau)\|e(t)\|^2 + 3\tau\|\tilde{f}(t)\|^2 \, ,
\end{aligned}$$

with $c_3 = 3(1/2 + c_2)$. Hence:

$$\|e(t+\tau)\|^2 \le \rho(e(0), \tilde{f}, t+\tau) e^{c_3 t} ,$$

where

$$\rho(e(0), \tilde{f}, t+\tau) = \|e(0)\|^2 + 3\tau \sum_{t'=0}^{t} \|\tilde{f}(t')\|^2 .$$

Lemma 1. Assume τ and N^{-1} are sufficiently small. $u(t')$ exists for all $t' \le t+\tau$, and satisfies

$$\|e(t')\|^2_{L^\infty} \le M_0 , \quad \forall t' = 0, \tau, \cdots, t+\tau,$$

with M_0 being a positive constant. Then we have for all $t' \le t+\tau$, that

$$\|e(t')\|^2 \le \rho(e(0), \tilde{f}, t') e^{c_3 t'}.$$

3.2. Existence and Convergence

Assume $\tau = o(N^{-1/2})$ and that N is sufficiently large. Furthermore, the initial error $e(0)$ and the truncation error $\tilde{f}(t)$ satisfy:

$$\rho(e(0), \tilde{f}, T) = \|e(0)\|^2 + 3\tau \sum_{\substack{t' \in S_\tau \\ t' \le T-\tau}} \|\tilde{f}(t)\|^2 \le \left(\frac{M_0 N^{-1/2}}{2}\right)^2 e^{-c_3 T} . \quad (3.5)$$

Suppose now the solution $u(t')$ at each time level $t' = 0, \tau, \cdots, t$ exists and satisfies

$$\|e(t')\|^2_{L^\infty} \le M_0 , \quad \forall t' = 0, \tau, \cdots, t . \quad (3.6)$$

Then we consider the solution of scheme (2.1) at time level $t+\tau$. We prove by an iteration method that there exists a solution $u(t+\tau)$ to (2.1), and show that (3.6) holds also for $t' = t+\tau$.

Consider the following iteration for solving $u(t+\tau)$:

$$\begin{cases} \frac{z^{k+1}-u(t)}{\tau} + P_N \Phi(\frac{z^k+u(t)}{2})_x + (\frac{z^{k+1}+u(t)}{2})_{xxx} = 0 , \quad k = 0,1,2,\cdots, \\ z^0 = u(t) . \end{cases} \quad (3.7)$$

We first prove the boundedness of the iteration sequence $\{z^k\}$. Let $e^k = z^k - U^N(t+\tau)$. Then we get from (3.1) and (3.7) that:

$$\begin{cases} \frac{e^{k+1}-e(t)}{\tau} + P_N(\tilde{\Psi})_x + (\frac{e^{k+1}+e(t)}{2})_{xxx} = -\tilde{f}(t) , \quad k = 0,1,2,\cdots, \\ e^0 = e(t) - \tau U_t^N(t) , \end{cases} \quad (3.8)$$

where

$$\tilde{\Psi} = \Phi(\frac{z^k + u(t)}{2}) - \Phi(\hat{U}^N(t)) .$$

Denote by $e_j^k, e_j(t), \tilde{\Psi}_j$, and $\tilde{f}_j(t)$ the j-th Fourier coefficients of $e^k, e(t), \tilde{\Psi}$, and $\tilde{f}(t)$, resp. Define a new norm $|\cdot|_\infty$ for $v \in V_N$ such that

$$|v|_\infty = \sum_{|j| \le N} |v_j| .$$

Obviously [13], for all $|j| \le N$,

$$|v_j| \le \|v\| \le \|v\|_{L^\infty} \le |v|_\infty \le \sqrt{2N+1}\|v\| . \qquad (3.9)$$

We prove by induction that:

$$|e^k|_\infty \le M_0 , \quad k = 0, 1, 2, \cdots, . \qquad (3.10)$$

Consider first the case $k = 0$. By the assumption (3.6), the *a priori* estimate holds for time level t. Thus we have from Lemma 1 that:

$$\|e(t)\|^2 \le \rho(e(0), \tilde{f}, t)e^{c_3 t} .$$

It follows from (3.5) that

$$\|e(t)\| \le \frac{1}{2} M_0 N^{-1/2} . \qquad (3.11)$$

Note $\tau = o(N^{-1/2})$ and (3.9), we have for suitably large N that:

$$|e^0|_\infty \le \sqrt{2N+1}\|e^0\| \le \sqrt{2N+1}(\|e(t)\| + \tau\|U_t^n(t)\|) \le M_0 , \qquad (3.12)$$

which completes the induction for $k = 0$.

Now we suppose (3.10) holds for all $k' = 0, 1, 2, \cdots, k$, and then examine $|e^{k+1}|_\infty$. By comparing the j-th Fourier coefficients on both sides of (3.8), we arrive at:

$$\frac{e_j^{k+1} - e_j(t)}{\tau} + ij\tilde{\Psi}_j - ij^3 \frac{e_j^{k+1} + e_j(t)}{2} = -\tilde{f}_j(t) .$$

It implies:

$$e_j^{k+1} = \frac{1 + i\tau j^3/2}{1 - i\tau j^3/2} e_j(t) - \frac{i\tau j}{1 - i\tau j^3/2}\tilde{\Psi}_j - \frac{\tau}{1 - i\tau j^3/2}\tilde{f}_j(t) .$$

By taking modulus on both sides of the above equation, we obtain:

$$|e_j^{k+1}| \le |e_j(t)| + \frac{\tau|j|}{\sqrt{1 + (\tau j^3/2)^2}}|\tilde{\Psi}_j| + \tau|\tilde{f}_j(t)|.$$

Summing up the above inequalities for all $|j| \le N$, we have

$$\begin{aligned} |e^{k+1}|_\infty &\le |e(t)|_\infty + \tau \sum_{|j|\le N} \frac{|j|}{\sqrt{1+(\tau j^3/2)^2}} |\tilde{\Psi}_j| + \tau|\tilde{f}(t)|_\infty \\ &\le |e(t)|_\infty + \tau \left[\sum_{|j|\le N} \frac{j^2}{1+(\tau j^3/2)^2}\right]^{1/2} \left[\sum_{|j|\le N} |\tilde{\Psi}_j|^2\right]^{1/2} + \tau|\tilde{f}(t)|_\infty \\ &\le |e(t)|_\infty + \tau \left[\sum_{|j|\le N} \frac{j^2}{1+(\tau j^3/2)^2}\right]^{1/2} \|\tilde{\Psi}\|^2 + \tau|\tilde{f}(t)|_\infty . \end{aligned} \qquad (3.13)$$

It is easy to see that:

$$\sum_{|j|\le N} \frac{j^2}{1 + (\tau j^3/2)^2} \le c_4 \int_0^\infty \frac{\xi^2}{1 + (\tau \xi^3/2)^2} d\xi \le c_5 \tau^{-1} .$$

On the other hand, by (3.6) and the induction assumption (3.10), we get that:

$$\begin{aligned} \|\tilde{\Psi}\| &\le \|\Phi_v{}'\,(\hat{U}^N(t)+\theta\tfrac{e^k+e(t)}{2})\|_{L^\infty}\|\tfrac{e^k+e(t)}{2}\| \\ &\le c_6(\|e^k\|+\|e(t)\|) \\ &\le c_6(|e^k|_\infty+|e(t)|_\infty)\ , \end{aligned}$$

where c_6 is a positive constant depending only on $\|\hat{U}^N(t)\|_{L^\infty}$ and M_0 . Hence (3.13) implies that:

$$|e^{k+1}|_\infty \le c_7\tau^{1/2}|e^k|_\infty + R(t)\ , \tag{3.14}$$

where

$$R(t) = (1+c_7\tau^{1/2})|e(t)|_\infty + \tau|\tilde{f}(t)|_\infty\ .$$

By (3.5), we have

$$\tau\|\tilde{f}(t)\| \le \frac{M_0N^{-1/2}}{2\sqrt{3}}\tau^{1/2}\ .$$

Assume now N is sufficiently large that the following is valid:

$$c_7\tau^{1/2} + \frac{N^{-1/2}\sqrt{2N+1}}{2}(1+c_7\tau^{1/2}+\frac{\tau^{1/2}}{\sqrt{3}}) \le 1\ .$$

Thus it follows from (3.9),(3.11),(3.14) and the induction assumption that:

$$\begin{aligned} |e^{k+1}|_\infty &\le c_7\tau^{1/2}M_0 + \sqrt{2N+1}\left[(1+c_7\tau^{1/2})\|e(t)\| + \tau\|\tilde{f}(t)\|\right] \\ &\le \{c_7\tau^{1/2} + \tfrac{N^{-1/2}\sqrt{2N+1}}{2}(1+c_7\tau^{1/2}+\tfrac{\tau^{1/2}}{\sqrt{3}})\}M_0 \\ &\le M_0\ , \end{aligned}$$

which completes the induction.

Lemma 2. Suppose $\tau = o(N^{-1/2})$ and N is sufficiently large, (3.5) and (3.6) hold. Then the iteration sequence $\{z^k\}$, defined in (3.7), satisfies

$$|z^k - U^N(t+\tau)|_\infty \le M_0\ , \quad k=0,1,2,\cdots . \tag{3.15}$$

Next we prove the convergence of the sequence $\{z^k\}$. By the iteration (3.7), we have

$$\frac{z^{k+1}-z^k}{\tau} + P_N\left[\Phi(\frac{z^k+u(t)}{2}) - \Phi(\frac{z^{k-1}+u(t)}{2})\right]_x + (\frac{z^{k+1}-z^k}{2})_{xxx} = 0\ , \quad k=0,1,2,\cdots .$$

Similarly to the derivation of (3.13), we arrive at:

$$|z^{k+1}-z^k|_\infty \le \tau\left(\sum_{|j|\le N}\frac{j^2}{1+(\tau j^3/2)^2}\right)^{1/2}\|\tilde{\Psi}^{(k)}\| \le c_5\tau^{1/2}\|\tilde{\Psi}^{(k)}\|\ , \tag{3.16}$$

where $\tilde{\Psi}^{(k)} = \Phi(\frac{z^k+u(t)}{2}) - \Phi(\frac{z^{k-1}+u(t)}{2})$. It can be bounded as:

$$\begin{aligned} \|\tilde{\Psi}^{(k)}\| &\le \|\Phi_v{}'\,(\tfrac{z^{k-1}+u(t)}{2}+\theta_1\tfrac{z^k-z^{k-1}}{2})\|_{L^\infty}\|\tfrac{z^k-z^{k-1}}{2}\| \\ &\le c_8\|z^k-z^{k-1}\| \end{aligned}$$

where c_8 depends only on $||U^N(t)||_{L^\infty}$,$||U^N(t+\tau)||_{L^\infty}$ and M_0. Thus we have from (3.16) that

$$|z^{k+1} - z^k|_\infty \le c_9 \tau^{1/2} |z^k - z^{k-1}|_\infty \ .$$

From the above inequality we conclude the convergence of $\{z^k\}$. We take $u(t+\tau) = \lim_{k\to\infty} z^k$, which is a solution to (2.1) and satisfies:

$$||e(t+\tau)||_{L^\infty} \le |e(t+\tau)|_\infty = \lim_{k\to\infty} |e^k|_\infty \le M_0 \ .$$

Lemma 3. If $\tau = o(N^{-1/2})$ and N is sufficiently large. Equations (3.5) and (3.6) are fulfilled. Then the iteration sequence $\{z^k\}$ converges to $u(t+\tau)$ with a compression factor $O(\tau^{1/2})$. Furthermore, $u(t+\tau)$ thus defined satisfies:

$$||u(t+\tau) - U^N(t+\tau)||_{L^\infty} \le M_0 \ .$$

Remark 1. In [7] the convergence property of the iteration procedure (3.7) was also investigated. However, the compression factor there obtained, which is of order $O(\tau^{\frac{2}{3}})$, is actually for each Fourier coefficient of $\{z^k\}$. The convolution effects of the nonlinear term were clearly neglected.

By the above lemmas, we can readily obtain the global existence and convergence.

Theorem 1. Suppose $\tau = o(N^{-1/2})$ and N is sufficiently large. (3.5) holds with a positive constant M_0. Then:

(*i*) there exists a solution to scheme (2.1) for each time level $t \in S_\tau$;

(*ii*) the iteration (3.7) for solving (2.1) possesses a compression factor of order $O(\tau^{1/2})$ in norm $|\cdot|_\infty$;

(*iii*) the solution obtained by (3.7) satisfies the error estimate

$$||u(t) - U^N(t)|| \le \rho(0, \tilde{f}, t) e^{c_4 t} \ , \quad \forall t \in S_\tau \ .$$

Remark 2. If the exact solution U of (1.1) is suitably smooth, *e.g.*, $U \in C^4(0,T;L^2) \bigcap C^2(0,T;H^{s+1})$, then we can easily show that $\rho(0,\tilde{f},T) = O(\tau^4 + N^{-2s})$, which implies the overall convergence rate for $u(t)$ is $O(\tau^2 + N^{-s})$.

ACKNOWLEDGEMENT

This work is supported by the Natural Science Foundation of China.

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Paper Received 13 January 1997; Revised 11 August 1997; Accepted 18 October 1997.