A GALERKIN METHOD FOR SINGULAR TWO POINT LINEAR BOUNDARY VALUE PROBLEMS

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الخلاصة :

نستخدم في هذا البحث طريقة التغيّر لدراسة مسألة الوجود والأُحادية لحلول مجموعة من مسائل القيم الحدّية ذات نقطتين شاذتين. بعد ذلك نستخدم طريقة (جالركن) مع دَوال أساسية خاصة لتفريد المسألة، ومِنْ ثَمَّ نُثبت تقارب حل المسألة المفَردَّة إلى نظيره للمسألة المتصلة ونعطي مُعَدَّل التقارب مستخدمين عيارات طاقة وانتظام مختلفة. وقد حصلنا على معدلات تقارب مُثلى.

ABSTRACT

In this paper we use variational techniques to address the problem of existence and uniqueness of solutions of a class of singular two-point boundary value problems. We then use a Galerkin method with special basis functions to discretize the problem. We prove convergence of the solution of the discrete problem to that of continuous problem and give the order of convergence in various energy and uniform norms. The orders of convergence obtained are optimal.

Subject Classifications: 65L15, 34E05

Keywords: Singular Differential Equations, Galerkin Method, Limit Point, Limit Circle, Order of Convergence

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1. INTRODUCTION

Many problems in physics and engineering give rise to second-order differential expression of the form:

$$\ell(y) = -\frac{1}{w}(py')' + qy \quad on \quad (0,1), \tag{1.1}$$

with boundary conditions. In general, one would be interested in solving the equation:

$$\ell(y) = f, \tag{1.2}$$

where y satisfies the boundary conditions. These equations may be regular or singular at each point of the interval $(0,1)$.

In the regular case, problem (1.2) is well understood and its numerical analysis has been extensively studied. Very successful software has been written for solving regular equations, of which we mention the programs SLEIGN [1] and DO2KEF [2]. Apart from a limited number of special cases, the errors involved in approximating singular problems are not yet well understood (see [3] p. 1665). In this work we provide some error estimates for approximating such problems. The trade-off between the roughness of the data and the rate of convergence of the numerical solution will be illustrated. (See also [4]).

Singular problems of the form (1.2) appear in many areas of applied mathematics; in transport processes [5], in the study of electrohydrodynamics [6], in the theory of thermal explosions [7], and in the separation of variables in partial differential equations [9]; just to name a few fields. The code SLEIGN2 [8] is written to compute the eigenvalues of the Sturm–Liouville problem in some singular cases. The Galerkin method for singular problems was considered in [10–12]. Special finite difference methods were considered in Chawla *et al.* [13]. The reader is referred to the references cited in the aforementioned articles for more details.

In this article the questions of existence, uniqueness, and regularity of sulutions of a class of singular two-point boundary value problems will be investigated to the extent needed for the error estimates of the numerical approximation. The variational setup of the problem will also be considered as well as the equivalence between the solution of the variational problem and the solution of the boundary value problem. Then we will investigate a Galerkin method, with special patch functions (considered earlier by Ciarlet *et al.* [10]), for the numerical approximation of the solution. We will derive error estimates in various energy and uniform norms and show how the accuracy of the approximation is affected by the norm in which we study convergence or, equivalently, by the regularity of the data. Extending these results to nonlinear problems will be the subject of a later paper.

2. PRELIMINARIES

For $r \geq 0$ on $I = (0,1)$ we let $L^r(I)$ denote the weighted Hilbert space with the inner product

$$\langle y, z\rangle_r = \int_I y(x)\overline{z}(x)r(x)dx. \tag{2.1}$$

We also let V_p be the Hilbert space consisting of functions $u \in L^2_w(I)$ which are locally absolutely continuous on I, $u(1) = 0$ and $u' \in L^2_p(I)$. The inner product on the space V_p is defined by

$$\langle y, z\rangle_{V_p} = \int_I y'(x)\overline{z}'(x)p(x)dx. \tag{2.2}$$

The notation $L_{loc}(I)$ is used to denote the space of functions $y \in L[\alpha,\beta]$ where $[\alpha,\beta]$ is any compact subinterval of I.

Throughout this paper we assume that the real valued functions p, q, w satisfy

$$q, w \in L^1(I) \tag{2.3}$$

$$p^{-1} \in L_{loc}(I),\ p^{-1} \notin L_{loc}([0,\alpha)) \text{ for any } \alpha > 0, \tag{2.4}$$

$$p, w \geq 0,\ \int_x^1 p^{-1} \in L^1_w(0,1), \text{ and} \tag{2.5}$$

$$\int_E w(x)dx = 0 \text{ implies } \int_E p(x)dx = 0 \text{ for } E \subseteq I \text{ measurable.} \tag{2.6}$$

Additional assumptions will be specified when needed. Assumption (2.4) means that ℓ is singular and 0 is a singular point for ℓ. See [14]. Under the above assumptions it can be shown [15] that the boundary condition at 0 may be taken as $\lim_{x\to 0}(py')(x) = 0$. For $q = 0$, assumptions (2.3),(2.5) allow for limit circle (LC) $\left(\int_x^1 p^{-1} \in L^2_w(0,1)\right)$ as well as a class of limit point (LP) cases for the operator ℓ [15]. Our goal is, thus, to investigate the problem (1.2) together with the boundary conditions

$$\left.\begin{array}{c}(py')(0^+) = 0\\ y(1) = 0\end{array}\right\} \tag{2.7}$$

both theoretically and numerically. Finally, we will make use of the operator $\widetilde{L}$:

$$D(\widetilde{L}) = \left\{y \in L^2_w(I)\ : \ell(y) \in L^2_w(I),\ y(1) = \lim_{x\to 0}(py')(x) = 0\right\}.$$

It is known that $\widetilde{L}$ is self-adjoint [15].

3. THE MAIN RESULTS

We study the theoretical and numerical aspects of the problem (1.2),(2.7) first for the case $q \equiv 0$ and then introduce the function q with nonzero values as a perturbation of the former case and strive to keep the assumptions on q minimal. This way we hope to be able to illustrate the effect of q on the analysis, the regularity, the error estimates, *etc.*

The operator $\widetilde{L}$ is real in the sense that $\overline{\widetilde{L}f} = \widetilde{L}\overline{f}\ \ \forall\ f \in D(\widetilde{L})$ as can be readily checked. Since the data functions we consider are all real valued, this means that it is enough to consider only real inner products. Therefore we will drop the conjugation overbars for the rest of this paper.

3.1. Existence, Uniqueness, and Regularity of the Solution

Various results about the existence, uniqueness, and regularity of the solution of (1.2),(2.7) are needed in order to obtain error estimates for the Galerkin method. We will state and discuss these results here. We will also have some comments about the conditions required for such results to follow. In particular we will discuss applicability of the boundary condition $u'(0) = 0$ which is most frequently used in the literature. The results listed below are organized into two categories: $q = 0$ and $q \neq 0$.

Theorem 3.1 (LC, $q = 0$) *For every $f \in L^2_w(0,1)$ the unique solution u of (1.2),(2.7) is absolutely continuous on $[0,1]$. The operator $\widetilde{L}^{-1} : L^2_w(0,1) \to C[0,1]$ is compact.*

Remark 3.1 *It follows from (2.3) that $C[0,1]$ is continuously embedded in $L^2_w(0,1)$. As a corollary of this and Theorem 3.1 we get that $\widetilde{L}^{-1} : L^2_w(0,1) \to L^2_w(0,1)$ is also compact. This is a well known result in the limit circle case.*

Theorem 3.2 *(a) If $f \in L^\infty_w(0,1)$ then (1.2),(2.7) has a unique solution which is absolutely continuous on $[0,1]$.*
(b) If $f \in L^2_w(0,1)$ then (1.2),(2.7) has a unique solution which is absolutely continuous on $(0,1]$.
(c) The operator $\widetilde{L}^{-1} : L^\infty_w(0,1) \to C[0,1]$ is compact.

Theorem 3.3 (LP1, $q = 0$) *For any $u \in D\left(\widetilde{L}\right)$ we have $u' \in L^2_p(0,1)$.*

Remark 3.2 *1. In general, the result of Theorem 3.3 cannot be sharpened in the sense that we may not have continuity of u' on $[0,1]$. For example, if in (1.2),(2.7) we take $p(x) = x, w(x) = 1, q(x) = 0$, and $f(x) = \ln x$ which is a limit circle case, then the solution $u(x) = -2 - x\ln x + 2x$ has an unbounded derivative at $x = 0$.*
2. If $p(x) = w(x)$, p is monotone increasing and $f \in L^\infty_w(0,1)$ then the boundary condition at $x = 0$ may be stated as $u'(0) = 0$. In particular this is true for $p(x) = w(x) = x^\alpha$ which is the case considered in the literature.
3. In the first part of Theorem 3.2, the condition $f \in L^\infty_w(0,1)$ cannot be relaxed to $f \in L^2_w(0,1)$. For example, if in (1.2), we take $p(x) = x^3, w(x) = x^{3/2}, q(x) = 0$, and $f(x) = \frac{3}{4x}$, then the solution to (1.2) is $u(x) = \frac{1}{\sqrt{x}} - 1$ which is unbounded at $x = 0$.
4. Theorem 3.3 means that the space V_p defined in Section 2 is the natural space to consider for the derivation of the solution and consequently for the set up of the Galerkin method.

The results for the case $q \neq 0$ will be obtained through the variational formulation of the problem (1.2),(2.7):

Find $u \in V_p$ such that

$$B(u,v) = \langle f, v\rangle_w \text{ for all } v \in V_p \tag{3.1}$$

where

$$B(u,v) = \langle u, v\rangle_{V_p} + \langle qu, v\rangle_w. \tag{3.2}$$

The following conditions are imposed on the function $q(x)$:

$$C_q := \int_0^1 |q(x)| \left(\int_x^1 \frac{ds}{p(s)}\right) w(x)dx < \infty, \tag{3.3}$$

$$\gamma := \inf_{u\in V_p} \frac{\int_0^1 q(x)u^2(x)w(x)dx}{\|u\|^2_{V_p}} > -1, \tag{3.4}$$

$$qu \in L^2_w(0,1) \text{ for all } u \in V_p. \tag{3.5}$$

Theorem 3.4 *Under the conditions (3.3),(3.4), the variational boundary value problem (3.1) has a unique solution* $u \in V_p$ i.e., *(1.2),(2.7) has a unique weak solution.*

Theorem 3.5 *Under the conditions (3.3)-(3.5) we have*
(1) The unique solution u *of (3.1) is also the solution of (1.2),(2.7).*
(2) (LC) If $f \in L^2_w(0,1)$ *then* u *is absolutely continuous on [0,1].*
(3) (LP1) If $f, q \in L^\infty_w(0,1)$ *then* u *is absolutely continuous on [0,1].*

Remark 3.3 *1. In the case (LC), (3.5) holds if* $q \in L^4_w(0,1)$. *(3.3) also holds in that case (because of (2.3)). In the case (LP), (3.3), (3.5) hold if* $q \in L^\infty_w(0,1)$.
2. In the case (LP), if $q \in L^\infty_w(0,1)$ *and* $q \geq 0$ *then all conditions (3.3)-(3.5) are satisfied. In particular this is true for* $p(x) = w(x) = x^\alpha,\ \alpha \geq 0$.

4. THE GALERKIN APPROXIMATION AND CONVERGENCE RESULTS

Let $\pi : 0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ be a mesh on the interval $[0,1]$ and for $i = 1, 2, \cdots N$ define the patch functions

$$r_i(x) = \begin{cases} r_i^-(x) & \text{if } x_{i-1} \leq x \leq x_i, \\ r_i^+(x) & \text{if } x_i \leq x \leq x_{i+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where

$$r_1^-(x) = 1,$$

$$r_i^-(x) = \frac{\int_{x_{i-1}}^{x} \frac{1}{p(s)} ds}{\int_{x_{i-1}}^{x_i} \frac{1}{p(s)} ds}, \quad i = 2, 3, \cdots, N$$

and

$$r_i^+(x) = \frac{\int_{x}^{x_{i+1}} \frac{1}{p(s)} ds}{\int_{x_i}^{x_{i+1}} \frac{1}{p(s)} ds}, \quad i = 1, 2, \cdots, N.$$

The above patch functions have been used by Ciarlet *et al.* [10]. Next we define the discrete subspace V_N of V_p by

$$V_N = \text{span } \{r_i\}_{i=1}^N.$$

The discrete version of the weak problem (3.1) reads:

Find $u^G \in V_N$ such that

$$B(u^G, v_N) = \langle f, v_N \rangle_w \text{ for all } v_N \in V_N. \tag{4.2}$$

It follows from (3.1) and (4.2) that

$$B(u - u^G, v_N) = 0 \quad \text{for all } v_N \in V_N.$$

Note that condition (3.4) implies that the bilinear form $B(u,v)$ is coercive which guarantees the existence of the solution u^G of (4.2). u^G is called the Galerkin approximation of (3.1) (and consequently of (1.2),(2.7)). We can now state our results on the convergence of the Galerkin solution u^G to the weak solution u of (3.1).

Theorem 4.1 (LC,LP1) *If the function q satisfies (3.3) and $f \in L^2_w(0,1)$, then*

$$\left\|u^G - u\right\|_{V_p} \leq C\sqrt{\ell(\pi_N)}\,\|f\|_w$$

where C depends only on the data and $\ell(\pi_N)$ is given by

$$\ell(\pi_N) = \max_{0\leq i\leq N} \int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{1}{p(t)}dt\right) w(s)ds \tag{4.3}$$

Theorem 4.2 (LC, LP1) *If $f, q \in L^\infty_w(0,1)$, then*

$$\left\|u^G - u\right\|_\infty \leq C\sqrt{\ell(\pi_N)}\,\|f\|_\infty .$$

Theorem 4.3 (LC) *If $f \in L^2_w(0,1)$ and $q \in L^4_w(0,1)$, then*

$$\left\|u^G - u\right\|_\infty \leq C\sqrt{\ell_1(\pi_N)}\,\|f\|_w$$

where

$$\ell_1(\pi_N) = \max\left\{\ell(\pi_N), \max_{0\leq i\leq N} \int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{1}{p(t)}dt\right)^2 w(s)ds\right\}. \tag{4.4}$$

Remark 4.1 *1. In the case of Theorem 4.2, if the additional assumption $q \geq 0$ is made, then the order of convergence improves to $\ell(\pi_N)$. See Remark 5.2.*
2. In the literature most authors treat the case $p(x) = w(x) = x^\alpha$. In [12], Jesperson obtains $O(h^2)$ convergence for the L^2_w norm and $O(h \log h)$ convergenc for the L^∞ norm under more restrictive assumptions on the mesh and the function q. Simillar results were obtained by Eriksson et al. [11]. In our case, the order of convergence $\ell(\pi)$ reduces to $O(h^2)$ in both norms.
3. In [10] Ciarlet et al. obtain the convergence estimate $\ell(\pi)$ under more general assumptions on q. However, in their case the function $\frac{1}{p}$ was assumed integrable on $[0,1]$. In our case, the loss of integrability of $\frac{1}{p}$ reduces the order of convergence to $\sqrt{\ell(\pi)}$ as indicated in Theorem 4.1. This result illustrates the effect of the strength of singularity on the order of convergence.
4. Comparing theorems 4.2,4.3 we see that relaxing the conditions on q and f reduces the order of convergence from $\sqrt{\ell(\pi)}$ to $\sqrt{\ell_1(\pi)}$.

5. PROOF OF THE RESULTS

5.1. Proof of Theorem 3.1

Our boundary value problem in this case reads

$$\left.\begin{array}{c} \ell(u) = -\frac{1}{w(x)}\left(p(x)u'(x)\right)' = f(x),\ 0 < x < 1 \\ \lim_{x\to 0^+} p(x)u'(x) = 0 \\ u(1) = 0 \end{array}\right\} \tag{5.1}$$

(5.1) gives the unique solution

$$u(x) = \int_x^1 \frac{1}{p(t)} \left(\int_0^t f(s)w(s)ds \right) dt. \tag{5.2}$$

For $x > 0$ we may integrate by parts to obtain

$$u(x) = \int_x^1 \frac{1}{p(t)} dt \int_0^x f(t)w(t)dt + \int_x^1 \left(\int_s^1 \frac{1}{p(t)} dt \right) f(s)w(s)ds. \tag{5.3}$$

Using the fact that

$$\int_x^1 \frac{1}{p(t)} dt \int_0^x f(t)w(t)dt \leq \int_0^x \left(\int_s^1 \frac{1}{p(t)} dt \right) |f(s)|\, w(s)ds \tag{5.4}$$

we obtain

$$|u(x)| \leq \|f\|_w \left\| \int_x^1 \frac{1}{p(t)} dt \right\|_w , \tag{5.5}$$

where $\|\cdot\|_w$ denotes the norm in $L^2_w(0,1)$. Thus

$$u(0) = \int_0^1 \left(\int_s^1 \frac{1}{p(t)} dt \right) f(s)w(s)ds \tag{5.6}$$

and (5.2) is an indefinite integral. Therefore u is absolutely continuous on [0,1].

The compactness of $\widetilde{L}^{-1}$ follows from a straightforward argument which uses the Arzela–Ascoli theorem and (5.4).

To complete the proof of Theorem 3.1 we show that $u' \in L^2_p(0,1)$.

$$\begin{aligned} \int_0^1 p\,|u'|^2\, dx &= \lim_{x \to 0+} p(x)u'(x)u(x)|_x^1 - \int_0^1 (pu')'\, u dx \\ &= \int_0^1 f(x)u(x)w(x)dx \\ &\leq \|f\|_w\, \|u\|_w , \end{aligned}$$

where we have used (5.6), (5.1) and the continuity of u.

Remark 5.1 *Since* $|u'(x)| \leq \frac{1}{p(x)} \left(\int_0^x w(s)ds \right)^{1/2} \|f\|_w$ *,then if*

$$\lim_{x \to 0+} \frac{1}{p(x)} \left(\int_0^x w(s)ds \right)^{1/2} < \infty \tag{5.7}$$

then $|u'(x)|$ *has a bounded limit at* $x = 0$*. If the limit in (5.7) is 0, then* $u'(x) \to 0$ *as* $x \to 0^+$*. This is the case for example for* $p(x) = x^\alpha$*,* $w(x) = x^{2\alpha - 1 + \epsilon}$ *for any* $\epsilon > 0$*. If* $f \in L^\infty(0,1)$*, then we get similar results for* $p(x) = w(x) = x^\alpha$*. However, in general,* $u'(0)$ *may not be zero.*

5.2. Proof of Theorem 3.2

The proofs of (a) and (c) are similar to the limit circle case except that we use Holder's inequality instead of Cauchy Schwartz inequality. To prove (b) we note first that (5.3) implies the absolute continuity on (0,1] of u. To show that $u \in L_w^2(0,1)$ we estimate the $L_w^2(0,1)$ norm of the two terms on the right of (5.3). The first term is estimated as follows:

$$\begin{aligned}
\int_0^1 \left(\int_0^x fw\right)^2 \left(\int_x^1 \frac{1}{p}\right)^2 wdx &\leq \|f\|_w^2 \int_0^1 \int_0^x w \left(\int_x^1 \frac{1}{p}\right)^2 wdx \\
&\leq \|f\|_w^2 \int_0^1 \left(\int_0^x \left(\int_s^1 \frac{1}{p}\right) w\right) \left(\int_x^1 \frac{1}{p}\right) wdx \\
&\leq \|f\|_w^2 \left(\int_0^1 \left(\int_x^1 \frac{1}{p}\right) w\right)^2 < \infty.
\end{aligned}$$

Similarly we can estimate the second term.

5.3. Proof of Theorem 3.3

The proof that $u' \in L_p^2(0,1)$ is the same as in the limit circle case.

5.4. Proof of Theorem 3.4

Before proving this theorem we state and prove the following lemmas; throughout which we assume that the conditions of Theorem 3.4 are satisfied.

Lemma 5.1 $\langle qu, v\rangle_w \leq C_q \|u\|_{V_p} \|v\|_{V_p}$.

Proof.

$$\begin{aligned}
&\int_0^1 q(t)w(t)u(t)v(t)dt \\
&= \int_0^1 q(t)w(t) \left(\int_t^1 u'(s)ds\right) \left(\int_t^1 v'(s)ds\right) dt \\
&= \int_0^1 q(t)w(t) \left(\int_t^1 \frac{1}{\sqrt{p(s)}}\sqrt{p(s)}u'(s)ds\right) \left(\int_t^1 \frac{1}{\sqrt{p(s)}}\sqrt{p(s)}v'(s)ds\right) \\
&\leq \|u\|_{V_p}\|v\|_{V_p} \int_0^1 |q(t)|w(t) \left(\int_t^1 \frac{1}{p(s)}ds\right) dt \\
&= C_q\|u\|_{V_p}\|v\|_{V_p}. \square
\end{aligned}$$

Corollary 5.1 *V_p is continuously embedded in $L_w^2(0,1)$.*

Proof. Putting $q \equiv 1$ in the previous lemma we obtain

$$\langle u, v\rangle_w \le C_1 \|u\|_{V_p} \|v\|_{V_p}$$

and so

$$\|u\|_w^2 \le C_1 \|u\|_{V_p}^2 . \square$$

It can now be readily checked that the bilinear form $B(\cdot,\cdot)$ given by (3.2) is continuous and V_p-elliptic and that $\ell(v) = \langle f, v\rangle_w$ is a linear bounded functional on V_p.

Theorem 3.4 now follows from the above lemmas and the Lax–Milgram theorem.

5.5. Proof of Theorem 3.5

We begin by collecting some facts about the relationships between the spaces $D\left(\widetilde{L}\right)$, V_p and $L_w^2(0,1)$. We recall here that assumptions (3.3)–(3.5) are in effect.

Lemma 5.2 *(a) V_p is continuously and densely embedded in $L_w^2(0,1)$*
(b) If $u \in D\left(\widetilde{L}\right)$ and $v \in V_p$ then $pu'v|_0^1 = 0$
(c) $D\left(\widetilde{L}\right)$ is dense in V_p.

Proof. (a) Follows from Corollary 5.1.

(b) Let $u \in D\left(\widetilde{L}\right)$ and $f \in L_w^2(0,1)$ be such that $pu' = -\int_0^x fw$, then

$$\begin{aligned} |(pu'v)(x)| &= \left|\int_0^x fw\right| \left|\int_x^1 v'\right| \le \int_0^x |f|\, w \int_x^1 |v'| \\ &\le \|f\|_w \|v\|_{V_p} \left(\int_0^x \left(\int_s^1 \frac{1}{p}\right) w\right)^{1/2} \end{aligned}$$

The above inequalities give the desired result.

(c) Let $v \in V_p$ be such that $\langle u, v\rangle_{V_p} = 0$ for all $u \in D\left(\widetilde{L}\right)$. Then

$$0 = \int_0^1 pu'v' = pu'v|_0^1 - \int_0^1 (pu')'\, v = -\int_0^1 (pu')'\, v = \langle \widetilde{L}u, v\rangle_w .$$

Since $\widetilde{L}$ is surjective, then $v = 0$ a.e.w. Hence by assumption (2.6) $v = 0$ a.e.p. Therefore $D\left(\widetilde{L}\right)$ is dense in V_p.$\square$

It thus follows from the Lax–Milgram theorem that (3.1) has a unique solution. To show that the weak solution is also the classical solution of (1.2) we introduce the operator S defined by

$$\begin{aligned} D(S) &= \left\{u \in V_p : v \mapsto B(u,v) \text{ is continuous in } L_w^2(0,1)\right\} \\ Su &= f \qquad \text{where} \\ B(u,v) &= \langle f, v\rangle_w \qquad \forall\, v \in V_p . \end{aligned}$$

A standard argument (*e.g.* see [16]) may now be used to show that $S = \widetilde{L}+q$ and the equivalence of the weak and the classical solutions. This proves part (1) of Theorem 3.5. To prove part (2) we notice that by the boundedness of the operator $\left(\widetilde{L}+q\right)^{-1}$ on $L^2_w(0,1)$ and assumption (3.3) it follows that the solution u of $\left(\widetilde{L}+q\right)u = f$ is in $D\left(\widetilde{L}\right)$. The desired result follows in the LC case from equations (5.2) and (5.6). Part (3) follows from the fact that $\widetilde{L}^{-1} : L^\infty_w(0,1) \to C[0,1]$ is compact (Theorem 3.2) and that adding the L^∞_w function q does not alter the domain of $\widetilde{L}$. We then recall (5.2) and (5.6) again.

5.6. Proof of Theorem 4.1

Standard proofs of theorems of the type of Section 4 usually use the Aubin–Nitsche trick and inequalities like Hardy's inequality. In our case either of these techniques can be used because either the solution u does not have enough smoothness properties or the inequalities are not sharp enough for our purposes. Our proofs will hinge on the knowledge of the closed form of the inverse of the matrix $A = (B(r_i, r_j))$ (with $q = 0$). To prove Theorem 4.1 we proceed by introducing a special interpolant u^I of the solution u of (3.1) (known as the V_N-interpolant of the solution [10]) . The error of the Galerkin approximation is compared to the error of the interpolant. Then the order of the error of the interpolant is worked out. This is done in Lemma 5.4. Theorem 4.1 will be a direct consequence of the aforementioned lemma.

We begin by introducing the V_N-interpolant u^I of the solution u

$$u^I(x) = \sum_{i=1}^{N} u_i r_i(x) \tag{5.8}$$

where $u_i = u(x_i)$ and r_i is given by (4.1), $i = 1, \ldots, N$. We note here in passing that u^I is the orthogonal projection of u with respect to the inner product $\langle \cdot, \cdot \rangle_{V_p}$:

$$\langle u - u^I, v_N \rangle_{V_p} = 0 \tag{5.9}$$

for all $v_N \in V_N$. The following two relations are also easily checked

$$B(u^G - u^I, v_N) = \langle q(u - u^I), v_N \rangle_w, \tag{5.10}$$

and

$$\langle u^G - u^I, v_N \rangle_{V_p} = \langle q(u - u^G), v_N \rangle_w, \tag{5.11}$$

for all $v_N \in V_N$.

Lemma 5.3 *Let u^G be the Galerkin approximation and u^I be the V_N-interpolant of the solution u of (3.1). Then*

$$\left\| u - u^G \right\|_{V_p} \le \left(1 + \frac{C_q}{1+\gamma}\right) \left\| u - u^I \right\|_{V_p} \tag{5.12}$$

where C_q and γ are given by (3.3) and 3.4).

Proof.

In (5.10) put $v_N = u^G - u^I$ and note that $|B(u,v)| \le (1 + C_q) \|u\|_{V_p} \|v\|_{V_p}$ and $|B(v,v)| \ge (1+\gamma) \|v\|^2_{V_p}$ $\forall v \in V_p$.□

Lemma 5.4 *Let $g = f - qu$ where u is the solution of (3.1). Then*

$$\left\|u - u^I\right\|_{V_p} \le 2\|g\|_w \sqrt{\ell(\pi)}.$$

Proof. For $x \in [x_i, x_{i+1}], i = 0, \ldots N$ we use the fact that $\sum_{i=1}^N r_i(x) = 1$ and integration by parts together with (5.2),(5.6) to obtain:

$$u(x) - u^I(x) = r_i^+(x)\int_{x_i}^x \left(\int_{x_i}^s \frac{dt}{p(t)}\right) g(s)w(s)ds + r_{i+1}^-(x)\int_x^{x_{i+1}}\left(\int_s^{x_{i+1}} \frac{dt}{p(t)}\right) g(s)w(s)ds. \tag{5.13}$$

Then:

$$\begin{aligned}
\left\|u - u^I\right\|_{V_p}^2 &= \sum_{i=0}^N \int_{x_i}^{x_{i+1}} \left|(u(x) - u^I(x))'\right|^2 p(x)dx \\
&= \int_0^{x_1} \frac{1}{p(x)}\left(\int_0^x g(s)w(s)ds\right)^2 dx + \\
&\quad \sum_{i=1}^N \left(\int_{x_i}^{x_{i+1}} \frac{1}{p}\right)^{-2} \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \times \\
&\quad \left\{\int_{x_i}^x \left(\int_{x_i}^s \frac{dt}{p(t)}\right) gwds - \int_x^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)}\right) gwds\right\}^2 dx \\
&\le \int_0^{x_1} \frac{1}{p(x)}\left(\int_0^x g^2(s)w(s)ds\right)\left(\int_0^x w(s)ds\right) dx + \\
&\quad \sum_{i=1}^N 2\left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-2} \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^x \int_{x_i}^s \frac{dt}{p(t)} g(s)w(s)ds\right)^2 dx + \\
&\quad \sum_{i=1}^N 2\left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-2} \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_x^{x_{i+1}} \int_s^{x_{i+1}} \frac{dt}{p(t)} g(s)w(s)ds\right)^2 dx.
\end{aligned}$$

Each term under the first summation can be majorized as follows:

$$\begin{aligned}
&\left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-2} \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^x \int_{x_i}^s \frac{dt}{p(t)} g(s)w(s)ds\right)^2 dx \\
\le &\left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-2} \left(\int_{x_i}^{x_{i+1}} \frac{dt}{p(t)}\right)^2 \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^x g(s)w(s)ds\right)^2 dx \\
= &\int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^x g(s)w(s)ds\right)^2 dx \\
\le &\int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^x g^2(s)(s)w(s)ds\right)\left(\int_{x_i}^x w(s)ds\right) dx
\end{aligned}$$

$$\leq \left(\int_{x_i}^{x_{i+1}} g^2(s)w(s)ds\right) \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^{x} w(s)ds\right) dx$$

$$= \left(\int_{x_i}^{x_{i+1}} g^2(s)w(s)ds\right) \int_{x_i}^{x_{i+1}} \left(\int_{x}^{x_{i+1}} \frac{ds}{p(s)}\right) w(x)dx,$$

and each term under the second summation can be majorized as follows:

$$\left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-2} \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x}^{x_{i+1}} \left(\int_{s}^{x_{i+1}} \frac{dt}{p(t)}\right) g(s)w(s)ds\right)^2 dx$$

$$\leq \left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-2} \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x}^{x_{i+1}} g^2(s)w(s)ds\right) \left(\int_{x}^{x_{i+1}} \left(\int_{s}^{x_{i+1}} \frac{dt}{p(t)}\right)^2 w(s)ds\right) dx$$

$$\leq \left(\int_{x_i}^{x_{i+1}} g^2(s)w(s)ds\right) \left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-1} \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \int_{x}^{x_{i+1}} \left(\int_{s}^{x_{i+1}} \frac{dt}{p(t)}\right) w(s)dsdx$$

$$\leq \left(\int_{x_i}^{x_{i+1}} g^2(s)w(s)ds\right) \left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-1} \int_{x_i}^{x_{i+1}} \left(\int_{s}^{x_{i+1}} \frac{dt}{p(t)}\right) w(s)ds \left(\int_{x_i}^{x_{i+1}} \frac{1}{p(x)}dx\right)$$

$$= \left(\int_{x_i}^{x_{i+1}} g^2(s)w(s)ds\right) \int_{x_i}^{x_{i+1}} \left(\int_{s}^{x_{i+1}} \frac{dt}{p(t)}\right) w(s)ds.$$

Therefore,

$$\begin{aligned}
\left\|u - u^I\right\|_{V_p}^2 &\leq \int_0^{x_1} g^2(s)w(s)ds \int_0^{x_1} \left(\int_x^{x_1} \frac{ds}{p(s)}\right) w(x)dx + \\
&\quad 4\sum_{i=1}^{N} \left(\int_{x_i}^{x_{i+1}} g^2(s)w(s)ds\right) \int_{x_i}^{x_{i+1}} \left(\int_{x}^{x_{i+1}} \frac{ds}{p(s)}\right) w(x)dx \\
&\leq 4\left(\max_{0\leq i\leq N} \int_{x_i}^{x_{i+1}} \left(\int_{x}^{x_{i+1}} \frac{ds}{p(s)}\right) w(x)dx\right) \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} g^2(s)w(s)ds \\
&= 4\left\|g\right\|_w^2 \max_{0\leq i\leq N} \int_{x_i}^{x_{i+1}} \left(\int_{x}^{x_{i+1}} \frac{ds}{p(s)}\right) w(x)dx. \square
\end{aligned}$$

Theorem 4.1 now follows from (3.3), Lemma 5.4 and the boundedness of the operator $\left(\widetilde{L} + q\right)^{-1}$. The last statement follows from the inequalities:

$$\begin{aligned}
\langle\left(\widetilde{L} + q\right) u, u\rangle_w &= \langle\widetilde{L}u, u\rangle_w + \langle qu, u\rangle_w = \|u\|_{V_p}^2 \left(1 + \frac{\langle qu, u\rangle_w}{\|u\|_{V_p}^2}\right) \\
&\geq (1+\gamma)\|u\|_{V_p}^2 \geq C\|u\|_w^2 .
\end{aligned}$$

5.7. Proof of Theorem 4.2

Lemma 5.5 $\left\|u-u^I\right\|_\infty \le C\ell(\pi_N)\left\|f\right\|_\infty$.

Proof. For any $x \in [x_i, x_{i+1}], i = 0, 1, \dots N$

$$u(x) - u^I(x) \le \int_{x_i}^{x_{i+1}} |g(s)| \left(\int_s^{x_{i+1}} \frac{dt}{p(t)}\right) w(s)ds$$

where $g = f - qu$. To see this we consider two cases: $i = 0$ and $i \ge 1$.

For $i = 0$ *i.e.*, for $x \in [0, x_1]$ we have

$$\begin{aligned}
u(x) - u^I(x) &= u(x) - u(x_1) \\
&= \int_x^{x_1} \frac{1}{p(s)} \int_0^s g(t)w(t)dt\, ds \\
&= \int_x^{x_1} \frac{ds}{p(s)} \int_0^x g(s)w(s)ds + \int_x^{x_1} g(s)w(s) \int_s^{x_1} \frac{dt}{p(t)} ds \\
&\le \int_0^x |g(s)|w(s) \int_s^{x_1} \frac{dt}{p(t)} ds + \int_x^{x_1} |g(s)|w(s) \int_s^{x_1} \frac{dt}{p(t)} ds \\
&= \int_0^{x_1} |g(s)| \int_s^{x_1} \frac{dt}{p(t)} w(s)ds. \qquad (5.14)
\end{aligned}$$

For $i = 1, \dots, N$, by (5.13) we have

$$\begin{aligned}
u(x) - u^I(x) &= \frac{\int_x^{x_{i+1}} \frac{ds}{p(s)}}{\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}} \int_{x_i}^x \left(\int_{x_i}^s \frac{dt}{p(t)}\right) g(s)w(s)ds \\
&\quad + \frac{\int_{x_i}^x \frac{ds}{p(s)}}{\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}} \int_x^{x_{i+1}} \int_s^{x_{i+1}} \frac{dt}{p(t)} g(s)w(s)ds \\
&\le \left(\int_x^{x_{i+1}} \frac{ds}{p(s)}\right) \int_{x_i}^x |g(s)|w(s)ds + \int_x^{x_{i+1}} \int_s^{x_{i+1}} \frac{dt}{p(t)} |g(s)|w(s)ds \\
&\le \int_{x_i}^x |g(s)|w(s) \int_s^{x_{i+1}} \frac{dt}{p(t)} ds + \int_x^{x_{i+1}} \int_s^{x_{i+1}} \frac{dt}{p(t)} |g(s)|w(s)ds \\
&= \int_{x_i}^{x_{i+1}} |g(s)| \int_s^{x_{i+1}} \frac{dt}{p(t)} w(s)ds \qquad (5.15)
\end{aligned}$$

The result follows again from the boundedness of the operator $\left(\widetilde{L} + q\right)^{-1}$ as an operator on $C[0,1]$. See Theorem 3.2 part (3) and the proof of part (3) of Theorem 3.5. □

In (5.11) taking $v_N = r_i$ for $i = 1, \cdots, N$, we obtain

$$\langle u^G - u^I, r_i\rangle_{V_p} = \langle q(u - u^G), r_i\rangle_w$$

$$\Rightarrow \sum_{j=1}^{N}(\alpha_j - u_j)\langle r_j, r_i\rangle_{V_p} = d_i.$$

This gives a system of equations

$$A\mathbf{e} = \mathbf{d} \tag{5.16}$$

where $A = (a_{ij}) = (\langle r_i, r_j\rangle_{V_p})$ is a symmetric and tridiagonal matrix given by:

$$a_{11} = \frac{1}{\int_{x_1}^{x_2}\frac{1}{p(s)}ds}, \tag{5.17}$$

$$a_{ii} = \frac{1}{\int_{x_{i-1}}^{x_i}\frac{1}{p(s)}ds} + \frac{1}{\int_{x_i}^{x_{i+1}}\frac{1}{p(s)}ds}, \quad i = 2, \cdots, N, \tag{5.18}$$

$$a_{i,i+1} = -\frac{1}{\int_{x_i}^{x_{i+1}}\frac{1}{p(s)}ds}, \quad i = 1, \cdots, N-1 \tag{5.19}$$

$\mathbf{e} = (e_i) = (\alpha_i - u_i)$ and $\mathbf{d} = (d_i)$ is given by:

$$d_1 = \int_{x_0}^{x_1} h(s)w(s)ds + \frac{\int_{x_1}^{x_2} h(s)w(s)\int_s^{x_2}\frac{dt}{p(t)}ds}{\int_{x_1}^{x_2}\frac{dt}{p(t)}}$$

and

$$d_i = \frac{\int_{x_{i-1}}^{x_i} h(s)w(s)\int_{x_{i-1}}^{s}\frac{dt}{p(t)}ds}{\int_{x_{i-1}}^{x_i}\frac{dt}{p(t)}} + \frac{\int_{x_i}^{x_{i+1}} h(s)w(s)\int_s^{x_{i+1}}\frac{dt}{p(t)}ds}{\int_{x_i}^{x_{i+1}}\frac{dt}{p(t)}}, \quad i > 1$$

where $h(s)$ stands for $q(s)(u(s) - u^G(s))$. The inverse of the matrix A, denoted by $B = (b_{ij})$, can be explicitly written as:

$$b_{ij} = \begin{cases} \int_{x_j}^{1}\frac{ds}{p(s)} & \text{if } i \le j \\ \int_{x_i}^{1}\frac{ds}{p(s)} & \text{if } i \ge j. \end{cases}$$

Therefore,

$$\begin{aligned} |e_i| &\le \sum_{j=1}^{N} b_{ij}|d_j| \\ &= \sum_{j=1}^{i}\int_{x_i}^{1}\frac{ds}{p(s)}|d_j| + \sum_{j=i+1}^{N}\int_{x_j}^{1}\frac{ds}{p(s)}|d_j| \\ &\le \sum_{j=1}^{N}\int_{x_j}^{1}\frac{ds}{p(s)}|d_j|. \end{aligned} \tag{5.20}$$

We see that:

$$\begin{aligned}
\int_{x_1}^{1} \frac{ds}{p(s)}|d_1| &\leq \int_{x_1}^{1} \frac{ds}{p(s)} \int_{x_0}^{x_1} |h(s)|w(s)ds + \int_{x_1}^{1} \frac{ds}{p(s)} \frac{\int_{x_1}^{x_2} |h(s)|w(s) \int_s^{x_2} \frac{dt}{p(t)} ds}{\int_{x_1}^{x_2} \frac{dt}{p(t)}} \\
&= \int_{x_1}^{1} \frac{ds}{p(s)} \int_{x_0}^{x_1} |h(s)|w(s)ds + \int_{x_1}^{x_2} \frac{ds}{p(s)} \frac{\int_{x_1}^{x_2} |h(s)|w(s) \int_s^{x_2} \frac{dt}{p(t)} ds}{\int_{x_1}^{x_2} \frac{dt}{p(t)}} \\
&\quad + \int_{x_2}^{1} \frac{ds}{p(s)} \frac{\int_{x_1}^{x_2} |h(s)|w(s) \int_s^{x_2} \frac{dt}{p(t)} ds}{\int_{x_1}^{x_2} \frac{dt}{p(t)}} \\
&\leq \int_{x_1}^{1} \frac{ds}{p(s)} \int_{x_0}^{x_1} |h(s)|w(s)ds + \int_{x_1}^{x_2} |h(s)|w(s) \int_s^{x_2} \frac{dt}{p(t)} ds + \\
&\quad \int_{x_2}^{1} \frac{ds}{p(s)} \int_{x_1}^{x_2} |h(s)|w(s)ds \\
&= \int_{x_1}^{1} \frac{ds}{p(s)} \int_{x_0}^{x_1} |h(s)|w(s)ds + \int_{x_1}^{x_2} |h(s)|w(s) \int_s^{1} \frac{dt}{p(t)} ds \\
&\leq \int_{x_0}^{x_1} |h(s)|w(s) \int_s^{1} \frac{dt}{p(t)} ds + \int_{x_1}^{x_2} |h(s)|w(s) \int_s^{1} \frac{dt}{p(t)} ds.
\end{aligned}$$

Also for $j = 2, \cdots, N$, by a similar approach, we have:

$$\begin{aligned}
\int_{x_j}^{1} \frac{ds}{p(s)}|d_j| &\leq \int_{x_j}^{1} \frac{ds}{p(s)} \int_{x_{j-1}}^{x_j} |h(s)|w(s)ds + \\
&\quad \int_{x_j}^{1} \frac{ds}{p(s)} \frac{\int_{x_i}^{x_{i+1}} |h(s)|w(s) \int_s^{x_{i+1}} \frac{dt}{p(t)} ds}{\int_{x_i}^{x_{i+1}} \frac{dt}{p(t)}} \\
&\leq \int_{x_{j-1}}^{x_j} |h(s)|w(s) \int_s^{1} \frac{dt}{p(t)} ds + \int_{x_j}^{x_{j+1}} |h(s)|w(s) \int_s^{1} \frac{dt}{p(t)} ds.
\end{aligned}$$

Substituting these two inequalities in (5.20), we obtain:

$$\begin{aligned}
|e_i| &\leq \int_{x_0}^{x_N} |h(s)|w(s) \int_s^{1} \frac{dt}{p(t)} ds + \int_{x_1}^{x_{N+1}} |h(s)|w(s) \int_s^{1} \frac{dt}{p(t)} ds \\
&\leq 2 \int_0^1 |h(s)|w(s) \int_s^{1} \frac{dt}{p(t)} ds \\
&= 2 \int_0^1 |q(s)(u(s) - u^G(s))|w(s) \int_s^{1} \frac{dt}{p(t)} ds.
\end{aligned}$$

Let $u - u^G = v$. Then:

$$\begin{aligned}
\max_{1\le i\le N} |\alpha_i - u_i| &\le 2\int_0^1 |q(s)| \left|\int_s^1 v'(t)dt\right| \int_s^1 \frac{dt}{p(t)} w(s)ds \\
&\le 2\int_0^1 |q(s)| \left(\int_s^1 p|v'|^2\right)^{1/2} \left(\int_s^1 \frac{dt}{p(t)}\right)^{1/2} \int_s^1 \frac{dt}{p(t)} w(s)ds \\
&\le 2\|v\|_{V_p} \int_0^1 |q(s)|w(s) \left(\int_s^1 \frac{dt}{p(t)}\right)^{3/2} ds \\
&= 2\left\|u - u^G\right\|_{V_p} \int_0^1 |q(s)| \left(\int_s^1 \frac{dt}{p(t)}\right)^{3/2} w(s)ds. \qquad (5.21)
\end{aligned}$$

Now:

$$\begin{aligned}
\left\|u - u^G\right\|_\infty &\le \left\|u - u^I\right\|_\infty + \left\|u^I - u^G\right\|_\infty \\
&\le \left\|u - u^I\right\|_\infty + 2\max_{1\le i\le N} |\alpha_i - u_i| \\
&\le \left\|u - u^I\right\|_\infty + 4C' \left\|u - u^G\right\|_{V_p} \qquad (5.22)
\end{aligned}$$

where $C' = \int_0^1 |q(s)| \left(\int_s^1 \frac{dt}{p(t)}\right)^{3/2} w(s)ds$. Since $q \in L_w^\infty(0,1)$ for the LC case then $C' < \infty$. For the LP1 case $\int_0^1 \left(\int_s^1 \frac{dt}{p(t)}\right)^{3/2} w(s)ds < \infty$ if $\int_s^1 \frac{1}{p} \in L_w^{3/2}(0,1)$. The result of the theorem, therefore, follows from Theorem 4.1 and Lemma 5.5.

Remark 5.2 *If as in Remark 4.1 we assume that $q \ge 0$, then (5.16) can be written in the form*

$$(\mathbf{A} + \mathbf{Q})\mathbf{e} = \mathbf{b}$$

where the elements a_{ij} of the matrix $\mathbf{A}$ are given by (5.17)-(5.19), $\mathbf{Q}$ is a tridiagonal matrix whose elements $q_{ij} = \langle qr_j, r_i\rangle_w$ are nonnegative, $\mathbf{b}$ is a vector with elements b_i given by

$$b_i = \langle q(u - u^I), r_i\rangle_w$$

and $\mathbf{e}$ is the vector $(e_i) = (\alpha_i - u_i)$. It is not hard to show that $\mathbf{A}$ is an $\mathbf{M}$-matrix (see Ortega[17]), $q_{ij} \le -a_{ij}$ $(i \ne j)$ for sufficiently small mesh size and that $\mathbf{A} + \mathbf{Q}$ is an $\mathbf{M}$-matrix with $(\mathbf{A} + \mathbf{Q})^{-1} \le \mathbf{A}^{-1}$. Thus $|\mathbf{e}| \le \mathbf{A}^{-1}|\mathbf{b}|$. A proof along the same lines of Theorem 4.2 would give

$$|e_i| \le 2\int_0^1 \left|q(s)\left(u(s) - u^I(s)\right)\right| \int_s^1 \frac{dt}{p(t)} w(s)ds.$$

Thus

$$\max_{1\le i\le N} |\alpha_i - u_i| \le 2C_q \left\|u - u^I\right\|_\infty.$$

Therefore,

$$\begin{aligned}\left\|u-u^G\right\|_\infty &\le \left\|u-u^I\right\|_\infty+\left\|u^G-u^I\right\|_\infty \\ &\le \left\|u-u^I\right\|_\infty+2\max_{1\le i\le N}|u_i-\alpha_i| \\ &\le (1+4C_q)\left\|u-u^I\right\|_\infty .\end{aligned}$$

The rate of convergence $\ell(\pi_N)$ now follows from Lemma 5.5.

Remark 5.3 *In the special case $p = w$ and p is increasing we can easily show that the method is $O(h^2)$ where* $h = \max_{0\le i\le N}\,(x_{i+1}-x_i)$.

5.8. Proof of Theorem 4.3

We first need the following lemma:

Lemma 5.6 $\left\|u-u^I\right\|_\infty \le C\max_{1\le i\le N}\left(\int_{x_i}^{x_{i+1}}\left(\int_s^{x_{i+1}}\frac{dt}{p(t)}\right)^2 w(s)\right)^{1/2}\|f\|_w .$

Proof. The proof follows from (5.14), (5.15) and the use of Cauchy–Schwartz inequality. □

The proof of the theorem is a direct consequence of this lemma, (5.22) and Theorem 4.1.

6. EXAMPLES

In this section we give some numerical examples to illustrate the generality of the method as well as verify the theoretical findings of the foregoing sections. In all these examples we take a uniform mesh size $h=\frac{b-a}{N+1}$ where $a=0$ and $b=1$. In Example 1 we take a LP1 case where the function $p(x)$ is not of the form x^α which is widely used in the literature. In this example f,q are continuous and $u'(0)=0$. In Example 2 we take again a LP1 case with $u'(0)\ne 0$ while f,q are continuous. In Example 3 we take a limit circle case with $f\in L^2_w$. In Example 4 we take a LP1 case with $w(x)=0$ on a set of positive measure. Finally in Example 5 we take a LP1 case with $f\in L^2_w$ such that the solution is unbounded at the singular point $x=0$. In all the examples except the last one the relative errors in the uniform norm are given. In the last example the relative error in V_p-norm is given. The theoretical order of convergence ($\sqrt{\ell(\pi_N)}$ or $\sqrt{\ell_1(\pi_N)}$) is also calculated for different values of h for each example.

Example 1 (LP1)

$$\begin{aligned} p(x) &= 1-e^{-x} \\ w(x) &= 1.0 \\ q(x) &= x \\ f(x) &= -x^3+x+2xe^{-x}-2e^{-x}+2 \\ u(x) &= 1-x^2 \\ \frac{\left\|u-u^G\right\|_\infty}{\|u\|_\infty} &= \begin{cases} 0.91671\times 10^{-2} & \text{when } h=0.1 \\ 0.91863\times 10^{-4} & \text{when } h=0.01\end{cases}\end{aligned}$$

$$\sqrt{\ell(\pi_N)} = \begin{cases} 0.3202 \text{ when } h = 0.1 \\ 0.10013 \text{ when } h = 0.01 \end{cases}$$

Example 2 (LP1)

$$\begin{aligned} p(x) &= x^2 \\ w(x) &= x \\ q(x) &= -x \\ f(x) &= -3x^3 + 4x^2 - 19x + 8 \\ u(x) &= 3x^2 - 4x + 1 \\ \frac{\|u - u^G\|_\infty}{\|u\|_\infty} &= \begin{cases} 0.37051 \quad \text{when } h = 0.1 \\ 0.039708 \quad \text{when } h = 0.01 \end{cases} \\ \sqrt{\ell(\pi_N)} &= \begin{cases} 0.22361 \text{ when } h = 0.1 \\ 0.070711 \text{ when } h = 0.01 \end{cases} \end{aligned}$$

Example 3 (LC)

$$\begin{aligned} p(x) &= x \\ w(x) &= 1 \\ q(x) &= x \\ f(x) &= x^2 \ln x - \ln x - 2 \\ u(x) &= x \ln x \\ \frac{\|u - u^G\|_\infty}{\|u\|_\infty} &= \begin{cases} 0.62537 \quad \text{when } h = 0.1 \\ 0.12518 \quad \text{when } h = 0.01 \end{cases} \\ \sqrt{\ell_1(\pi_N)} &= \begin{cases} 0.44721 \text{ when } h = 0.1 \\ 0.14142 \text{ when } h = 0.01 \end{cases} \end{aligned}$$

Example 4 (LP1)

$$p(x) = x, \quad q(x) = 0.0, \quad f(x) = 10.0$$

$$w(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{3}, \\ 0 & \text{if } \frac{1}{3} < x < \frac{2}{3}, \\ 1 & \text{if } \frac{2}{3} \le x \le 1. \end{cases} \tag{6.1}$$

$$u(x)=\begin{cases} -10x+\frac{20}{3}+\frac{10}{3}\ \ln\left(\frac{4}{3}\right) & \text{if } 0\le x\le \frac{1}{3}, \\ -\frac{10}{3}\ln x+\frac{10}{3}+\frac{20}{3}\ln\left(\frac{2}{3}\right) & \text{if } \frac{1}{3}\le x\le \frac{2}{3}, \\ \frac{10}{3}\ln x+10(1-x) & \text{if } \frac{2}{3}\le x\le 1. \end{cases} \tag{6.2}$$

$$\begin{aligned} \frac{\left\|u-u^G\right\|_\infty}{\|u\|_\infty} &= 0.04371 \ \text{ when } \ h=0.0333 \\ \sqrt{\ell(\pi_N)} &= 0.18257 \ \text{ when } \ h=0.0333 \end{aligned}$$

Example 5 (LP1)

$$\begin{aligned} p(x) &= x^3 \\ w(x) &= x^{\frac{3}{2}} \\ q(x) &= x \\ f(x) &= \sqrt{x}-x+\frac{3}{4x} \\ u(x) &= \frac{1}{\sqrt{x}}-1 \\ \frac{\left\|u-u^G\right\|_{V_p}}{\|u\|_{V_p}} &= 0.05773, \ \ h=0.01 \\ \sqrt{\ell(\pi_N)} &= 0.28284, \ \ h=0.01. \end{aligned}$$

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