COMPLEXITY OF IDENTIFYING A FEASIBLE BASIC VECTOR INVOLVING SPECIFIC VARIABLES IN A TRANSPORTATION PROBLEM

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1. INTRODUCTION

A fairly common class of problems that arise in modeling or solving resource allocation problems relates to examining and/or identifying particular structural properties in simple linear programming problems. One such problem, posed by Akinc [1], was encountered in the analysis of an m-dimensional knapsack problem. The problem can be described as follows:

Given the vectors $\mathbf{a} = (a_i, i = 1, \ldots, m)$ and $\mathbf{b} = (b_j, j = 1, \ldots, n)$ with positive integer elements, and the system of linear constraints:

$$\sum_{i=1}^{m} x_{i,j} = b_j, \; j = 1, \ldots, n$$

$$\sum_{j=1}^{n} x_{i,j} + s_i = a_i, \; i = 1, \ldots, m$$

$$x_{i,j}, s_i \geq 0$$

Is there a feasible basic vector for the above system in which all the variables, $(s_i, i = 1, \ldots, m)$, are basic? If so, how can such a basic vector be identified? In this paper, we will demonstrate that this problem is NP-Complete.

2. DISCUSSIONS AND RESULTS

The above system of constraints is trivially transformed into an equivalent m by $n + 1$ balanced transportation problem. The last column (column $n + 1$) in the two-dimensional (array) representation of this problem contains the slack variables s_1 through s_m. The problem then becomes that of identifying a feasible basic vector for a balanced transportation problem which includes all the variables in a given column of the corresponding transportation array as basic. The following theorem establishes that this problem is NP-Complete by showing that a special case of it is the *Equal Sums Partition Problem*, which is known to be NP-Complete [2].

Theorem. The problem of checking whether there exists a feasible basic vector to a balanced transportation problem in which all the variables in a specific column of the associated transportation array are basic, is NP-Complete.

Proof. That the problem is in NP is clear, since the desired feasible basic vector, if one exists, can be identified by enumerating a finite number of feasible bases. That the problem is NP-Complete is shown as follows:

Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a vector of positive integers, where $\Sigma_{j=1}^{n} d_j = W$ is an even number. The *Equal Sums Partition Problem* is the problem of partitioning the elements of $\mathbf{d}$ into two equal sum subsets. It is, of course, known to be NP-Complete [2].

Now, consider the 3 by $(n + 1)$ balanced transportation problem whose supply and demand vectors are $(W/2, W/2, 1)$ and $(d_1, d_2, \ldots, d_n, 1)$, respectively.

Claim. There exists a feasible basic vector for this transportation problem which includes all the variables in the "$n + 1$"st column of the corresponding array if and only if there exists an equal sums partition of the elements in $\mathbf{d}$.

Proof. The "if" part of the claim can be shown easily. To see this, suppose (P, Q) is an equal sums partition of $\mathbf{d}$. Then, the basic vector composed of $(x_{1,j}, j \in P)$, $(x_{2,j}, j \in Q)$, and $(x_{1,n+1}, x_{2,n+1}, x_{3,n+1})$ has the desired property.

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* This paper pertains to a problem studied during 1984 at the University of Michigan.

The "only if" part of the claim is shown as follows: Suppose that $\mathbf{x}$ is a feasible basic vector for the above transportation problem satisfying the desired property. This implies that $\mathbf{x}$ contains at most one variable from each of the first n columns of the transportation array as basic. Otherwiser $\mathbf{x}$ would contain a θ-loop (*i.e.*, variables associated with a dependent subset of cells in the transportation array), which is contrary to the assumption that $\mathbf{x}$ is a basic vector. Let $\bar{\mathbf{x}}$ be the Basic Feasible Solution (BFS) associated with $\mathbf{x}$. Now, there are two possible cases:

Case 1: $\bar{x}_{3,n+1} = 1$. In this case, the remaining basic variables with non-zero value in each of the first two rows of the transportation array identify the desired partition.

Case 2: $\bar{x}_{3,n+1} = 0$. In this case, the feasibility of $\bar{\mathbf{x}}$ implies that for some j from 1 to n, d_j must have been 1, and hence, $\bar{x}_{3,j} = 1$. Moreover, for some i from 1 to 2, $\bar{x}_{i,n+1}$ must also be equal to 1. Therefore, bringing $x_{i,j}$ into this basic vector to replace $\mathbf{x}_{3,j}$ leads to a new feasible basic vector for this transportation problem that satisfies the desired property and the condition discussed in Case 1, thereby leading to the desired partition.

This completes the proof of the above claim, which establishes that the "Equal Sums Partition Problem" is a special case of the problem posed in the statement of this theorem. The truth of the theorem follows immediately. □

The above theorem, in turn, establishes that the problem posed by Akinc is also NP-Complete, since it is essentially equivalent to the problem addressed within the theorem.

3. CONCLUDING REMARKS

An interesting follow-up to the problem addressed here is to examine the complexity of identifying a feasible basic vector for a transportation problem that includes an arbitrary subset (of cardinality two or more) of the variables. In the case of general LPs, the problem of identifying a feasible basic vector involving even two specified variables as basic seems to be hard, as there are no known efficient algorithms to solve it [3]. However, whether this is also true for transportation problems remains an open question.

REFERENCES

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