A CONSTRAINED ASSIGNMENT PROBLEM

Atidel Ben Hadj-Alouane*

and Katta G. Murty†

Department of Industrial and Operations Engineering
The University of Michigan
1205 Beal Avenue
Ann Harbor, MI 48109, U.S.A.

الخلاصة :

تظهر مسألة « التّعيين المقيّد » في الكثير من التطبيقات العمليّة ، ومنها توزيع طلبة الطّب إلى مقيمين في أيّ مستشفى ، أنّ النّموذج الرّياضي لهذه المسألة هو برنامج بالأعداد الصّحيحة هو (صفر – واحد) . ونثبت أنّ البرنامج الخطيّ المقابل لهذه المسألة يمكن تمثيله بجريان في « شبكات معمّمة » .

ABSTRACT

A constrained assignment problem that arises in assigning medical students to internships is considered. A similar problem arises in many other applications. It is a 0-1 integer programming problem. We show that the LP relaxation for this problem has a generalized network flow model.

† *e-mail:* katta murty@ub.cc.umich.edu

* To whom correspondence should be addressed. (*e-mail*: atidel@caen.engin.umich.edu)

A. B. Hadj-Alouane and K. G. Murty

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1. INTRODUCTION

Every large medical college faces, every year, the problem of assigning their students to internships at different types of facilities. Typically, there are three types of facilities, (1) speciality, (2) rehabilitation, and (3) general acute; and each student must intern for one term in a facility of each type.

Each year, they get a new batch of students. Suppose a batch has T students (T is typically ≤ 50). This batch will have a period of three consecutive terms over which they have to complete their internships.

$k = 1, 2, 3$ denote the types of facilities. n_1, n_2, and n_3 are the numbers of the three types of facilities. Let $P_{i,k}$ denote the ith facility of type k, $i = 1, \ldots, n_k$, $k = 1, 2, 3$. The index $r = 1, 2, 3$ denotes the term during the period. The index $j = 1, \ldots, T$ denotes the various students.

Each facility specifies the maximum number of interns it can take each term. $U(i,k;r)$ denotes this number for $P_{i,k}$ in term r, it is called the capacity of $P_{i,k}$ in term r.

Students usually express their preferences for interning in specified facilities during certain terms. The schools normally encourage this and try to make assignments so as to maximize the number of students receiving their preferred facility in the term they asked for it. The manner in which the students express their preferences may differ slightly from one school to another, but typically, each student states three choices in decreasing order of preference (*1st*, *2nd*, and *3rd* choices). Each choice mentions a specific facility that the student would like to intern in, in each term of the period, so that all types appear over the period. For example, listing the desired facilities in terms 1, 2 and 3 in that order, the three choices for a student may be the following.

1^{st} choice	$(P_{3,2}, P_{1,1}, P_{7,3})$
2^{nd} choice	$(P_{6,3}, P_{4,2}, P_{3,1})$
3^{rd} choice	$(P_{2,1}, P_{1,2}, P_{2,3})$

We assume that this preference data is provided by each student. We define the decision variables $x(j;i,k;r)$ for $j = 1, \ldots, T$; $i = 1, \ldots, n_k$, $k = 1, 2, 3$; $r = 1, 2, 3$ by

$$x(j;i,k;r) = \begin{cases} 1 & \text{if student } j \text{ is assigned to } P_{i,k} \text{ in term } r \\ 0, & \text{otherwise.} \end{cases}$$

Since the aim is to maximize the number of students who get their most preferred choices as far as possible, we define the objective function to maximize to be

$$Z(x) = \sum_{j,i,k,r} c(j;i,k;r)x(j;i,k;r)$$

where

$$c(j;i,k;r) = \begin{cases} w_1, w_2 \text{ or } w_3 & \text{if } P_{i,k} \text{ is the first, the second or the third choice of student } j \text{, for term } r \\ 0, & \text{otherwise.} \end{cases}$$

where w_1, w_2, and w_3 are positive weights satisfying $w_1 > w_2 > w_3$. One possible selection for these weights is $w_1 = 3$, $w_2 = 2$, $w_3 = 1$.

The constraints to be satisfied by the decision variables are

$$\sum_{k=1}^{3} \sum_{i=1}^{n_k} x(j;i,k;r) = 1, \text{ for each } j, r \qquad (1)$$

$$\sum_{j=1}^{T} x(j;i,k;r) \leq U(i,k;r), \text{ for each } r, i, k \qquad (2)$$

$$\sum_{i=1}^{n_k} \sum_{r=1}^{3} x(j;i,k;r) = 1, \text{ for each } j, k \qquad (3)$$

Constraint (1) guarantees that each student gets assigned to *exactly one* facility every term of the period. Constraint (2) guarantees that the number of students assigned to any facility in any term, is less than or equal to its capacity in that term. Constraint (3) guarantees that every student interns *one* term in each type of facility, during the period.

Therefore, the internship assignment problem is to find $x = (x(j;i,k;r))$ to maximize $Z(x)$ subject to (1), (2), (3) and the 0–1 restriction on each decision variable $x(j;i,k;r)$. This is a 0–1 integer programming problem. Such problems are usually solved by branch and bound methods, using the LP relaxation (the linear program obtained by replacing the 0–1 restriction on each variable $x(j;i,k;r)$ by $0 \leqslant x(j;i,k;r) \leqslant 1$) for bounding strategy. If $\bar{x} = (\bar{x}(j;i,k;r))$ is an optimum solution for the relaxation, and it so happens that all the $\bar{x}(j;i,k;r)$ are 0 or 1, then of course it is an optimum solution for the original integer program. If some of the $\bar{x}(j;i,k;r)$ are fractional, we could then branch and proceed with the branch and bound approach; but in the practical world, simple heuristic techniques are applied on $\bar{x}$ to check if it can be rounded into a reasonable solution for the original student internship assignment problem, before deciding to proceed with the branch and bound approach.

The LP relaxation is itself a large linear program for a typical school because of the large number of variables and constraints, and hence, may be hard to solve by available general purpose linear programming algorithms such as the simplex method. In this note we show that the LP relaxation can be transformed into a generalized network flow problem [1–3]. There are several practically efficient special algorithms for solving large scale generalized network flow problems; thus, this transformation makes it possible to solve the LP relaxation efficiently.

2. The Generalized Network Formulation for the LP Relaxation

We construct a directed generalized network $G = (\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ = the set of nodes, $\mathcal{A}$ = the set of arcs, are specified below. The data on the arcs and the interpretation of the flow on each arc is also described.

$\mathcal{N}$ contains a source node s and a sink node t.

For each $k = 1, 2, 3$, $i = 1, \ldots, n_k$, $r = 1, 2, 3$, there is a node $P_{i,k}(r)$ in $\mathcal{N}$, corresponding to facility i of type k in term r.

For each $j = 1, \ldots, T$, $k = 1, 2, 3$, $r = 1, 2, 3$, there is a node (j,k,r) in $\mathcal{N}$ corresponding to student j in type k and term r.

For each $j = 1, \ldots, T$, $k = 1, 2, 3$, there is a node (j,N_k) in $\mathcal{N}$ corresponding to the assignment of student j to type k.

For each $j = 1, \ldots, T$, $r = 1, 2, 3$, there is a note (j,M_r) in $\mathcal{N}$ corresponding to the assignment of student j in term r.

For each $j = 1, \ldots, T$, there are two nodes j_1 and j_2 in $\mathcal{N}$. j_1 helps to count the number of types of facilities in which student j interned over the period. j_2 helps to count the number of terms over the period, during which student j has been assigned to some facility.

There is an arc $(s, P_{i,k}(r))$ in $\mathcal{N}$ with lower bound 0 for flow, capacity $U(i,k;r)$, objective coefficient 0, and multiplier 1, for each i, k, and r. The flow on this arc represents the total number of students assigned for internship at $P_{i,k}$ in term r.

There is an arc $(P_{i,k}(r),(j,k,r))$ in $\mathcal{A}$ with lower bound 0 for flow, capacity 1, objective coefficient $c(j;i,k;r)$ defined earlier, and multiplier 2, for each j, i, k, and r. The flow on this arc represents the decision variable $x(j;i,k;r)$ defined earlier. The reason for making the multiplier on this arc to be 2 is the following. If there is a flow of one unit on this arc, it implies that student j is assigned to $P_{i,k}$ for internship during term r. By the time this flow arrives at the head node (j,k,r) on this arc, it becomes two units because of the multiplier being 2. One of these units will flow towards node (j,N_k) and another towards node (j,M_r) along arcs defined below. This helps to keep count of the types of facilities where student j has interned, and the terms during which this student had some internship assignment, separately.

There are arcs $((j,k,r),(j,N_k))$ and $((j,k,r),(j,M_r))$ in $\mathcal{A}$, with lower bound 0 for flow, capacity 1, objective coefficient 0, and multiplier 1, for each j, k, and r. Any positive flow, say y, arriving at node (j,k,r) through the arc $(P_{i,k}(r), (j,k,r))$ will be split among these two arcs $((j,k,r), (j,N_k))$ and $((j,k,r), (j, M_r))$. If $y = 2$, it will have to split evenly along these two arcs, this makes it possible for us to keep track of the types of facilities, and the terms in which student j has performed internship over the period, separately.

There are arcs $((j, N_k),j_1)$ and $((j,M_r),j_2)$ in $\mathcal{A}$, with the lower bound for flow and capacity both equal to 1, objective coefficient 0, and multiplier 1, for each j, k, and r. The flow on $((j,N_k),j_1)$ represents the total number of assignments for internships for student j at facilities of type k, and that on arc $((j,M_r),j_2)$ represents the total number of internship assignments for this student at all facilities during term r. Because of the lower bounds and capacities defined here, the flows will satisfy constraints (1) and (3) described earlier.

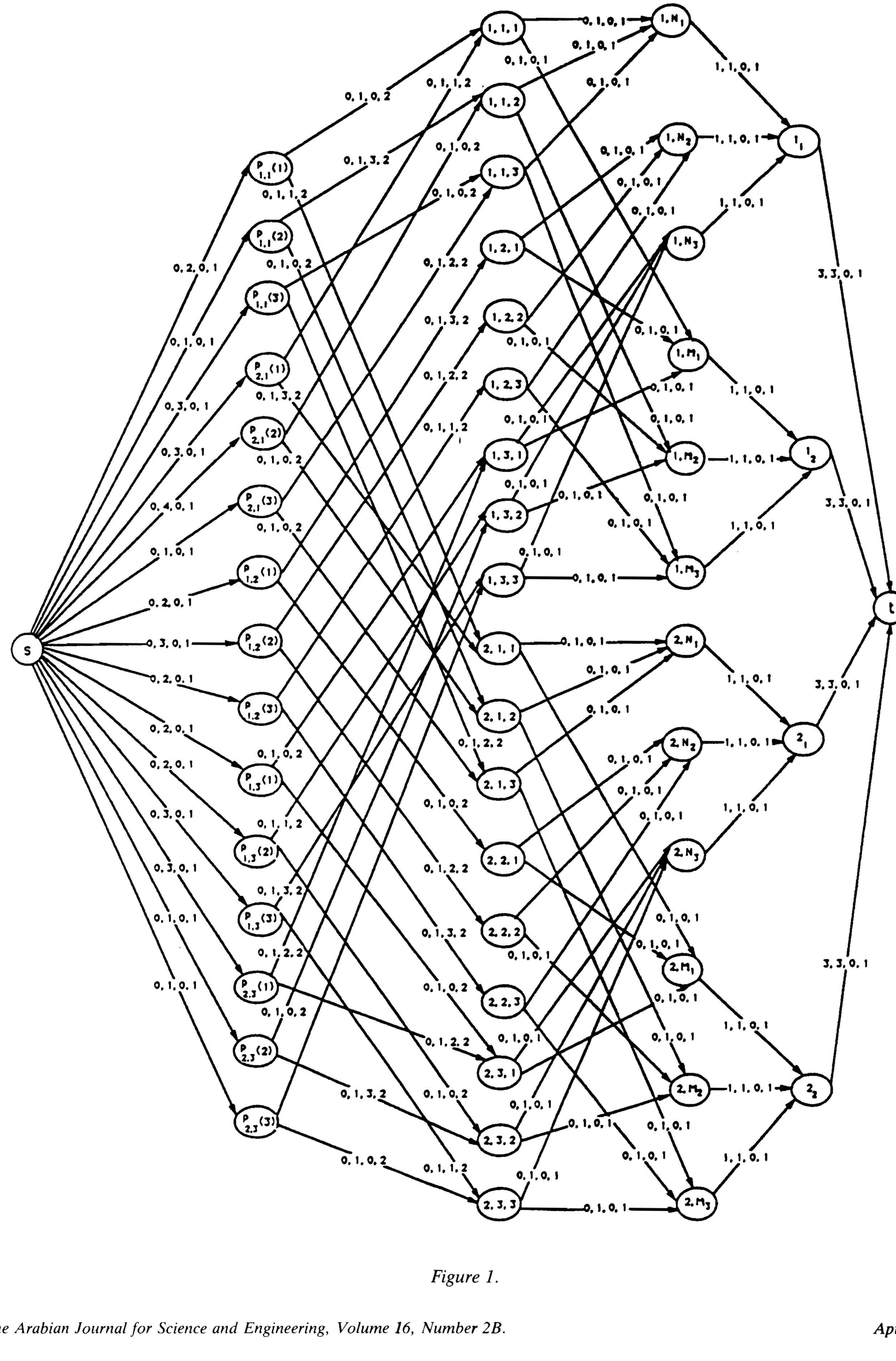
s
t
1, 1, 1
1, 1, 2
1, 1, 3
1, 2, 1
1, 2, 2
1, 2, 3
1, 3, 1
1, 3, 2
1, 3, 3
2, 1, 1
2, 1, 2
2, 1, 3
2, 2, 1
2, 2, 2
2, 2, 3
2, 3, 1
2, 3, 2
2, 3, 3
3, 3, 0, 1
1, 1, 0, 1
0, 1, 0, 1

Figure 1.

There are arcs (j_1,t) and (j_2,t) in $\mathcal{A}$, with lower bound for flow and capacity both equal to 3, objective coefficient 0, and multiplier 1, for each j. The lower bounds and the capacities defined on these arcs guarantee that in any integer feasible flow vector, student j interns at three types of facilities in all, and for the three terms, during the period.

Except for the arcs $(P_{i,k}(r),(j,k,r))$ which have multipliers equal to 2, all other arcs in G have multipliers of 1. Because of these arcs with multipliers different from 1, G is a generalized network.

The problem of finding a feasible flow vector in G maximizing the objective function is equivalent to the LP relaxation. So, by maximizing the objective function in G, we can find an optimum solution $\bar{x} = (\bar{x}(j;i,k;r))$ to the LP relaxation, if one exists. If $\bar{x}$ is integer, then it is actually an optimum solution to the original integer program, and therefore defines an optimum assignment of students to facilities for internship. If $\bar{x}$ is not integral, one proceeds to the heuristic rounding or the branching operation as discussed earlier. On the other hand, if there is no feasible integral flow in G, the original integer program must be infeasible, this could happen if the number of students exceeds the capacity of existing facilities, or for other similar reasons.

G has $17T + 3n + 2$ nodes and $26T + 3n(T + 1)$ arcs, where $n = n_1 + n_2 + n_3$, is the total number of facilities of all types. In most medical schools T is ≤ 50, and n is ≤ 10, so the network G is of reasonable size in these applications. We provide a small numerical example. It is very simple, in fact an optimum solution where each student gets his (or her) first choice every term, can be easily determined by observation. However, our main purpose is to illustrate the construction of the network G. The number of students is $T = 2$, the number of facilities in each of the three types are as follows: $n_1 = 2$, $n_2 = 1$, and $n_3 = 2$. The capacities of the facilities, $U(i,k;r)$ are given below.

		$U(i,k;r)$				
	$(i,k) \rightarrow$	(1,1)	(2,1)	(1,2)	(1,3)	(2,3)
	1	2	3	2	2	3
$r =$	2	1	4	3	2	1
	3	3	1	2	3	1

The choices of the two students, for the three terms, listed in that order, are given below.

	Student 1	Student 2
1*st* choice	$(P_{1,2}, P_{1,1}, P_{1,3})$	$(P_{2,1}, P_{2,3}, P_{1,2})$
2*nd* choice	$(P_{2,3}, P_{1,2}, P_{2,1})$	$(P_{2,3}, P_{1,2}, P_{1,1})$
3*rd* choice	$(P_{2,1}, P_{1,3}, P_{1,2})$	$(P_{1,1}, P_{1,2}, P_{1,3})$

The network G corresponding to this example is given in Figure 1. The data displayed on the arcs is, from left to right, the lower bound for flow on it, the capacity, the objective coefficient, and the multiplier. In determining the objective coefficients, we have taken weights w_1, w_2 and w_3 associated with the first, second, and the third choices to be 3, 2, and 1 as described earlier.

A similar problem arises in organizations which handle people that need to undergo different types of training, or training at different locations. The same problem arises in organizations where workers are rotated among different shifts, or different types of work, on a periodic basis. This model is expected to be useful in all such situations. For some other network models of scheduling or assignment problems see references [4–8].

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