QUASI COMPACTNESS WITH RESPECT TO AN IDEAL

M. E. Abd El-Monsef*, E. F. Lashien, and A. A. Nasef

Tanta University
Tanta, Egypt

الخلاصـــة :

يُعـرَّف المثالى I على المجموعة X بأنّـه تجـمّـع المجموعات الجزئية المغلق تحت تأثير عمليتي الاحتواء والاتحاد المحدود . ولقد أدخل (نيوكمب) عام ١٩٦٧ و (رانسن) عام ١٩٧٢ مفهوم الأصماط I كتعميم للأصماط والذي يتطلب أنّـه لأيِّ غطاءٍ مفتوح للفراغ توجد عائلة جزئية تغطي الفراغ ماعدا مجموعـةٍ من المثالي .

تقدِّم في هذا البحث وندرس نوعين من المفاهيم كتعميم للأصماط ؛ الأول شبه الاصماط I ، والثاني شبه الأصماط المعدود I . وقد تم تقوية بعض النتائج في البحثين [١ ، ٢] من قائمة المراجع .

ABSTRACT

Given a nonempty set X, an ideal I on X is a collection of subsets of X closed under finite union and subset operations. Newcomb (1967) and Rančin (1972) defined a generalization of compactness (I-compactness) which requires that an open cover of a space have a finite subcollection which covers all the space except for a set in the ideal. In this paper we introduce and study two different notions of generalized compactness namely, quasi I-compactness and countable quasi I-compactness. Classical results concerning quasi H-closed and lightly compact spaces are obtained by letting $I\,.\,\{\phi\}$. Some results in [1, 2] are improved.

*Address for correspondence:
Department of Mathematics
Faculty of Science
Tanta University
Tanta, Egypt

QUASI COMPACTNESS WITH RESPECT TO AN IDEAL

1. INTRODUCTION

The concept of compactness modulo an ideal was introduced by Newcomb [3] and Rančin [4], and studied by Hamlett, Rose, and Janković in [1] and [2]. Newcomb also defined the concept of countable compactness modulo an ideal in [3]. The aim of this paper is to introduce and study quasi compact and countably quasi compact spaces *via* ideals as a generalization of I-compact and countably I-compact spaces. The concepts of quasi H-closed, H-closed, and lightly compact spaces are special cases.

Throughout the present paper, (X,τ) and (Y,σ) (or simply X and Y) denote topological spaces on which no separation axiom is assumed unless explicitly stated. A subset S of a topological space is said to be regular open (resp. regular closed, preopen [5]) if $Int(Cl(S)) = S$(resp. $Cl(Int(S)) = S$, $S \subseteq Int(Cl(S))$), where $Cl(S)$ (resp. $Int(S)$) denotes the closure (resp. interior) of S. The complement of S will be denoted by $(X\text{-}S)$.

Given a nonempty set X, a collection I of subsets of X is called an ideal [6] if:

(1) $A \in I$ and $B \subseteq A$ implies $B \in I$ (heredity), and
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$ (finite additivity).

If $X \notin I$ then I is called a proper ideal. We will denote by (X,τ,I) a topological space (X,τ) and an ideal I of subsets of X. Given a space (X,τ,I), we denote by $\tau^*(I)$ the topology generated by the basis $\beta(I,\tau) = \{U\text{-}E\colon U \in \tau,\ E \in I\}$ [7]. A bijection $f\colon (X,\tau,I) \to (Y,\sigma,J)$ is called a $*$-homeomorphism if $f\colon (X,\tau^*) \to (Y,\sigma^*)$ is a homeomorphism [8]. A space (X,τ) is said to be extremally disconnected if $Cl(U) \in \tau$ for every $U \in \tau$. Recall that a space (X,τ) is said to be quasi H-closed, abbreviated QHC, iff every open cover of X has a finite subcollection which covers a dense subset of X. A space is said to be H-closed iff it is Hausdorff and QHC. We will say that a space (X,τ) lightly compact iff for every countable open cover $\{U_\alpha\colon \alpha \in \nabla\}$ of X there exists a finite subcollection $\{U_{\alpha i}\colon i = 1,2,\dots,n\}$ such that $X = Cl(\cup\{U_{\alpha i}\colon i = 1,2,\dots,n\})$. A space (X,τ,I) is said to be I-compact [3] resp. countably I-compact [3]) iff for every open (resp. countable open) cover $\{U_\alpha\colon \alpha \in \nabla\}$ of X there exists a finite subfamily $\{U_{\alpha i}\colon i = 1,2,\dots,n\}$ such that $X\text{-} \cup\{U_{\alpha i}\colon i = 1,2,\dots,n\} \in I$.

2. QUASI I-COMPACT SPACES

Definition 2.1. A space (X,τ,I) is said to be quasi I-compact (abbreviated as QI-compact) if for every open cover $\{U_\alpha\colon \alpha \in \nabla\}$ of X, there exists a finite subcollection $\{U_{\alpha i}\colon i = 1,2,\dots,n\}$ such that $X\text{-} \cup\{Cl(U_{\alpha i})\colon i = 1,2,\dots,n\} \in I$.

Remark 2.1. The class of QI-compact spaces contains the class of I-compact spaces and the reverse does not hold.

Example 2.1. Let $X = [0,1]$ be the closed unit interval in the real line and let τ be the topology generated by using the usual subspace topology and the rationals as a subbase. One can deduce that (X,τ) is QI_f-compact but not I_f-compact, where I_f denotes the ideal of finite subsets of X.

The following two immediate theorems are stated without proof.

Theorem 2.1. Let (X,τ,I) be a space, then we have:

(*a*) (X,τ) is $Q\{\phi\}$-compact iff (X,τ) is QHC.

(*b*) If (X,τ) is Hausdorff, then (X,τ) is $Q\{\phi\}$-compact iff (X,τ) is H-closed.

(*c*) If (X,τ) is E.D., then (X,τ) is QI-compact iff (X,τ) is I-compact. □

Theorem 2.2. (X,τ) is QHC iff (X,τ,I_f) is QI_f-compact. □

Corollary 2.3. If (X,τ) is Hausdorff, then (X,τ) is H-closed iff (X,τ) is QI_f-compact. □

Recall that (X,τ) is a Baire space [9] iff $I_m \cap \tau = \phi$, where I_m denotes the ideal of meager (first category) subsets of X.

Corollary 2.4. If (X,τ) is a space, consider the following:

(*i*) (X,τ) is QI_f-compact.
(*ii*) (X,τ) is QHC.
(*iii*) (X,τ) is I_n-compact, where I_n denotes the ideal of nowhere dense subsets of X.
(*iv*) (X,τ) is H-closed.
(*v*) (X,τ) is I_m-compact.

Then we have:

(1) The properties from (*i*) to (*iii*) are equivalent.

(2) The properties (i), (iii), and (iv) are equivalent if (X,τ) is Hausdorff.

(3) The properties (i), (ii), and (v) are equivalent if (X,τ) is a Baire space.

(4) The properties from (i) to (v) are equivalent if (X,τ) is Baire space and Hausdorff.

Proof.

(1) Follows from Theorem 2.2 and Corollary 1.5 (1) of [1].

(2) This follows immediately from Corollary 2.3 and Corollary 1.5 (2) [1].

(3) The proof is immediate from Theorem 2.2 and Corollary 1.6 of [1].

(4) The result follows immediately from Corollary 2.3 and Corollary 1.6 of [1]. □

Theorem 2.5. Let (X,τ,I) be a space. Then the following are equivalent:

(a) (X,τ) is QI-compact.

(b) For every regular open cover $\{U_\alpha: \alpha \in \nabla\}$ of X, there exists a finite subfamily $\{U_{\alpha i}: i=1,2,\ldots.,n\}$ such that $X\text{-} \cup \{Cl(U_{\alpha i}): i=1,2,\ldots.,n\} \in I$.

(c) For each family $\{F_\alpha: \alpha \in \nabla\}$ of closed (regular closed) sets of X for which $\cap\{F_\alpha: \alpha\in\nabla\}=\phi$, there exists a finite subfamily $\{F_{\alpha i}: i=1,2,\ldots.,n\}$ such that $\cap\{Int(F_{\alpha i}): i=1,2,\ldots.,n\}\in I$.

Proof.

$(a)\rightarrow(b)$: Straightforward.

$(b)\rightarrow(a)$: Let $\{U_\alpha: \alpha\in\nabla\}$ be an open cover of X, then $\{Int(Cl(U_\alpha)): \alpha\in\nabla\}$ is a regular open cover of X, then there exists a finite subfamily $\{Int(Cl(U_{\alpha i})): i = 1, 2,\ldots., n\}$ such that $X\text{-}\cup\{Cl(Int(Cl(U_{\alpha i}))): i=1,2,\ldots.,n\}\in I$. This implies, $X\text{-}\cup\{Cl(U_{\alpha i}): i=1,2,\ldots.,n\}\in I$.

$(a)\rightarrow(c)$: Let $\{F_\alpha: \alpha\in\nabla\}$ be a family of closed sets for which $\cap\{F_\alpha: \alpha\in\nabla\}=\phi$. Then $\{X\text{-}F_\alpha: \alpha\in\nabla\}$ is an open cover of X; by (a) there exists a finite subfamily $\{X\text{-}F_{\alpha i}: i=1,2,\ldots.,n\}$ such that $X\text{-}\cup\{Cl(X\text{-}F_{\alpha i}): i=1,2,\ldots.,n\}\in I$. Hence $X\text{-}\cup\{(X\text{-}Int\ F_{\alpha i}): i=1,2,\ldots.,n\}\in I$. This implies, $X\text{-}(X\text{-}\cap\{Int\ F_{\alpha i}: i=1,2,\ldots.,n\})\in I$. Thus $\cap\{Int(F_{\alpha i}): i=1,2,\ldots.,n\}\in I$.

$(c)\rightarrow(a)$: Let $\{U_\alpha: \alpha\in\nabla\}$ be an open cover of X. Then $\{X\text{-}U_\alpha: \alpha\in\nabla\}$ is a collection of closed sets and $\cap\{(X\text{-}U_\alpha): \alpha\in\nabla\}=\phi$. Hence there exists a finite subcollection $\{(X\text{-}U_{\alpha i}): i=1,2,\ldots.,n\}$ such that $\cap\{Int(X\text{-}U_{\alpha i}): i = 1, 2,\ldots., n\}\in I$, $\cap\{(X\text{-}Cl(U_{\alpha i})): i = 1, 2,\ldots., n\}\in I$. Thus, $X\text{-}\cup\{Cl(U_{\alpha i}): i=1,2,\ldots.,n\}\in I$. □

Theorem 2.6. A space (X,τ,I) is QI-compact iff for each preopen cover $\{U_\alpha: \alpha\in\nabla\}$ of X, there exists a finite subcollection $\{U_{\alpha i}: i=1,2,\ldots.,n\}$ such that $X\text{-}\cup\{Cl(U_{\alpha i}): i=1,2,\ldots.,n\}\in I$.

Proof. Sufficiency is obvious. To show necessity, assume (X,τ,I) is QI-compact and let $\{U_\alpha: \alpha\in\nabla\}$ be a preopen cover of X. So, $\{Int(Cl(U_\alpha)): \alpha\in\nabla\}$ is an open cover of X; from the hypothesis, there exists a finite subcollection $\{Int(Cl(U_{\alpha i})): i=1,2,\ldots.,n\}$ such that $X\text{-}\cup\{Cl(Int(Cl(U_{\alpha i}))): i=1,2,\ldots.,n\} = X\text{-}\cup\{Cl(U_{\alpha i}): i=1,2,\ldots.,n\}\in I$. □

Theorem 2.7. If (X,τ,I) is QI-compact, and J is an ideal on X with $J\supseteq I$, then (X,τ,J) is QJ-compact.

Proof. This is obvious. □

The following two theorems are slight improvements of Theorems 1.3 and 1.4 of reference [1].

Theorem 2.8. If (X,τ,I_c) is QI_c-compact, where I_c denotes the ideal of countable subsets of X, then (X,τ) is Lindelöf.

Proof. Suppose that $\{U_\alpha: \alpha\in\nabla\}$ is an open cover of X, then from the hypothesis there exists a finite subfamily $\{U_{\alpha i}: i=1,2,\ldots.,n\}$ such that $X\text{-}\cup\{Cl(U_{\alpha\, i}): i = 1, 2,\ldots., n\}\in I_c$, *i.e.* $X\text{-}\cup\{Cl(U_{\alpha i}): i=1,2,\ldots.,n\}$ has a countable subcover. □

Given a space (X,τ,I), I is said to be τ-boundary [3] if $I\cap\tau=\{\phi\}$.

Theorem 2.9. Let (X,τ,I) be a space with I_n the ideal of nowhere dense subsets of X.

(a) If (X,τ,I) is QI-compact, and I is τ-boundary, then (X,τ) is QHC.

(b) If $I\supseteq I_n$ and (X,τ) is QHC, then (X,τ) is QI-compact.

Proof.

(a) Let $\{U_\alpha: \alpha\in\nabla\}$ be an open cover of X, there exists a finite subcollection $\{U_{\alpha i}: i=1,2,\ldots.,n\}$ such that $X\text{-}\cup\{Cl(U_{\alpha i}): i=1,2,\ldots.,n\}=E\in I$, since I is τ-boundary, then $Int(E) = \phi$ and hence (X, τ) is QHC.

(*b*) It follows immediately. □

Theorem 2.10. Let (X, τ, I) be a space. If (X, τ^*) is QI-compact, then (X, τ) is QI-compact.

Proof. Follows from the fact that $\tau^* \supseteq \tau$. □

Remark 2.2. [10]. Let (X, τ, I) be a space, then $\tau^*(I, \tau) = \tau$ iff every member of I is τ-closed.

Remark 2.3. Let (X, τ, I) be a space such that every member of I is τ-closed. Then (X, τ) is QI-compact iff (X, τ^*) is QI-compact.

3. PRESERVATION BY FUNCTIONS

The following lemma will be used in the sequel.

Lemma 3.1. [3]. Let f: $(X, \tau, I) \rightarrow (Y, \sigma)$ be a function. Then $f(I) = \{f(E)$: $E \in I\}$ is an ideal on Y. □

It is well known that the image of a compact space is compact under a continuous function. This result is generalized as follows.

Theorem 3.2. If f: $(X, \tau, I) \rightarrow (Y, \sigma)$ is a continuous surjection, and (X, τ) is QI-compact, then (Y, σ) is $Qf(I)$-compact.

Proof. Let $\{V_\alpha$: $\alpha \in \nabla\}$ be a σ-open cover of Y, then $\{f^{-1}(V_\alpha)$: $\alpha \in \nabla\}$ is a τ-open cover of X, from assumption, there exists a finite subcollection $\{f^{-1}(V_{\alpha i})$: $i = 1, 2, \ldots, n\}$ such that X- $\cup$ $\{Cl(f^{-1}(V_{\alpha i}))$: $i = 1, 2, \ldots, n\} \in I$ implies, Y- $\cup$ $\{Cl(V_{\alpha i})$: $i = 1, 2, \ldots, n\} \in f(I)$. Therefore (Y, σ) is $Qf(I)$-compact. □

Theorem 3.3. Let f: $(X, \tau, I) \rightarrow (Y, \sigma, f(I))$ be a *-homeomorphism such that every member of I is τ-closed. Then (X, τ) is QI-compact iff (Y, σ) is $Qf(I)$-compact.

Proof: Necessity. Assume that (X, τ) is QI-compact, and let $\{V_\alpha$: $\alpha \in \nabla\}$ be a σ-open cover of Y. Then $\{f^{-1}(V_\alpha)$: $\alpha \in \nabla\}$ is a τ^*-open cover of X, from Remark 2.3, there exists a finite subcollection $\{f^{-1}(V_{\alpha i})$: $i = 1,2,\ldots,n\}$ such that X- $\cup \{Cl(f^{-1}(V_{\alpha i}))$: $i = 1,2,\ldots,n\} = E \in I$. Consequently, Y- $\cup \{Cl(V_{\alpha i})$: $i = 1,2,\ldots,n\} = f(E) \in f(I)$ and it is shown that (Y, σ) is $Qf(I)$-compact.

Sufficiency. Assume that (Y, σ) is $Qf(I)$-compact and let $\{U_\alpha$: $\alpha \in \nabla\}$ be a τ-open cover of X. Then $\{f(U_\alpha)$: $\alpha \in \nabla\}$ is a σ^*-open cover of Y, and there exists a finite subcollection $\{f(U_{\alpha i})$: $i = 1,2,\ldots,n\}$ such that Y- $\cup \{Cl f(U_{\alpha i})$: $i = 1,2,\ldots,n\} = f(E) \in f(I)$. Then X- $\cup \{Cl((U_{\alpha i})$: $i = 1,2,\ldots,n\} \subset f^{-1}[Y$- $\cup \{Cl (f(U_{\alpha i}))$: $i = 1,2,\ldots,n\}] = E \in I$, thus (X, τ) is QI-compact. □

Definition 3.1. [11]. A function f: $(X, \tau, I) \rightarrow (Y, \sigma)$ is said to be pointwise I-continuous (PI C) if f: $(X, \tau^*) \rightarrow (Y, \sigma)$ is continuous.

Clearly continuous functions are PI C (since $\tau^* \supseteq \beta \supseteq \tau$).

Theorem 3.4. Let f: $(X, \tau, I) \rightarrow (Y, \sigma)$ be a surjection. If f is PI C and (X, τ) is QI-compact, then (Y, σ) is $Qf(I)$-compact.

Proof. The result follows immediately from Theorem 3.2 and Remark 2.3. □

Ideals are not as well behaved with respect to function inverses as the following example shows.

Example 3.1. [1]. Let X and Y be the reals with the usual topology and let I be the ideal on Y of all subsets of the unit interval [0, 1]. Define f: $X \rightarrow Y$ by $f(x) = |x|$. Observe that $[½, ¾] \subseteq f^{-1}([0, 1])$ but $[½, ¾] \neq f^{-1}(A)$ for any $A \subseteq [0, 1]$. Thus the collection $f^{-1}(I) = \{f^{-1}(E)$: $E \in I\}$ is not hereditary and hence not an ideal.

Lemma 3.5. [1]. If f: $(X, \tau) \rightarrow (Y, \sigma, J)$ is an injection, then $f^{-1}(J)$ is an ideal on X. □

Theorem 3.6. Let f: $(X, \tau) \rightarrow (Y, \sigma, J)$ be an open bijection. If (Y, σ, J) is QJ-compact, then (X, τ) is $Qf^{-1}(J)$-compact.

Proof. We observe that f^{-1}: $(Y, \sigma, J) \rightarrow (X, \tau)$ is a continuous surjection and by applying Theorem 3.2 and Lemma 3.5, we have the result. □

Theorem 3.7. Let (X_α, τ_α) be a family of spaces and let I be an ideal on $(\Pi X_\alpha, \Pi\, \tau_\alpha)$. If ΠX_α is QI-compact, then each space (X_α, τ_α) is $QP_\alpha(I)$-compact, where P_α is the projection map in coordinate α.

Proof. Follows immediately from Theorem 3.2 and the fact that each P_α is a continuous surjection. □

4. SETS QI-COMPACT RELATIVE TO A SPACE

Definition 4.1. A subset S of a space (X, τ, I) is said to be QI-compact relative to X if for every open cover $\{U_\alpha$: $\alpha \in \nabla\}$ of S, there exists a finite subcollection $\{U_{\alpha i}$: $i = 1,2,\ldots,n\}$ such that S - $\cup \{Cl(U_{\alpha i})$: $i = 1,2,\ldots,n\} \in I$.

Recall that $S \subseteq (X, \tau)$ is an H-subset if every open cover of S contains a finite subcollection whose closures cover S.

The following results are immediate and the obvious proofs are omitted.

Theorem 4.1. A subset S of a space (X, τ, I) is an H-subset iff it is $Q\{\phi\}$-compact iff it is QI_f-compact. □

Theorem 4.2. For a subset S of a space (X, τ, I), the following are equivalent:

(*a*) S is QI-compact relative to X,

(*b*) For every cover $\{V_\alpha: \alpha \in \nabla\}$ of S by preopen sets of X, there exists a finite subfamily $\{V_{\alpha i}: i = 1,2,\dots,n\}$ such that $S\text{-} \cup \{Cl(V_{\alpha i}): i = 1,2,\dots,n\} \in I$. □

Theorem 4.3. Any H-subset is a QI_n-compact subset. □

Theorem 4.4. If $S_k, k = 1,2$ are QI-compact sets relative to a space (X, τ, I), then $S_1 \cup S_2$ is QI-compact relative to X.

Proof. Let $\{V_\alpha: \alpha \in \nabla\}$ be an open cover of $S_1 \cup S_2$. Then it is an open cover of S_k for $K = 1,2$. Since S_k is QI-compact relative to X, then there exists a finite subcollection $\{V_{\alpha i}: i = 1, 2,\dots, n\}$ such that $S_k\text{-} \cup \{Cl(V_{\alpha i}): i = 1, 2,\dots, n\} \in I$ for $K = 1, 2$. Therefore, $S_1 \cup S_2\text{-} \cup \{Cl(V_{\alpha i}): i = 1, 2,\dots, n\} \in I$. So, $S_1 \cup S_2$ is QI-compact relative to X. □

Corollary 4.5. The intersection of two open sets having QI-compact complements is also open having QI-compact complement.

Proof. Follows directly from Theorem 4.4. □

If I is an ideal on X and $S \subseteq X$, we denote the restriction of I to S by $I|S = \{E \cup S: E \in I\}$. It is easily seen that $I|S$ is an ideal [1].

Theorem 4.6. Let (X, τ, I) be a space, if $(S, \tau|S)$ is $QI|S$-compact, then S is QI-compact, for every $S \subseteq X$.

Proof. Let $\{U_\alpha: \alpha \in \nabla\}$ be a τ-open cover of S. Then $\{U_\alpha \cap S: \alpha \in \nabla\}$ is a $\tau|S$-open cover of S. There exists then a finite subfamily $\{U_{\alpha i} \cap S: i = 1,2,\dots,n\}$ such that $S\text{-} \cup \{Cl(U_{\alpha i} \cap S): i = 1, 2,\dots, n\} \in I|S \subseteq I$. Hence S is QI-compact. □

5. COUNTABLY QI-COMPACT SPACES

Definition 5.1. A space (X, τ, I) is said to be countably QI-compact if for every countable open cover $\{V_\alpha: \alpha \in \nabla\}$ of X, there exists a finite subfamily $\{V_{\alpha i}: i = 1,2,\dots,n\}$ such that $X\text{-} \cup \{Cl(V_{\alpha i}): i = 1,2,\dots,n\} \in I$.

From the above definition, we deduce the following theorem.

Theorem 5.1. For a space (X, τ, I), the following are equivalent:

(*a*) (X, τ) is lightly compact.

(*b*) (X, τ, I) is countably QI-compact, where $I = \{\phi\}$.

(*c*) (X, τ, I) is countably QI_f-compact. □

Remark 5.1.

(*a*) Every QI-compact space is countably QI-compact.

(*b*) Every countably I-compact is countably QI-compact.

Question. The authors need an example for countably QI-compact and not for QI-compact.

Theorem 5.2. If (X, τ, I) is countably QI-compact and Lindelöf, then (X, τ, I) is QI-compact. □

Theorem 5.3. Let (X, τ, I) be a space. The following are equivalent:

(*a*) (X, τ) is countably QI-compact.

(*b*) For every countable regular open cover $\{U_\alpha: \alpha \in \nabla\}$ of X, there exists a finite subfamily $\{U_{\alpha i}: i = 1,2,\dots,n\}$ such that $X\text{-} \cup \{Cl(U_{\alpha i}): i = 1,2,\dots,n\} \in I$.

(*c*) For every countable family $\{F_\alpha: \alpha \in \nabla\}$ of closed sets such that $\cap \{F_\alpha: \alpha = 1,2,\dots,\infty\} = \phi$ there exists a finite subfamily $\{F_{\alpha i}: i = 1,2,\dots,n\}$ such that $\cap \{Int(F_{\alpha i}): i = 1,2,\dots,n\} \in I$.

Proof. The proof is similar to that of Theorem 2.5 and is thus omitted. □

The following two theorems are slight improvements of Theorem 2.4 [2].

Theorem 5.4. If (X, τ, I) is countably QI-compact and I is τ-boundary, then (X, τ) is lightly compact. □

Theorem 5.5. Let (X,τ,I) be a space. If $I \supseteq I_n$ and (X,τ) is lightly compact, then (X,τ) is countably QI-compact.

Proof. Let $\{U_\alpha : \alpha \in \nabla\}$ be a countable open cover of X. From hypothesis, there exists a finite subfamily $\{U_{\alpha i}: i = 1,2,\ldots,n\}$ such that $X = Cl(\cup U_{\alpha i}: i = 1,2,\ldots,n)$. Thus $X - \cup\{Cl(U_{\alpha i}): i = 1,2,\ldots,n\} \subseteq X - \cup\{(U_{\alpha i}): i = 1,2,\ldots,n\} \in I_n \subseteq I$. □

The following theorem is an improvement of Corollary 2.5 [2]

Theorem 5.6. Let (X,τ) be a space. Then (X,τ) is lightly compact iff (X,τ) is countably QI_n-compact.

Proof. Follows from the fact that I_n is τ-boundary and by applying Theorem 5.5. □

Corollary 5.7. If (X,τ) is a completely regular T_1-space, then (X,τ) is pseudocompact iff (X,τ) is countable QI_n-compact.

Proof. It is well known [12] that in a completely regular T_1-space, pseudocompactness is equivalent to light compactness. The result then follows from Theorem 5.6. □

Theorem 5.8. Let (X,τ) be a Baire space. Then the following are equivalent:

(*a*) (X,τ) is countably QI_m-compact.

(*b*) (X,τ) is lightly compact.

(*c*) (X,τ) is countably QI_n-compact.

Proof. Follows from the definition of Baire space and Theorems 5.5 and 5.6. □

Theorem 5.9. Let (X,τ,I) be a space. If (X,τ^*) is countably QI-compact, then (X,τ) is countably QI-compact.

Proof. The result follows immediately by from the observtion that $\tau^*(I) \supseteq \tau$. □

The obvious proofs of the following theorems are omitted.

Theorem 5.10. If (X,τ,I) is countably QI-compact and J is an ideal on X such that $J \supseteq I$, then (X,τ,J) is countably QJ-compact. □

Theorem 5.11. Let $f: (X,\tau,I) \to (Y,\sigma)$ be a continuous surjection. If (X,τ,I) is countably QI-compact, then (Y,σ) is countably $Qf(I)$-compact. □

Theorem 5.12. If $f: (X,\tau) \to (Y,\sigma,J)$ is an open bijection and (Y,σ) is countably QJ-compact, then (X,τ) is countably $Qf^{-1}(J)$-compact. □

Theorem 5.13. Let $\{(X_\alpha,\tau_\alpha): \alpha \in \nabla\}$ be a family of spaces with I an ideal on $(\Pi X_\alpha, \Pi\,\tau_\alpha)$. If ΠX_α is countably QI-compact, then each factor (X_α,τ_α) is $QP_\alpha(I)$-compact, where P_α is the projection map in coordinate α.

Proof. The result follows from Theorem 5.11, since each P_α is a continuous surjection. □

ACKNOWLEDGEMENT

We would like to thank the referees for valuable comments and suggestions.

REFERENCES

[1] T. R. Hamlett and D. Janković, "Compactness with Respect to an Ideal", *Boll. U.M.I.*, **7(4-B)** (1990), p. 849.

[2] T. R. Hamlett, D. Janković, and D. Rose, "Countable Compactness with Respect to an Ideal", *Math. Chron.*, **20** (1991), p. 109.

[3] R. L. Newcomb, "Topologies Which are Compact Modulo an Ideal", *Ph.D. Dissertation, University of California at Santa Barbara,* 1967.

[4] D. V. Rančin, "Compactness Modulo an Ideal", *Soviet Math. Dokl.*, **13(1)** (1972), p. 193.

[5] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El Deeb, "On Precontinuous and Weak Precontinuous Mappings", *Proc. Math. and Phys. Soc., Egypt,* **53** (1982), p. 47.

[6] K. Kuratowski, *Topologies I.* : Warszawa, 1933.

[7] R. Vaidyanathaswamy, *Set Topology.* New York: Chelsea Publishing Company, 1960.

[8] T. R. Hamlett and D. Rose, "*-Topological Properties", *Internat. J. Math. and Math. Sci.*, **13(3)** (1990), p. 507.

[9] R. C. Haworth and R. A. Mc Coy, "Baire Spaces", *Dissertationes Mathematicae,* **CXLI** (1977), p. 1.

[10] P. Samuels, "A Topology Formed from a Given Topology and an Ideal", *J. London Math. Soc.*, **2(10)** (1975), p. 409.

[11] K. Kaniewski and Z. Piotrowski, "Concerning Continuity Apart from a Meager Set", *Proc. Math. Soc.*, **98** (1986), p. 324.

[12] S. Willard, *General Topology.* London: Addison–Wesley, 1970.

Paper Received 2 November 1991; Revised 27 May 1992.