APPROXIMATE SOLUTIONS OF GENERAL NONLINEAR BOUNDARY VALUE PROBLEMS USING SUBDIVISION TECHNIQUES

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الخلاصـة :

نستخدم مجموعـة دوال أُنتجتْ باستخدام منوال تقسيمي منتظم لتوصيف منوال علـى درجـة عالية من الدقة لحساب حلٍّ تقريبي للمعضلة العاملة للشروط الحدّية لمعادلة تفاضلية ذات نقطتين حدّيتيْن. وهذا الأسلوب مختلف عن طريقة الفروق الحدية وعن طريقة العناصر الحدية إذْ يُـنتج حلاً تقريبياً غير مُـتعدد الحدود / وغير لُسَـيني، لكنه تفاضلي مُتصل. أما الأفكار الرئيسة في هذا البحث فهي: إنتاج دوال أُساسية، وتنظيـم الـعُـقد، ومعالجة الحدود. ثم نورد عدة أُمثلة حسابيـة لأنماط مختلفة لمعاضل غير خطية ثنائية الشرط الحدي للدلالة على سرعة تقارب ودقة المنـوال المقترح. ويعتبر هذا البحث استكمالاً لعمل سابق يُعنى بحل معاضل القيمة الحدية الخطية ومعاضل القيمة الحدية ذات إزاحة (argument) منحرفة.

ABSTRACT

A special class of basis functions generated by uniform subdivision algorithms is used to formulate a high accuracy algorithm for the computation of approximate solutions of general two point boundary value problems of differential equations with or without deviating arguments. This approach, which is different from the traditional finite difference or finite element method, produces non-polynomial/non-spline type, but continuous and differentiable approximate solutions to the boundary value problems provided the parameters of the algorithm are chosen appropriately. The main ideas of the method are generation of basis functions, node collocation, and boundary treatments. Numerical examples of various types of non-linear two-point boundary value problems are included to show the fast convergence and high accuracy of the algorithm. This paper is a further development of our previous work for solving linear boundary value problems and boundary value problems with deviating arguments.

Keywords: Boundary value problem, approximate solution, subdivision algorithm, collocation method, refinable function.

AMS(MOS) Classification: 65L10, 65L60.

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1. INTRODUCTION

In our previous work [1, 2], we constructed a numerical algorithm to solve linear boundary value problems and non-linear boundary value problems with deviating arguments. It has been found that this method produces quite good results when compared with the finite difference and spline methods. Therefore, in this paper, we consider the following general second-order nonlinear differential equation with a deviating argument $g(t)$:

$$\begin{cases} a(t)x''(t) + b(t)x'(t) = f(t, x(t), x(g(t)), x'(t)), & t \in (t_a, t_b), \\ x(t) = \psi_0(t), \quad t \le t_a \quad \text{and} \quad x(t) = \psi_1(t), & t \ge t_b, \end{cases} \tag{1.1}$$

where the coefficients $a(t)$, $b(t)$, the so-called *deviating argument* $g(t)$, and the boundary functions $\psi_0(t)$ and $\psi_1(t)$ are all known continuous functions and, we also assume that $f \in C([t_a, t_b] \times R^3, R)$, $g \in C[t_a, t_b]$. Furthermore, we define:

$$c := \min\left\{ \inf_{t_a \le t \le t_b} g(t),\ t_a \right\}, \qquad d := \max\left\{ \sup_{t_a \le t \le t_b} g(t),\ t_b \right\}. \tag{1.2}$$

If $c = t_a$ and/or $d = t_b$ then, ψ_0 and/or ψ_1 are interpreted as constants. It should be noted that this BVP (1.1) may have a singularity since $a(t)$ can have a zero in the interval $[t_a, t_b]$. A function $x \in C[c, d] \cap C^2[t_a, t_b]$ is called a solution of the BVP (1.1) if it satisfies all the equations in (1.1).

Due to the non-linear and/or deviating nature of the BVP (1.1), it is well-known that the construction of an exact solution of (1.1) is much more difficult than for the boundary-value problems of ordinary differential equations. As a result, there are many papers discussing the solutions of this type of problems [3–11]. Even when the differential equation in (1.1) is linear, its solution is not simply some linear combination of a particular solution with a nontrivial solution of the homogeneous equation. This is because, in general, the space of linearly independent solutions of homogeneous equations is of infinite dimension. Therefore, the practical shooting-type methods discussed in [12] could be ineffective. Based on a theory of differential inequalities De Nevers and Schmitt in [9] demonstrated that the shooting method could be applied provided that $g(t) \le t$, *i.e.*, Equation (1.1) is only of *delay type*. Another difficulty in developing a numerical procedure to solve the BVP (1.1) numerically is the fact that in general the solutions are only of class $C^2[t_a, t_b]$ (cf. [3, 10, 11, 13]). Thus, the well known spline method could introduce unnecessary restrictions. Having these limited continuity assumptions, Reddien and Travis introduced some projection type methods by using polynomial splines [10]. Agarwal and Chow in [13] investigated the problem from a finite difference approach and derived an $\mathcal{O}(h^2)$ algorithm for the numerical computation of its approximate solutions in which both the approximation and the convergence of the algorithm were discussed. Due to the deviating property of the problem, all these numerical algorithms can be used to compute some discrete values of the approximate solutions.

In this paper, the collocation method developed for solving linear two point boundary value problems in [1, 2] is generalized to find approximate solutions of problem (1.1). The main advantages of using this collocation method are the continuity and finite differentiability of the approximation solutions (C^2), its local approximation property, and its high order of convergence. In fact, it has been proven that if a quintic approximation method with proper boundary treatments is used, then the order of approximation is $\mathcal{O}(h^4)$ provided that the functions

$a(t)$, $b(t)$, $g(t)$, $f(t,u,v,w)$, and the exact solution $x(t)$ of problem (1.1) is C^6. However, if $x(t)$ is C^2, then the order of approximation of the method is reduced to the worst case $\mathcal{O}(h^3)$.

The paper is organized as follows. In Section 2, some results about subdivision algorithms and the basis functions are reviewed. In Section 3, a numerical method to solve (1.1) using the refinable basis functions is formulated and its approximation and convergence property discussed. The approximation properties of the algorithm is given in Section 4. Some numerical examples are given in Section 5.

2. PRELIMINARIES

2.1. Existence and Uniqueness of Solutions

From the papers [8–11, 13], we know that the boundary value problem (1.1) has a unique solution $x(t)$ if the right hand side function $f(t,u,v,w)$ satisfies the Lipschitz continuity condition for the variables u, v, and w, and all the corresponding Lipschitz constants are small enough. More details about the results on the existence and uniqueness of solutions of (1.1) can be found in these papers. For convenience, in the following sections, we assume that the BVP problem (1.1) has a unique C^2 solution.

It should be noted that if

$$a(t_a) = 0, \quad a(t_b) = 0 \quad \text{or} \quad a(t_c) = 0, \quad t_c \in (t_a,\ t_b)$$

then, the BVP (1.1) has a singularity at the corresponding point t_a, t_b, or t_c. Therefore, special treatment at this singular point is necessary. More details about the results on the existence and uniqueness of solutions of singular BVP can be found in [6]. Our treatments at the singular point for approximate solutions will be discussed in Section 3.

2.2. The Approximate Function

The following is an important result on the basis functions that are used to construct the approximate solutions of problem (1.1). A proof of it and a detailed description of these functions generated by uniform subdivisions can be found in [14–16] and the references therein.

Proposition 2.1. Let $\phi(t)$, $t \in \mathbf{R}$, be the fundamental solution of the two-scale functional equation:

$$\begin{aligned} \phi(t) = \phi(2t) + \frac{1}{256}\Big\{ & 150(\phi(2t-1) + \phi(2t+1)) \\ & - 25(\phi(2t-3) + \phi(2t+3)) \\ & + 3(\phi(2t-5) + \phi(2t+5))\Big\}, \quad t \in \mathbf{R}. \end{aligned} \tag{2.1}$$

Then $\phi(t) \in C^2(\mathbf{R})$ and it can be generated by a stepwise interpolatory subdivision algorithm and

$$\phi(t) = \phi(-t), \qquad \phi(t) = 0, \quad t \notin (-5,5), \qquad \phi(i) = \delta_{0,i}, \quad i \in \mathbf{Z}. \tag{2.2}$$

Furthermore, the derivatives of $\phi(t)$ at integer points have the following values:

$$\begin{cases} \phi'(0) = 0, & \phi''(0) = -\frac{295}{56}, \\ \phi'(\pm 1) = \mp\frac{272}{365}, & \phi''(\pm 1) = \frac{356}{105}, \\ \phi'(\pm 2) = \pm\frac{53}{365}, & \phi''(\pm 2) = -\frac{92}{105}, \\ \phi'(\pm 3) = \mp\frac{16}{1095}, & \phi''(\pm 3) = \frac{4}{35}, \\ \phi'(\pm 4) = \mp\frac{1}{2920}, & \phi''(\pm 4) = \frac{3}{560}. \end{cases} \tag{2.3}$$

From Equation (2.1) and the above values of $\phi^{(m)}(i)$, $m = 0,1,2$, $i \in \mathbf{Z}$, all the values of $\phi^{(m)}(t)$, $m = 0,1,2$, at dyadic points $\{\pm i2^{-k},\ i,k \in \mathbf{Z}_+\}$ can be obtained by using the corresponding subdivision process (2.4) below. ■

Proposition 2.2. Let $Y_i^k := \phi(i2^{-k})$, $i \in \mathbf{Z}$, $k = 0,1,2,3,\cdots$, then, $\{Y_i^k\}$ satisfy the following stepwise interpolatory subdivision algorithm:

$$\begin{cases} Y_{2i}^{k+1} = Y_i^k, \\ Y_{2i+1}^{k+1} = \frac{150}{256}\left(Y_i^k + Y_{i+1}^k\right) - \frac{25}{256}\left(Y_{i-1}^k + Y_{i+2}^k\right) \\ \qquad\qquad + \frac{3}{256}\left(Y_{i-2}^k + Y_{i+3}^k\right). \end{cases} \tag{2.4}$$

It has been shown that the subdivision algorithm (2.4) produces C^2 interpolatory functions. The subdivision process to approximate the function $\phi(t)$ by a sequence of piecewise linear functions $\{\phi^k(t)\}$ is shown in Figure 1. ■

Remark 2.3. Other similar interpolatory basis functions of different smoothness that are defined by stationary subdivision algorithms can be used to solve higher order boundary value problems [14, 15]. It is known that these functions are closed related to wavelets.

3. APPROXIMATE SOLUTIONS BY THE COLLOCATION METHOD

3.1. The Collocation Method

For convenience of mathematical formulation, we assume, without loss of generality, that the interval of interest $[t_a,\ t_b]$ is the standard interval:

$$t_a := 0, \qquad t_b := 1.$$

Let N be a positive integer ($N \geq 4$) and $h := \frac{1}{N}$. Let $t_i := ih$, $i = 0,1,2,\cdots,N$, be a uniform partition of the interval [0, 1], and let

$$z(t) := \sum_{i=-4}^{N+4} z_i \phi\left(\frac{t-t_i}{h}\right), \qquad 0 \leq t \leq 1 \tag{3.1}$$

be an approximate solution of (1.1), where $\{z_i\}$ are the unknowns to be determined by (1.1) and some extrapolation methods to be described later. The collocation method, together with the boundary conditions and some boundary treatments to be discussed, is given by setting

$$a(t_i)z''(t_i) + b(t_i)z'(t_i) = f(t_i, z_i, y_i, z_i'), \quad i = 0,1,2,\cdots,N \tag{3.2}$$

where for convenience the following notations are used:

$$z_0 = x_0 = \psi_0(0), \quad z_N = x_N = \psi_1(1), \quad z_i' = z'(t)|_{t=t_i},$$

$$y_i = \begin{cases} \psi_0(g(t_i)) & \text{if} \quad g(t_i) \leq 0, \\ \psi_1(g(t_i)) & \text{if} \quad g(t_i) \geq 1, \\ z(g(t_i)) & \text{if} \quad 0 < g(t_i) < 1. \end{cases} \tag{3.2a}$$

The relation (3.2) is equivalent to the following nonlinear system of $N+1$ equations with $(N+9)$ unknowns $\{z_i\}$:

$$\mathbf{Az} = \mathbf{F}(\mathbf{z}) \tag{3.3}$$

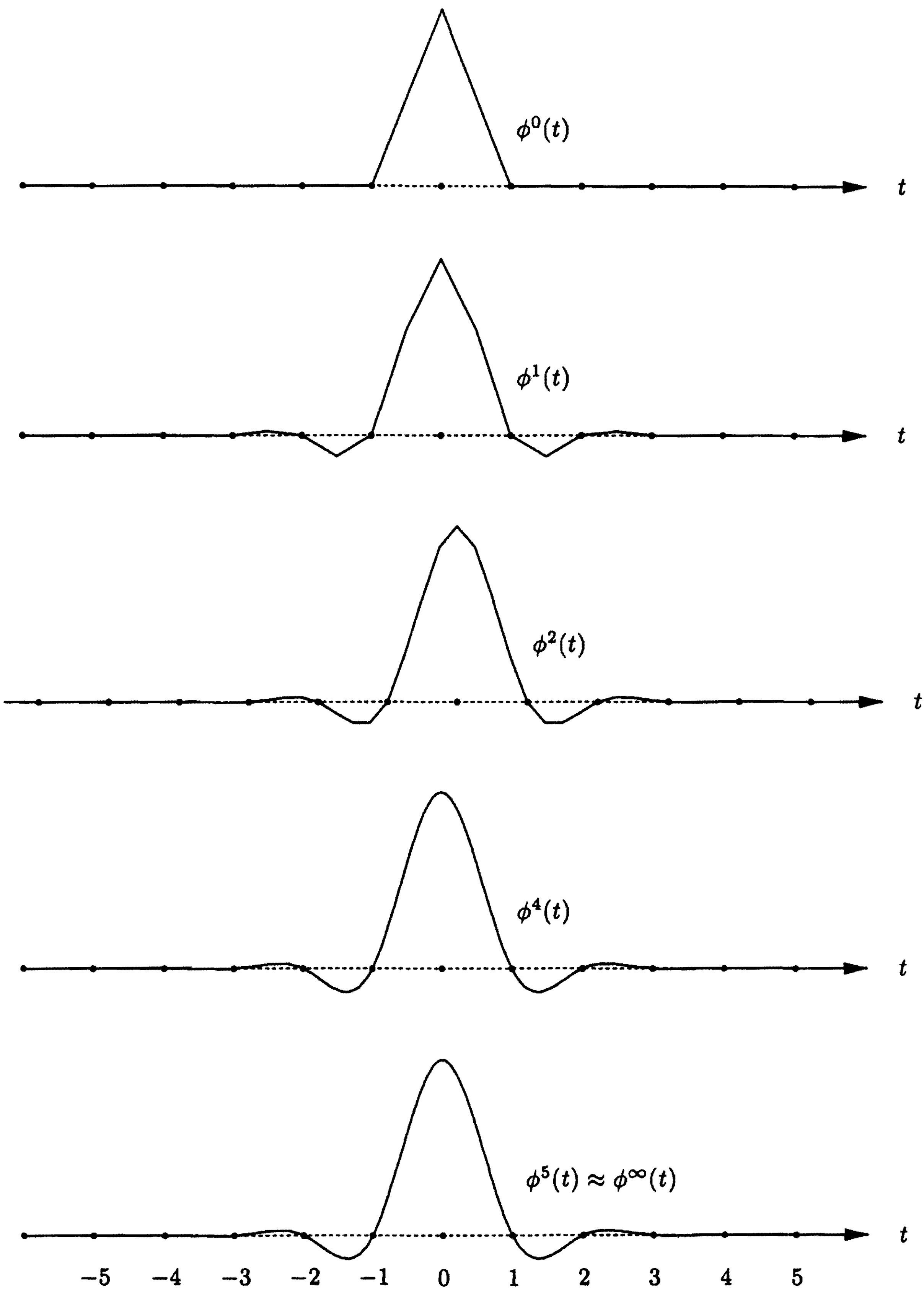

Figure 1. The Generation of the Basis Function $\phi(t) := \phi^\infty(t)$ by Subdivision.

where $\mathbf{A}$ is a banded matrix of order $(N+1)\times(N+9)$, $\mathbf{z}$ is the unknown vector of order $N+9$ and, $\mathbf{F}(\mathbf{z})$ is the right hand side vector of order $N+1$ dependent of $\mathbf{z}$. The matrix $\mathbf{A}$ and vectors $\mathbf{z}$ and $\mathbf{F}(\mathbf{z})$ are defined as

$$\mathbf{A} := \mathbf{P}_a\mathbf{A}_2 + h\mathbf{P}_b\mathbf{A}_1 \tag{3.4}$$

where $\mathbf{P}_a, \mathbf{P}_b \in \mathbf{R}^{(N+1)\times(N+1)}$ are diagonal matrices and $\mathbf{A}_1, \mathbf{A}_2 \in \mathbf{R}^{(N+1)\times(N+9)}$. Explicitly, these matrices and vectors are given by:

$$\mathbf{P}_a := \mathrm{Diag}(a_0, a_1, \cdots, a_{N-1}, a_N); \quad \mathbf{P}_b := \mathrm{Diag}(b_0, b_1, \cdots, b_{N-1}, b_N);$$

$$\mathbf{A}_2 := \begin{pmatrix} \phi''_4 & \phi''_3 & \phi''_2 & \phi''_1 & \phi''_0 & \phi''_{-1} & \phi''_{-2} & \cdots & 0 & 0 & 0 \\ 0 & \phi''_4 & \phi''_3 & \phi''_2 & \phi''_1 & \phi''_0 & \phi''_{-1} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \phi''_4 & \phi''_3 & \phi''_2 & \phi''_1 & \phi''_0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi''_4 & \phi''_3 & \phi''_2 & \phi''_1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \phi''_{-3} & \phi''_{-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \phi''_{-2} & \phi''_{-3} & \phi''_{-4} \end{pmatrix}; \tag{3.4a}$$

$$\mathbf{A}_1 := \begin{pmatrix} \phi'_4 & \phi'_3 & \phi'_2 & \phi'_1 & \phi'_0 & \phi'_{-1} & \phi'_{-2} & \cdots & 0 & 0 & 0 \\ 0 & \phi'_4 & \phi'_3 & \phi'_2 & \phi'_1 & \phi'_0 & \phi'_{-1} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \phi'_4 & \phi'_3 & \phi'_2 & \phi'_1 & \phi'_0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi'_4 & \phi'_3 & \phi'_2 & \phi'_1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \phi'_{-3} & \phi'_{-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \phi'_{-2} & \phi'_{-3} & \phi'_{-4} \end{pmatrix};$$

$$\begin{cases} \mathbf{z} := (z_{-4}, \ z_{-3}, \ z_{-2}, \ \cdots, \ z_{N+4})^T; \\ \mathbf{F}(\mathbf{z}) := (h^2 f(t_0, z_0, y_0, z'_0), \cdots, h^2 f(t_N, z_N, y_N, z'_N))^T \end{cases} \tag{3.4b}$$

where the following notations for $j \in \mathbf{Z}$ are used:

$$\begin{cases} \phi''_j := \phi''(j); \quad \phi'_j := \phi'(j); \quad z_j := z(t_j); \quad y_j := z(g(t_j)); \\ z'_j := z'(t_j) = \dfrac{1}{8760h}\Big(- 6528(z_{j+1} - z_{j-1}) + 1272(z_{j+2} - z_{j-2}) \\ \qquad\qquad - 128(z_{j+3} - z_{j-3}) - 3(z_{j+4} - z_{j-4})\Big). \end{cases} \tag{3.4c}$$

3.2. Boundary and Singularity Treatments

To obtain a unique approximate solution of (1.1) from (3.3), we have to use the given boundary conditions (cf. (1.1)):

$$z_0 = x_0 = \psi_0(0), \qquad z_N = x_N = \psi_1(1). \tag{3.5}$$

It should be noted that, for singular problems with $a(0) = 0$, for example, the boundary condition $z'_0 = 0$ will be exactly the same as the first equation in (3.3). Therefore an independent equation for the approximation of z_0 has to be formulated so that a proper closed form of non-linear system can be formed. By using different extrapolation methods, the following formulae for coping with approximation and singularity are proposed in [1, 6, 17, 18]:

$$\begin{cases} z_0 := \dfrac{1}{75}\Big(144z_1 - 108z_2 + 48z_3 - 9z_4\Big) - \dfrac{12}{125}z^{(5)}(\xi)h^5; \\ z_0 := \dfrac{1}{147}\Big(360z_1 - 450z_2 + 400z_3 - 225z_4 + 72z_5 - 10z_6\Big) \\ \qquad - \dfrac{20h^7}{343}z^{(7)}(\xi). \end{cases} \tag{3.5a}$$

A similar technique is applied at all the other singular node and/or points to obtain consistent solutions. It is clear that the first formula in (3.5a) is of order 5 and the second one is of order 7. Thus, in order to keep our method uniform, the second formula is recommended for practical applications since our method is an $\mathcal{O}(h^6)$ method.

A general method to accommodate this under-determined system (3.3) and (3.5) is extrapolation at the end points by use of the Lidstone polynomial interpolation formulae of degree 3 or 5 or above. Therefore, some boundary treatments, for example, the three boundary treatments BT1, BT2, and BT3 of order six at each of the end points should be used (cf. [1, 2]). If the BT1 treatment of *extrapolation of* $z(t)$ is used, then the following 6 boundary equations are obtained in [1]

$$\begin{cases} z_{-3} - 6z_{-2} + 15z_{-1} - 20z_0 + 15z_1 - 6z_2 + z_3 = 0, \\ z_{-2} - 6z_{-1} + 15z_0 - 20z_1 + 15z_2 - 6z_3 + z_4 = 0, \\ z_{-1} - 6z_0 + 15z_1 - 20z_2 + 15z_3 - 6z_4 + z_5 = 0, \\ z_{N-5} - 6z_{N-4} + 15z_{N-3} - 20z_{N-2} + 15z_{N-1} - 6z_N + z_{N+1} = 0, \\ z_{N-4} - 6z_{N-3} + 15z_{N-2} - 20z_{N-1} + 15z_N - 6z_{N+1} + z_{N+2} = 0, \\ z_{N-3} - 6z_{N-2} + 15z_{N-1} - 20z_N + 15z_{N+1} - 6z_{N+2} + z_{N+3} = 0. \end{cases} \tag{3.5b}$$

Combining (3.3) with (3.5), (3.5b) (and a proper one from (3.5a) for problems with a singularity), a nonlinear system of $N+9$ equations with $N+9$ unknowns $\{z_i\}$ is formed, in which $N+1$ equations are from (3.3), one or two equations from the given boundary conditions (3.5), six from (3.5b), and if necessary, one or more from the proposed formula (3.5a). This nonlinear system is denoted by

$$\mathbf{B}\mathbf{z} = \mathbf{R}(\mathbf{z}) \tag{3.6}$$

where, for non-singular problems, the matrix $\mathbf{B}$ and the right hand side vector $\mathbf{R}(\mathbf{z})$ are given by:

$$\mathbf{R}(\mathbf{z}) := (0,0,0,x_0,\mathbf{F}^T(\mathbf{z}),x_N,0,0,0)^T, \quad \mathbf{B} := (\mathbf{B}_0^T,\ \mathbf{A}^T,\ \mathbf{B}_1^T)^T. \tag{3.6a}$$

In (3.6a) the matrix $\mathbf{A}$ is defined by (3.4) and the matrices $\mathbf{B}_0, \mathbf{B}_1 \in \mathbf{R}^{4\times(N+9)}$ are formed by (3.5) and (3.5b), and appear as:

$$\mathbf{B}_0 := \begin{pmatrix} 0 & 1 & -6 & 15 & -20 & 15 & -6 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -6 & 15 & -20 & 15 & -6 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -6 & 15 & -20 & 15 & -6 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & -6 & 15 & -20 & 15 & -6 & 1 & \cdots & 0 \end{pmatrix};$$

$$\mathbf{B}_1 := \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 1 & -6 & 15 & -20 & 15 & -6 & 1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & -6 & 15 & -20 & 15 & -6 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -6 & 15 & -20 & 15 & -6 & 1 & 0 \end{pmatrix}.$$

For problems with one or two singular points at the end points, the last row of $\mathbf{B}_0$ and/or the first row of $\mathbf{B}_1$ should be modified accordingly by use of (3.5a). The solvability of the nonlinear system (3.6) and an iterative algorithm for solving it are discussed next.

Remark 3.1. Other boundary treatments BT2 or BT3 in [1, 2] may also be used. For compatibility, sixth order approximations at the ends and singular points as well are recommended to form a closed system similar to (3.6). However, for different boundary conditions and boundary treatments, the matrices $\mathbf{B}_0$ and $\mathbf{B}_1$ in (3.6) and the vector $\mathbf{F}(\mathbf{z})$ should be modified accordingly.

3.3. Existence of the Approximate Solutions

From intensive computations and numerical experiments, the above matrix $\mathbf{B}$ (cf. (3.6)) appears to be non-singular and relatively well conditioned in all our cases. For large N, *i.e.*, small mesh size h, the matrix $\mathbf{B}$ is dominated by the first part $\mathbf{P}_a\mathbf{A}_2$. Therefore, similar to the discussions in [1, 2], $\mathbf{B}$ can be approximated by the main part of it, *i.e.*, the square matrix $\mathbf{C}$ and $\mathbf{D}$ of order $(N+1)$:

$$\mathbf{C} := \mathbf{P}_a\Phi_2, \quad \mathbf{D} := \mathbf{P}_a\Phi_2 + h\mathbf{P}_b\Phi_1 \tag{3.7}$$

where Φ_1 and Φ_2 are square matrices determined by the basis function $\phi(t)$ only:

$$\Phi_1 := \{\phi'_{i-j}\} \in \mathbf{R}^{(N+1)\times(N+1)}, \qquad \Phi_2 := \{\phi''_{i-j}\} \in \mathbf{R}^{(N+1)\times(N+1)}. \tag{3.7a}$$

That is, $\mathbf{D}$ is the sub-matrix formed from $\mathbf{B}$ by deleting the first and last four rows and columns. Thus all the boundary and singularity treatments at the end points discussed above do not have any influence on $\mathbf{C}$ and $\mathbf{D}$.

Remark 3.2. From our numerical experiments, the matrices $\mathbf{C}$ and $\mathbf{D}$ are always non-singular, even for the singular BVP with $a(t) = t^\gamma(1-t)^\delta$, $\gamma,\ \delta \in [0,\ 1]$, and $b(t) \equiv \alpha$, $\alpha \in \mathbf{R}$. In such a case, the condition numbers of $\mathbf{C}$ and $\mathbf{D}$ increase as N increases. In fact, for large N, the matrix $\mathbf{B}$ is very similar to $\mathbf{C}$ and $\mathbf{D}$. By assuming the type one boundary treatments (3.5b), the following estimates for the condition numbers of $\mathbf{D}$ have been obtained:

$$\mathrm{Cond}_2(\mathbf{C}) < \mathrm{Cond}_2(\mathbf{D}) \approx \frac{1}{10}(6\beta^2 + 9\beta + 14)N^2, \qquad \beta := \max\{\gamma,\ \delta\}. \tag{3.8}$$

Remark 3.3. For the singular case when $a(t) = t^\gamma$, $\gamma \in [0,\ 1]$ and $b(t) \equiv 1$, $t \in [0,\ 1]$, by intensive numerical computations for many different values of γ and N , we have obtained the following estimates for the coefficient matrix $\mathbf{B}$ for large N

$$\begin{cases} \|\mathbf{B}^{-1}\|_\infty & \approx 25(\gamma+1)\|\mathbf{D}^{-1}\|_\infty & \approx 50(\gamma+1)^2N^2; \\ \mathrm{Cond}_2(\mathbf{B}) & \approx 10(\gamma+1)\mathrm{Cond}_2(\mathbf{D}) & \approx 10(\gamma+1)^2N^2. \end{cases} \tag{3.8a}$$

From the assumption that the BVP (1.1) has a unique solution and that $\mathbf{B}$ is non-singular and the fact that the right hand side vector $\mathbf{R}(\mathbf{z})$ is a multiple of h^2 (excluding the boundary conditions (3.5) and boundary/singularity treatments), the existence of a solution of (3.6) can be established.

Proposition 3.4. Suppose the coefficient matrix $\mathbf{B}$ is nonsingular for $h \to 0$, then the non-linear system (3.6) of the collocation method has a unique solution $\mathbf{z}^*$ provided that all the partial derivatives $\|f_u\|_\infty$, $\|f_v\|_\infty$, $\|f_w\|_\infty$ and the mesh size h are small enough. ■

3.4. The Iterative Algorithm and its Convergence

In order to find the approximate solution (3.1), the values of $\{z_i\}$ must be computed from the nonlinear system (3.6). The following iterative algorithm is proposed to obtain a numerical solution of this system (3.6) which is composed of three steps listed below.

(i). Initial Approximation to $\mathbf{z}^*$

The initial approximating solution $\mathbf{z}^0$ is chosen to be the solution of the following linear system

$$\mathbf{B}\mathbf{z} = \mathbf{F}^0, \tag{3.9}$$

where for a regular BVP with BT1 treatments

$$\begin{cases} \mathbf{F}^0 := (0,0,0,x_0,f_0,\cdots,f_N,x_N,0,0,0,)^T, & \\ f_i := h^2 f(t_i,l_i,y_i,d_i), & i=0,1,2,\cdots,N, \\ l_i := x_0 + ih(x_N - x_0), & i=0,1,2,\cdots,N, \\ y_i := \tilde{z}(g(t_i)), & i=0,1,2,\cdots,N, \\ d_i := \tilde{z}'(t_i), & i=0,1,2,\cdots,N, \\ \tilde{z}(t) := \sum_{j=-4}^{N+4} l_j \phi(\frac{t-t_j}{h}). & \end{cases} \tag{3.9a}$$

The linear system (3.9) comes from the linear approximation of function $\mathbf{R}(\mathbf{z})$ using (3.5). For other boundary/singularity treatments, the above $\mathbf{F}^0$ and $\mathbf{B}$ should be modified accordingly. For example, for BVPs with a singularity at $t = 0$, and the corresponding boundary conditions at this ends being given by

$$z'(0) = 0, \qquad z(1) = x_N, \tag{3.9b}$$

the value $\{l_i\}$ in (3.9a) should be replaced by

$$l_i := x_N, \qquad i = 0,1,2,\cdots,N \tag{3.9c}$$

and all the others are kept the same. For BVPs with singularities at $t = 0$ and $t = 1$, and the corresponding boundary conditions at this ends being given by

$$z'(0) = 0, \qquad z'(1) = 0, \tag{3.9d}$$

the value $\{l_i\}$ in (3.9a) should be replaced by zero or

$$l_i := \tfrac{1}{2}(\psi_0(0) + \psi_1(1)), \qquad i = 0,1,2,\cdots,N \tag{3.9e}$$

if the right hand side of (3.9e) is well defined, and all the others in (3.9a) are kept the same.

(ii). Iterative Procedure for Refinement

The subsequent approximations $\{\mathbf{z}^{(k)}\}$ to the solution $\mathbf{z}^*$ of (3.6) are obtained by using the following simple iteration scheme, which in fact is solving similar linear systems:

$$\mathbf{B}\mathbf{z}^{(k+1)} = \mathbf{R}(\mathbf{z}^{(k)}), \qquad k = 0,1,2,\cdots \tag{3.10}$$

where for all $k = 0,1,2,3,\cdots$,

$$\begin{cases} \mathbf{z}^{(k)} := (z^{(k)}(t_{-4}), z^{(k)}(t_{-3}), \cdots, z^{(k)}(t_{N+4}))^T, \\ z^{(k)}(t) := \sum_{j=-4}^{N+4} z_j^{(k)} \phi(\frac{t-t_j}{h}). \end{cases} \tag{3.10a}$$

(iii). Stopping Criteria

For a given tolerance ϵ_{tol}, the iterative process (3.10) stops at the k-th iteration if any one of the following conditions is satisfied

$$\begin{cases} C_1: & \|\mathbf{z}^{(k-1)} - \mathbf{z}^{(k)}\|_\infty & \leq \epsilon_{tol}; \\ C_2: & \|\mathbf{B}\mathbf{z}^{(k)} - \mathbf{R}(\mathbf{z}^{(k)})\|_\infty & \leq \epsilon_{tol}; \\ C_3: & \|\mathbf{B}\mathbf{z}^{(k)} - \mathbf{B}\mathbf{z}^{(k-1)}\|_\infty & \leq \epsilon_{tol}; \\ C_4: & \|\mathbf{z}^{(k-1)} - \mathbf{z}^{(k)}\|_\infty & \leq \|\mathbf{z}^{(k)}\|_\infty \epsilon_{tol}. \end{cases} \tag{3.10a}$$

Remark 3.5. If the sequence $\{\mathbf{z}^{(k)}\}$ generated by the iterative scheme (3.9) and (3.10) converges to $\mathbf{z}^*$, then the limit $\mathbf{z}^*$ satisfies (3.6). However, it should be noted that even for well-posed BVPs, the convergence of the iterative process is not guaranteed unless $\mathbf{B}$ is nonsingular and h is small.

Remark 3.6. In the iteration process (3.9) and (3.10) for solving $\mathbf{z}^{(k)}$, the exact value of the deviating argument $y_i^k = z^k(g(t_i))$ are used for all i since $g(t)$ and $\phi(t)$ are defined for all values. Therefore, when convergence occurs, $z(t)$ will satisfy the discrete system (3.2). That is, there is no discritization error in the collocation method (3.2), which is different from the finite difference methods [9, 12, 13, 17].

4. ERROR ESTIMATES

First, the following result on the approximation properties of $\phi(t)$ is needed. A proof of it can be found in [1, 14].

Proposition 4.1. Suppose $\mathbf{F}(t)$, $t \in \mathbf{R}$, is a regular and C^6 curve in $\mathbf{R}^m$, $m \geq 2$. Let $\mathbf{P}(t)$, $t \in \mathbf{R}$, be the limit curve generated by scheme (2.4) from the initial data $\mathbf{P}_i := \mathbf{F}(ih)$, $i \in \mathbf{Z}$, $0 < h < 1$. Then, on any finite interval $[a, b]$, we have the following estimate:

$$\|h^j \mathbf{F}^{(j)}(ht) - \mathbf{P}^{(j)}(t)\|_\infty = \mathcal{O}(h^{6-j}), \qquad j = 0, 1, 2. \qquad \blacksquare \tag{4.1}$$

From the above approximation properties of the basis function $\phi(t)$, it can be shown that the collocation method (3.2) with quintic precision treatments at the end points and proper treatments at all the singular points/nodes has at least the power of approximation $\mathcal{O}(h^4)$. More precisely, similar to the proof in [1], the following result can be proved.

Proposition 4.2. Suppose the right hand side function $f(t, u, v, w)$ and the given functions $a(t)$, $b(t)$, $\psi_0(t)$, $\psi_1(t)$, and $g(t)$ are sufficiently differentiable and the exact solution $x(t)$ of the BVP (1.1) is of class $C^6[c, d]$. Let $\{z_i\}$ be obtained by solving (3.3) with sixth order boundary and/or singular point treatments. Then if all the partial derivatives $\|f_u\|_\infty$, $\|f_v\|_\infty$, $\|f_w\|_\infty$ and the mesh size h are all small enough, we have:

$$\|x^{(j)}(t) - z^{(j)}(t)\|_\infty = \mathcal{O}(h^{4-j}), \quad j = 0, 1, 2. \tag{4.2}$$

Furthermore, if fourth order boundary and/or singular point treatments are used, then

$$\|x^{(j)}(t) - z^{(j)}(t)\|_\infty = \mathcal{O}(h^{2-j}), \quad j = 0, 1, 2. \qquad \blacksquare \tag{4.3}$$

Remark 4.3. In the papers [12, 13, 17], only linear and quadratic convergent finite difference algorithms were constructed. The higher order convergence of our method lies in the good approximation properties of the basis function $\phi(t)$ (cf. Propositions 4.1). In their formulations, they could only obtain $x(g(t_i)) - z(g(t_i)) = \mathcal{O}(h^2)$ instead of our estimate (4.1) since linear interpolants were used in their approximation process.

Remark 4.4. The error estimates (4.2) and (4.3) are valid only if all the functions g, f are sufficiently smooth and the partial derivatives and h are small and $x(t) \in C^6[c,d]$. Otherwise, the order of approximation could be much lower. This can be seen clearly from our examples.

5. NUMERICAL EXAMPLES AND CONCLUDING REMARKS

In this section, by using the collocation method described in Section 3 with the tolerance value $\epsilon_{tol} = 10^{-12}$, and the sixth order boundary treatment BT1 at the end points (cf. (3.5b)), and/or the second formula in (3.5a) at the singular points; we have solved the following 14 boundary value problems (some with singularity/singularities or deviating argument) and obtained the discrete values $\{z_i\}$ when convergence occurs with respect to the criterion C_1. Three types of example BVPs have been considered. Examples 1 to 5 are just nonlinear boundary value problems without any singularity or deviating argument. Examples 6 to 10 are nonlinear boundary value problems with one or two singularity points at the end point(s). Finally, Examples 11 to 14 are nonlinear boundary value problems with a deviating argument.

In our computations, we assumed the standard interval [0, 1] and used double precision arithmetic. It is found that for all the Examples, these numerical values are very close to the corresponding exact solutions and our method produces much better results than many other methods given in [6, 9–13, 16–19]. The numerical results and/or maximum errors of these example problems are summarized in Tables 1–3.

5.1. Examples of Nonlinear Boundary Value Problems

$$\begin{array}{llcll}
\text{Ex1.} & x''(t) & = & x^3(t) - \frac{x'(t)}{1+t}, & x(0)=1, \quad x(1)=0.5. \\
\text{Ex2.} & x''(t) & = & 2\sqrt{e^{2t} - x^2(t)}, & x(0)=0, \quad x(1)=e\sin 1. \\
\text{Ex3.} & x''(t) & = & -\sqrt{1-(x'(t))^2}, & x(0)=0, \quad x(1)=\sin 1. \\
\text{Ex4.} & x''(t) & = & \frac{1}{3(1+t)}[(2-t-t^2)e^{2x(t)}+1], & x(0)=0, \quad x(1)=-\log 2. \\
\text{Ex5.} & x''(t) & = & \frac{1}{2}[e^{2x(t)} + (x'(t))^2], & x(0)=0, \quad x(1)=-\log 2.
\end{array}$$

Table 1. Errors of Numerical Solutions for Examples 1 to 5 with $h = 10^{-1}$.

i	t_i	$\text{Ex1}(t_i)h^{-6}$	$\text{Ex2}(t_i)h^{-6}$	$\text{Ex3}(t_i)h^{-6}$	$\text{Ex4}(t_i)h^{-6}$	$\text{Ex5}(t_i)h^{-6}$
0	0.0	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.1	-35.5550	5.2505	0.0479	-7.8841	-8.3879
2	0.2	-51.2277	9.4316	0.1195	-11.8695	-12.5724
3	0.3	-65.9792	13.4779	0.2373	-15.8221	-16.3438
4	0.4	-70.8289	16.3881	0.3625	-17.6572	-17.7668
5	0.5	-66.1341	17.6025	0.4588	-17.1843	-16.8506
6	0.6	-55.5831	16.9028	0.4963	-15.0377	-14.3729
7	0.7	-42.1011	14.2867	0.4549	-11.8196	-11.0118
8	0.8	-27.7279	10.2245	0.3431	-8.0438	-7.3081
9	0.9	-14.1912	5.8505	0.2135	-4.2303	-3.7745
10	1.0	0.0000	0.0000	0.0000	0.0000	0.0000

5.2. Examples of Boundary Value Problems with Singularities

Ex6. $x''(t) + \frac{2}{t}x'(t) = 4t^2(5+4t^4), \quad x'(0)=0,\ x(1)=e.$

Ex7. $x''(t) + \frac{2}{t}x'(t) = 3\cos t - t\sin t, \quad x'(0)=0,\ x(1)=\cos 1 + \sin 1.$

Ex8. $x''(t) + \frac{2}{t}x'(t) = -2(e^{x(t)} + e^{x(t)/2}), \quad x'(0)=0,\ x(1)=0.$

Ex9. $x''(t) + \frac{2}{t}x'(t) = -x^5(t), \quad x'(0)=0,\ x(1)=\frac{\sqrt{3}}{2}.$

Ex10. $x''(t) + \frac{3x'(t)}{2\sqrt{t(1-t)}} = \frac{5}{4}(3-4t)(1-4t)\sqrt{t(1-t)} + \frac{15}{4}(1-2t)x^{\frac{2}{5}}(t),$
$x(0)=0,\ x(1)=0.$

Table 2. The Maximum Errors in Examples 6 to 10.

	$h=1/8$	$h=1/16$	$h=1/32$
Ex6.	1.72E(−2)	2.80E(−3)	2.66E(−4)
Ex7.	1.09E(−5)	1.08E(−6)	7.89E(−8)
Ex8.	1.20E(−3)	1.07E(−4)	8.03E(−6)
Ex9.	6.60E(−6)	7.23E(−7)	5.33E(−8)
Ex10.	$h=1/10$	$h=1/50$	$h=1/100$
Ex10.	4.30E(−3)	2.29E(−5)	4.25E(−6)

Table 3(a). The Maximum Errors of Numerical Solutions of Example 11.

$h=0.1$	$h=0.02$	$h=0.01$
0.1803E(−4)	0.8055E(−7)	0.7406E(−8)

Table 3(b). The Numerical Solution of Example 12 for $h=1/1000$.

t	$z(t)$	t	$z(t)$
0.0	−0.500000000000000	0.6	−2.143702697670429
0.1	−0.972570724582965	0.7	−2.068766795053254
0.2	−1.372983258931085	0.8	−1.803972902999028
0.3	−1.693641049002736	0.9	−1.296982814629590
0.4	−1.930498429831545	1.0	−0.500000000000000
0.5	−2.082883825998235		

Table 3(c). Maximum Errors of Numerical Solutions of Examples 13 & 14.

	h	1/16	1/32	1/64	1/128	1/256
Ex13.		6.9E(−4)	1.9E(−4)	5.2E(−5)	1.3E(−05)	3.4E(−06)
Ex14.	$\alpha=4.0$	3.3E(−7)	2.2E(−8)	1.4E(−9)	8.6E(−11)	3.5E(−12)
Ex14.	$\alpha=4.0$	4.2E(−7)	2.8E(−8)	1.8E(−9)	1.1E(−10)	3.9E(−12)

5.3. Examples of Boundary Value Problems with Deviating Arguments

$$\text{Ex11.} \quad x''(t) = x(t^2), \qquad x(0) = 0, \quad x(1) = 1.$$

$$\text{Ex12.} \quad x''(t) = \tfrac{1}{4}\left(-\sin x(t) - 16(2t+1)x(t-\tfrac{1}{2}) + 32t\right),$$

$$x(t) = 2t - \tfrac{1}{2}, \quad t \le 0, \quad x(1) = -\tfrac{1}{2}.$$

$$\text{Ex13.} \quad x''(t) = x(|t-\tfrac{1}{2}|) + |t-\tfrac{1}{2}|\left(6 - (t-\tfrac{1}{2})^2\right), \quad x(0) = x(1) = 1.$$

$$\text{Ex14.} \quad x''(t) = \left(x\left(\tfrac{t}{\alpha}\right)\right)^{\alpha}, \qquad x(t) = e^t, \quad t \notin (0,1).$$

Remark 5.1. From the above numerical results, clearly the errors in all the Examples with smooth solutions have the estimated form (4.2): $\text{Error}(t) = \mathcal{O}(h^4)$. For Examples with solutions of lower order regularities, the orders of the errors are slightly lower. For all the used values of the step size h in the above Examples, our method produces much better results than that of the methods in [6, 9, 13, 17, 19]. Moreover, our solutions are only C^2 and all the derivatives of the solutions can be obtained directly from (3.1) and (2.3) without any other errors.

Remark 5.2. In all our computations, the iterative algorithm (3.9) and (3.10) converges very fast even for quite large mesh size h. In fact, for all the Examples, we have

$$\|\mathbf{z}^{(16)} - \mathbf{z}^{(15)}\|_\infty < 10^{-12}.$$

For the problems without any singularity, the convergence occurs after only 3, 4, or 5 iterations.

Remark 5.3. For all the Examples, the approximate solution is approaching the exact one from one side, *i.e.*, the errors are all positive or negative. Thus there may be some mathematical explanation for this monotone approximation phenominon.

Remark 5.4. Higher order boundary value problems can be solved similarly using higher order interpolatory subdivision schemes. The derivatives of the corresponding cardinal function at integers are given by formulae similar to (2.2), (2.3) and (2.4). For more details, the reader is refered to [14, 15].

Remark 5.5. Since the basis function $\phi(t)$ is locally supported, therefore, even smaller errors could be expected at points further away from the boundary points if the solution is quite smooth but some lower order boundary treatments are used.

Remark 5.6. The basis function $\phi(t)$ has many nice properties and can be used to solve many other proctical problems such as approximate solutions of integral euqtions (cf. [20]) and fast integral transforms.

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