ON NEWTON'S METHOD UNDER MILD DIFFERENTIABILITY CONDITIONS

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الخلاصـــة :

يُقدّم هذا البحث شروطَ تقاربِ طريقة معاودة نيوتن لحلِّ معادلة مؤثر غير خطية في فراغ (بناخ) . والافتراض الوحيد في البحث هو أنَّ مشتقات (فريشيه) للمؤثر غير الخطي تحقق اتصال (هولدر) . كما ويحتوي البحث على بعض أمثلة حيث أنَّ الفرضيَة العادية لطريقة نيوتن لا تتحقق . ولكنَّ الفرضية المقترحة يمكن تحقيقها .

ABSTRACT

We provide sufficient conditions for the convergence of Newton's iteration to a solution of nonlinear operator equation in Banach space. We assume only that the Fréchet-derivative of the nonlinear operator is Hölder continuous. Some examples are provided where the usual hypotheses for the application of Newton's method are not satisfied but ours are.

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INTRODUCTION

The Newton–Kantorovich method, namely

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \tag{1}$$

has been used extensively to solve the nonlinear operator equation

$$F(x) = 0 \tag{2}$$

in a Banach space X [4–6] (and the references there).

Using some ideas of Altman [3], we generalize his results, assuming only that the Fréchet-derivative $F'(x)$ is Hölder (c, p) continuous on X (to be made precise later).

If $p = 1$ and the inverse of $F'(x)$ exists on X then our results reduce to the ones obtained by Kantorovich and others [1], [3], [4].

Some examples are also provided.

Let X and Y be two Banach spaces and let L_1 be a continuous linear operator mapping X onto Y. Denote by e_{L_1} the set of all solutions of the equation $L_1 x = 0$. We divide the space X into classes, and we say that x_1 and x_2 belong to the same class $\bar{X}$, if $x_1 - x_2 \in e_{L_1}$. This quotient space X/e_{L_1} is a Banach space with the norm $\|\bar{X}\| = \inf\|x\|$, $x \in \bar{X}$. The operator L_1 gives rise to an operator $L : X/e_{L_1} \to Y$ which is bijective and $L\bar{X} = L_1 x$ for $x \in \bar{X}$.

We now state the lemmas whose proof can be found in [3].

Lemma 1. Let L_1 and L_2 be two linear operators mapping X onto Y. If

$$\|L_2 - L_1\| < \frac{1}{\|\bar{L}_1^{-1}\|}, \tag{3}$$

then

$$\|\bar{L}_2^{-1}\| \le \frac{\|\bar{L}_1^{-1}\|}{1 - \|\bar{L}_1^{-1}\|\,\|L_2 - L_1\|}, \tag{4}$$

where $\bar{L}_1$ and $\bar{L}_2$ denote the adjoints of L_1 and L_2 respectively.

Let

$$P = A\bar{X} \tag{5}$$

be the linear transformation of X/e_{L_2} onto Y, induced by the operator L_2. Then easily

$$\|A^{-1}\| = \|\bar{L}_2^{-1}\|. \tag{6}$$

Now, let $y = F(x)$ be a nonlinear continuous operator on Y. We suppose that $F(x)$ is Fréchet-differentiable in a certain closed sphere $S(x_o, r)$ with center x_o and of radius $r > 0$. We suppose also that the Fréchet-derivative $F'(x)$ is a linear operator onto Y for every $x \in S(x_o, r)$.

Denote by e_x the set of all solutions z of the equation $F'(x)z = 0$ and consider the quotient space X/e_x. Let us assume that the following is satisfied:

For every $x \in S(x_o, r)$ the norm of $l \in X/e_x$ is reached at a point $z \in l$, *i.e.*, there exists an element $z \in l$ such that

$$\|z\| = \|l\| = \inf\|z'\|, \; z' \in l. \tag{7}$$

We can now define an iteration for solving (2). We denote by A_n the linear operator defined on X/e_{x_n} and induced by the linear transformation $F'(x_n)$ and put $e_n = e_{x_n}$, $n = 0, 1, 2, \ldots$.

Set

$$\bar{x}_1 = \bar{x}_0 - A_o^{-1}F(x_o),$$

where $x_o \in \bar{x}_o$ and $\bar{x}_o$, $\bar{x}_1 \in X/e_o$, and choose an element x_1 in $\bar{x}_1$ such that

$$\|x_1 - x_o\| = \|\bar{x}_1 - \bar{x}_o\|.$$

If the approximate solutions $x_1, \ldots, x_n$ are already defined, then we put

$$\bar{x}_{n+1} = \bar{x}_n - A_n^{-1}F(x_n), \tag{8}$$

where $\bar{x}_n \in X/e_n$ and $x_n \in \bar{x}_n$. Further, we choose elements x_{n+1} and $\bar{x}_{n+1} \in X/e_n$ such that

$$\|x_{n+1} - x_n\| = \|\bar{x}_{n+1} - \bar{x}_n\|. \tag{9}$$

It is well known that at branch and limit points of nonlinear functional equations the first Fréchet-derivative is singular and an interest in the computation of such solution points (see, for example [7] and the references therein) has provided some of the motivation for the construction of operators A_n in this paper.

Conditions for the convergence of iterations (8) and (9), which are also sufficient conditions for the existence of a solution of (2) will be given in the main theorem that follows. But first we need the following:

Definition 1. Assume that F is Fréchet-differentiable and $F'(x)$ is the first Fréchet-derivative at a point x. We say that the Fréchet-derivative is Hölder continuous over a domain R if for some $c>0$, $p\in[0,1]$, and all $x, y\in R$

$$\|F'(x)-F'(y)\|\le c\|x-y\|^p . \tag{10}$$

In this case, we say that $F'(x)\in H_R(c,p)$.

We will need the following lemma ([4], p. 142).

Lemma 2. Let $F: X\to Y$ and $\tilde{D}\subseteq X$. Assume $\tilde{D}$ is open and that $F'(x)\in H_{\tilde{D}_o}(c,p)$ for some convex $\tilde{D}_o\subseteq\tilde{D}$. Then for all $x, y\in\tilde{D}_o$

$$\|F(x)-F(y)-F'(x)(x-y)\|\le\frac{c}{p+1}\|x-y\|^{p+1}. \tag{11}$$

MAIN RESULTS

Definition 2. Define the functions g_1, g_2, and g_3 by,

$g_1(d) = g_1z(d) = vz^{p+1}+z^p-v$,
where $z = d^{1/(p+1)}$, $v = (p+1)^{p/(p+1)}$,

$g_2(d) = d^p+vd-v$,

$$g_3(p) = 1-\sqrt{\left(\frac{1}{(p+1)v}\right)} .$$

Claim. There exists d, with $0<d<1$ such that

$$g_1(d)\le 0 \tag{12}$$

$$g_2(d)\le 0 \tag{13}$$

and

$$d\le g_3(p) . \tag{14}$$

Note:

$g_1(0) = -v<0$,
$g_1'(z) = (p+1)vz^p+pz^{p-1}>0,\ z\in(0,+\infty)$.

Therefore, g_1 is increasing on $(0,+\infty)$. But,

$$g_1(1) = 1>0 .$$

That is, there exists, $0<d_1<1$ such that

$$g_1(d)\le 0 \text{ for all } d\in(0,d_2) .$$

Similarly there exists $0<d_2<1$ such that

$$g_2(d)\le 0 \text{ for all } d\in(0,d_1) .$$

Set,

$$d_3 = \min(d_o, d_1, d_2, g_3(p)),\ d_o = \frac{v-1}{v} . \tag{15}$$

Then

$$g_1(d)\le 0 ,$$
$$g_2(d)\le 0 ,$$

and

$$d\le g_3(p)<1 \text{ for all } d\in(0,d_3) .$$

By the choice of d above, it is possible to choose w, y such that

$$\frac{1}{(1-d)v}\le w\le\min(1,(p+1)(1-d)), \tag{16}$$

$$\frac{1}{(p+1)(1-d)}\le y<\frac{1}{w} . \tag{17}$$

Theorem. Let us assume the following conditions are satisfied:

(a) For the Fréchet-derivative $F'(x_o)^{-1}$ there exists $B_o>0$ such that

$$\|F'(x_o)^{-1}\|\le B_o ; \tag{18}$$

(b) the first approximate solution x_1 satisfies

$$\|x_1-x_o\|\le k_o ; \tag{19}$$

(c) for $R=S(x_o,r)$ the Fréchet-derivative $F'(x)\in H_R(c,p)$ with $x\in R$, for some $p\in[0,1]$;

(d) the constants B_o, k_o, c satisfy the inequality

$$h_o = cB_ok_o^p\le d\le d_o=\frac{v-1}{v}<1 , \tag{20}$$

and

$$\frac{k_o}{1-ywD^p}\le r , \tag{21}$$

where $d\in(0,d_3)$ and d_3, y, w, v, are as in definition 2 with $D=d^{1/p}$.

Then (2) has a solution $x^*\in S(x_o,r)$. The sequence $\{x_n\}$, $n=0,1,2,\ldots$ defined by (8) and (9), remains in $S(x_o,r)$ and converges to x^*, with

$$\|x^*-x_n\|\le\frac{k_o(yw)^nD^{(p+1)^n}}{1-ywD^p},\ n=0,1,2,\ldots . \tag{22}$$

Moreover if

$$cB_o(3r)^p<1 \tag{23}$$

then

$$\|x_n - x^*\| \le \frac{cB_o}{(p+1)[1-cB_o(2r+\|x_n-x_o\|)^p]}\|x_n-x_{n-1}\|^{p+1}, \quad n = 1, 2, \ldots. \tag{24}$$

Proof. We will follow the standard inductive Newton–Kantorovich proof used also in ([3], p. 792) and ([4], p. 143) with some modifications. By (8), we have

$$A_n(\bar{x}_{n+1}-\bar{x}_n) = -F(x_n), \; n = 0, 1, 2, \ldots. \tag{25}$$

Hence, applying the definition of the transformation A_n, we obtain

$$F'(x_n)(x_{n+1}-x_n) = -F(x_n). \tag{26}$$

Using (11) and the approximation

$$F(x_n) = F(x_n)-F(x_{n-1})-F'(x_{n-1})(x_n-x_{n-1}) \tag{27}$$

we derive

$$\|F(x_n)\| \le \frac{c}{p+1}\|x_n-x_{n-1}\|^{p+1}. \tag{28}$$

By (8), (9), and (28), we obtain

$$\|x_{n-1}-x_n\| \le \frac{c}{p+1}\|A_n^{-1}\|\cdot\|x_n-x_{n-1}\|^{p+1}.$$

We shall now estimate the norm $\|A_n^{-1}\|$ using lemma 1 (5) and (6) to obtain

$$\|A_n^{-1}\| \le \frac{\|\bar{F}'(x_{n-1})^{-1}\|}{1-\|\bar{F}'(x_{n-1})^{-1}\|\|F'(x_n)-F'(x_{n-1})\|} \le \frac{B_{n-1}}{1-B_{n-1}ck_{n-1}^p}$$

where

$$\|A_{n-1}^{-1}\| = \|\bar{F}'(x_{n-1})^{-1}\| \le B_{n-1},$$

$$\|x_n-x_{n-1}\| \le k_{n-1},$$

and

$$h_{n-1} = B_{n-1}ck_{n-1}^p.$$

Using

$$\|x_{n+1}-x_n\| \le k_n,$$

$$B_n = \frac{B_{n-1}}{1-h_{n-1}},$$

and

$$k_n = \frac{1}{p+1}\cdot\frac{h_{n-1}}{1-h_{n-1}}\cdot k_{n-1},$$

we obtain

$$h_n = B_nck_n^p = \left(\frac{1}{p+1}\right)^p\cdot\left(\frac{h_{n-1}}{1-h_{n-1}}\right)^{p+1}, \quad n = 0, 1, 2, \ldots.$$

We now note the following:

Claim 1. For, $h_o \le d < 1$, assuming $h_{n-1} \le d$ we can show

$$h_n \le d.$$

To show,

$$\left(\frac{1}{p+1}\right)^p\left(\frac{h_{n-1}}{1-h_{n-1}}\right)^{p+1} \le d$$

it suffices to show

$$g_1(d) = g_1(z) \le 0,$$

$$\text{with } z = d^{1/(p+1)}, \; v = (p+1)^{p/(p+1)},$$

which is true for $d \in (0, d_3)$.

Claim 2. For w, given by (16),

$$h_n \le wh_{n-1}^{p+1}.$$

$$\left(\frac{1}{p+1}\right)^p\left(\frac{h_{n-1}}{1-h_{n-1}}\right)^{p+1} \le wh_{n-1}^{p+1},$$

assuming

$$h_{n-1} \le d,$$

it suffices to show

$$w \ge \frac{1}{(1-d)v},$$

which is true by the choice of w.

For consistency,

$$h_n \le wh_{n-1}^{p+1} \le wd^{p+1} \le d,$$

or

$$w \le \frac{1}{d^p}.$$

That is, we must have

$$\frac{1}{(1-d)v} \le w \le \frac{1}{d^p},$$

which is true by the choice of w and the fact that

$$g_2(d) \le 0 \text{ for } d \in (0, d_3).$$

Claim 3. For y, given by (17)

$$k_n = \frac{1}{p+1} \frac{h_{n-1}k_{n-1}}{1-h_{n-1}} \le y h_{n-1} k_{n-1}. \quad (29)$$

To show (29) it is enough to choose

$$(1-d)(p+1)y \ge 1 ,$$

which is true by the choice of y.

Now,

$$h_n \le w h_{n-1}^{p+1} \le w(h_{n-2}^{p+1})^{p+1} \le \dots \le w(h_o)^{(p+1)^n}$$

and

$$\begin{aligned} k_n &\le y h_{n-1} k_{n-1} \le y^2 h_{n-1} h_{n-2} k_{n-2} \\ &\le y^n h_{n-1} h_{n-2} \dots h_o k_o \\ &\le (yw)^n d^{[(p+1)^n-1]/p} k_o . \end{aligned}$$

Now, using a standard argument as in ([4], p.143), we can show by induction that the approximate solution x_n and the corresponding numbers B_n, k_n, and h_n can be defined for every n, and, moreover conditions $(a)-(d)$ are satisfied.

Moreover,

$$\begin{aligned} \|x_{n+q}-x_n\| &\le k_{n+q-1} + k_{n+q-2} + \dots + k_n \\ &\le (yw)^{n+q-1} d^{[(p+1)^{n+q-1}-1]/p} k_o \\ &+ (yw)^{n+q-2} d^{[(p+1)^{n+q-2}-1]/p} k_o \\ &+ \dots + (yw)^n d^{[(p+1)^n-1]/p} k_o . \\ &\le D^{(p+1)^n-1} k_o (yw)^n [(yw)^{q-1} D^{(p+1)^{q-1}} \\ &+ (yw)^{q-2} D^{(p+1)^{q-2}} + \dots + 1] \\ &\le D^{(p+1)^n} k_o (yw)^n [(yw)^{q-1} D^{p(q-1)} \\ &+ (yw)^{q-2} D^{p(q-2)} + \dots + 1] \end{aligned}$$

(since for $D<1$, $p \in [0,1]$, $D^{(p+1)^{q-1}} \le D^{p(q-1)}$)

$$\le D^{(p+1)^n} k_o (yw)^n \left[\frac{1-(ywD^p)^n}{1-ywD^p}\right]. \quad (30)$$

The right hand side of (30) and the choice of y, w, d show that the sequence $\{x_n\}$, $n=0,1,2,\dots$ is a Cauchy sequence and as such it converges to some $x^* \in X$. Letting $q \to \infty$ in (30), we obtain (22).

By (30), setting $n=0$ and $q=n$, we can easily obtain that

$$\|x_n - x_o\| \le r ,$$

that is $x_n \in S(x_o, r)$, $n=0,1,2,\dots$ if (21) holds.

By the continuity of $F(x)$ and since the norms of the operators $F'(x_n)$ are bounded, it follows that x^* is a solution of (2).

Moreover, $x^* \in S(x_o, r)$ since $S(x_o, r)$ is closed and (22) holds.

Let us now observe that

$$F(x_n) = F(x_n) = F(x^*) = Q_n(x_n - x^*) , \quad (31)$$

where

$$Q_n = \int_o^1 F'(x^* + t(x_n - x^*))\,dt, \ n = 0,1,2,\dots .$$

We want to prove that the linear operator Q_n is invertible for all n.

To this effect we note that according to (10) and (23) we have:

$$\begin{aligned} &\|F'(x_o)^{-1}\left(\int_o^1 [F'(x^*+t(x_n-x^*)) - F'(x_o)]\,dt\right)\| \\ &\le cB_o \int_o^1 \|x^* + t(x_n - x^*) - x_o\|^p dt \\ &\le cB_o(2\|x_n - x^*\| + \|x_n - x_o\|)^p \\ &\le cB_o(3r)^p < 1 . \end{aligned}$$

By virtue of Banach's lemma, B_n is invertible and the following norm estimation holds:

$$\|Q_n^{-1}\| \le \frac{B_o}{1-cB_o(2r+\|x_n-x_o\|)^p} . \quad (32)$$

Finally from (31), (28), and (32) we deduce (24), since

$$\|x_n - x^*\| \le \|Q_n^{-1}\| \cdot \|F(x_n)\| .$$

That completes the proof of the theorem.

A theorem similar to the above can easily be stated here for the modified Newton's iteration

$$x_{n+1} = x_n - F'(x_o)^{-1} F(x_n), \ n = 0,1,2,\dots .$$

However, we leave that to the motivated reader. We now provide some examples.

APPLICATIONS

Example 1.

Consider the function G defined on $[0,b]$ by

$$G(t) = \tfrac{2}{3} t^{3/2} + t - 3$$

for some $b > 0$.

Let $\|\,\|$ denote the max norm on $\mathbb{R}$, then

$$\|G''(t)\| = \max_{t\in[0,b]} |\tfrac{1}{2}t^{-1/2}| = \infty ,$$

which implies that the basic hypothesis in [3] and [4] for the application of Newton's method is not satisfied for finding a solution of the equation

$$G(t) = 0 . \tag{33}$$

However, it can easily be seen that $G'(t)$ is Hölder continuous on $[0, b]$ with

$$c = 1 \text{ and } p = \tfrac{1}{2} .$$

Therefore, under the assumptions of the theorem, iteration (1) will converge to a solution t^* of (33).

A more interesting nontrivial application is given by the following example.

Example 2.

Consider the differential equation

$$x''+x^{1+p} = 0,\ p\in[0,1] \tag{34}$$

$$x(0) = x(1) = 0 .$$

We divide the interval $[0,1]$ into n subintervals and we set $h = 1/n$. Let $\{v_k\}$ be the points of subdivision with

$$0 = v_0 < v_1 < \dots < v_n = 1 .$$

A standard approximation for the second derivative is given by

$$x_i'' = \frac{x_{i-1}-2x_i+x_{i+1}}{h^2} ,\ x_i = x(v_i) ,\ i = 1, 2, \dots, n-1 .$$

Take $x_0 = x_n = 0$ and define the operator $F: \mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ by

$$F(x) = H(x)+h^2\,\phi(x) \tag{35}$$

$$H = \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ 0 & & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix} ,$$

$$\phi(x) = [x_1^{1+p}, x_2^{1+p}, \dots, x_{n-1}^{1+p}]^{tr} ,$$

and

$$x = [x_1, x_2, \dots, x_{n-1}]^{tr} .$$

Then

$$F'(x) = H+h^2(p+1)\begin{bmatrix} x_1^p & & & & 0 \\ & x_2^p & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & x_{n-1}^p \end{bmatrix}$$

Newton's method cannot be applied to the equation

$$F(x) = 0 . \tag{36}$$

We may not be able to evaluate the second Fréchet-derivative since it would involve the evaluation of quantities of the form x_i^{-p} and they may not exist.

Let $x\in\mathbb{R}^{n-1}$, $H\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ and define the norms of x and H by

$$\|x\| = \max_{1\le j\le n-1} |x_j|$$

$$\|H\| = \max_{1\le j\le n-1} \sum_{k=1}^{n-1} |h_{jk}| .$$

For all $x, z\in\mathbb{R}^{n-1}$ for which $|x_i|>0$, $|z_i|>0$, $i = 1, 2, \dots, n-1$ we obtain, for $p = \tfrac{1}{2}$ say,

$$\begin{aligned}\|F'(x)-F'(z)\| &= \|\mathrm{diag}\{(1+\tfrac{1}{2})h^2(x_j^{1/2}-z_j^{1/2})\}\| \\ &= \tfrac{3}{2}h^2 \max_{1\le j\le n-1} |x_j^{1/2}-z_j^{1/2}| \\ &\le \tfrac{3}{2}h^2\,[\max|x_j-z_j|]^{1/2} \\ &= \tfrac{3}{2}h^2\|x-z\|^{1/2} .\end{aligned}$$

Given $z_0\in\mathbb{R}^{n-1}$ Newton's method consists of solving

$$F'(z_n)(z_n-z_{n+1}) = F(z_n),\ n = 0, 1, 2, \dots$$

as a system of linear equations.

We choose $n = 10$ which gives 9 equations. Since a solution would vanish at the end points and be positive in the interior a reasonable choice of initial approximation seems to be $130\times\sin\pi x$. This gives us the following vector:

$$z_0 = \begin{bmatrix} 4.01524\text{ E}+01 \\ 7.63785\text{ E}+01 \\ 1.05135\text{ E}+01 \\ 1.23611\text{ E}+02 \\ 1.29999\text{ E}+02 \\ 1.23675\text{ E}+02 \\ 1.05257\text{ E}+02 \\ 7.65462\text{ E}+01 \\ 4.03495\text{ E}+01 \end{bmatrix}$$

After four iterations we get a vector

$$z_4 = \begin{bmatrix} 3.35740 \text{ E} + 01 \\ 6.52027 \text{ E} + 01 \\ 9.15664 \text{ E} + 01 \\ 1.09168 \text{ E} + 02 \\ 1.15363 \text{ E} + 02 \\ 1.09168 \text{ E} + 02 \\ 9.15664 \text{ E} + 01 \\ 6.52027 \text{ E} + 01 \\ 3.35740 \text{ E} + 01 \end{bmatrix} .$$

We choose z_4 as our x_o for the theorem. We get the following results:

$B_o = \|F'(x_o)^{-1}\| = 2.55882 \text{ E} + 01$,

$K_o = \|x_1 - x_o\| = 9.15311 \text{ E} - 05$,

$c = 3/2\, h^2 = 0.0015$,

$p = 1/2$,

$v = 1.44714243$,

$d_o = 0.126419535.$

Choose $d = 4\text{E}-03$ to get from (20), thus

$h_o = 3.672107076 \text{ E}-03 < 4\text{E}-03\, d_o < 1.$

Finally choose $y = 1$ and $w = 0.9$ to obtain from (21)

$$r \geq 9.18618024 \text{ E}-05 = r_o .$$

With the above values it can easily be seen that all the hypotheses of the theorem are satisfied. Therefore the sequence of iterates remains in $S(x_o, r_o)$ and converges to a nontrivial solution x^* of Equation (36).

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