BAYESIAN ESTIMATION FOR A GENERALIZED GAMMA DISTRIBUTION

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الخلاصـــة :

تمّ في هذا البحث دراسة تقدير بِبيز لِمَعْلَم (بارامتر) القياس في تعميم توزيع جاما حيث أعتُبر هذا المَعلَم عشوائياً وفُرض خضوعه لتوزيع أوّلي للأنواع التالية :
التوزيع المتجانس ، والتوزيع الأسي ، وتوزيع جاما المقلوب .
كما تم ايجاد تقدير ببيز لدالة الفاعلية تحت الفروض السابقة بالاضافة الى ايجاد تقديرات ببيز لَمعلَم القياس ولدالة الفاعلية في حالة «الجهل التام» عن التوزيع الأوّلي وفي حالة «أشباه التوزيعات» .
واحتوى البحث على ملخص للنتائج السابقة المتعلقة ببعض التوزيعات المشهورة كحالات خاصة لنموذج تعميم جاما .

ABSTRACT

In this paper we consider the generalized gamma distribution and derive the Bayes' estimators of its random scale parameter and reliability function with respect to uniform, exponential, and inverted gamma priors.

The estimators are provided also in case of prior ignorance and quasi-densities. Results for special cases of the model which include some well-known distributions are summarized.

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1. INTRODUCTION

Let X be a random variable (RV) having generalized gamma distribution (GGD) (c.f. Lajkó [1]):

$$f(x|\theta, p) = \frac{(1/\theta)^p |B'(x)| [B(x)]^{p-1}}{\Gamma(p)} \exp\left(\frac{-B(x)}{\theta}\right), \text{for } x \in (c,d)$$
$$= 0, \text{ otherwise,} \quad (1.1)$$

where $B(x)$ is a continuously differentiable function from the interval (c,d) onto $(0,\infty)$ such that $B'(x) \neq 0$ for $x \in (c,d)$ and θ, $p > 0$ are arbitrary constants.

Stacy and Mirham's [2] generalized gamma and their Table 1 of special cases all belong to the family (1.1). The GGD and its special cases are of interest for applications in reliability and life testing (c.f. Englehardt and Bain [3], and Cross and Clark [4] for applications of gamma distribution).

Our aim in this paper is to consider a Bayesian approach to reliability and life parameter estimation in the GGD which includes many well-known failure distributions such as the exponential, Weibull, gamma, Rayleigh, and inverse Rayleigh distributions. Our Bayesian analysis of the GGD model is carried out under the assumption that p is known while θ is a realization of an RV Θ having any of the following prior densities:

$$g_1(\theta) = \frac{(a-1)(\alpha\beta)^{a-1}}{\beta^{a-1}-\alpha^{a-1}} \frac{1}{\theta^a}, \quad 0 < \alpha \leq \theta \leq \beta$$
$$= 0, \text{ otherwise} \quad (1)$$

$$g_2(\theta) = \frac{1}{\lambda} e^{-\theta/\lambda}, \quad 0 < \theta < \infty, \ \lambda > 0 \quad (2)$$

$$g_3(\theta) = \frac{e^{-\mu/\theta} (\mu/\theta)^{b+1}}{\mu\Gamma(b)}, \quad 0 < \theta < \infty, \ \mu, \ b > 0 \quad (3)$$

These three priors were used by Bhattacharya [5] for a Bayesian analysis of the exponential distribution $(1/\theta)e^{-x/\theta}$.

Justification and support of Bayesian analysis have been advanced by many authors. (c.f. references [6, 7, and 8]).

The choice of uniform prior is justified if prior information concerning the range of the parameter is provided. If no information about the parameter is available then quasi-density prior may be used. The inverted gamma density (3) is the natural conjugate family for (1.1).

In seeking an estimate of θ from a Bayesian point of view we consider a decision function $w(\mathbf{x})$ where $\mathbf{x} = (x_1,\ldots,x_n)$ denotes a realization of a random sample $X_1,\ldots,X_n$, and assume a squared error loss function:

$$L(\theta, w(\mathbf{x})) = (\theta - w(\mathbf{x}))^2. \quad (1.2)$$

2. SCALE PARAMETER AS A SUBJECTIVE RANDOM VARIABLE

Consider a random sample of n items whose life times are described by a GGD (1.1) where p is known and θ is assumed to be a realization of a random variable Θ.

We assume the general uniform density (1) as the prior density of θ. Let $(x_1,x_2,\ldots,x_n) = \mathbf{x}$ denote the observed life times of the test items. The likelihood of the sample is

$$L(\mathbf{x}|\theta) = \frac{\theta^{-np}}{\Gamma^n(p)} \prod_{i=1}^{n} |B'(x_i)| [B(x_i)]^{p-1} \exp\left(-\sum_{i=1}^{n} B(x_i)/\theta\right),$$
$$\text{all } x_i \in (c,d). \quad (2.1)$$

Letting $S_n = \sum_{i=1}^{n} B(x_i)$ and using Bayes' theorem we obtain the posterior density of Θ as:

$$h_1(\theta|\mathbf{x}) = C(1/\theta)^{np+a}\exp(-S_n/\theta); \ \alpha \leq \theta \leq \beta \quad (2.2)$$

where, with $\gamma(m, z) = \int_0^z t^{m-1} e^{-t} dt$,

$$C^{-1} = \frac{\gamma\left(np+a-1, \frac{S_n}{\alpha}\right) - \gamma\left(np+a-1, \frac{S_n}{\beta}\right)}{S_n^{\,np+a-1}}$$

Writing $\gamma*(m,z) = \gamma(m,z/\alpha) - \gamma(m,z/\beta)$, and using the squared error loss function given in (1.2), we find the Bayes' estimator of θ to be the posterior mean given by $\theta* = \int_\alpha^\beta \theta h_1(\theta|\mathbf{x})\, d\theta$, which reduces to:

$$\theta* = \frac{\gamma*(np+a-2, S_n)}{\gamma*(np+a-1, S_n)} \times S_n, \quad np+a > 2 \quad (2.3)$$

Also the variance of $\theta*$ will be the posterior variance of θ given by Equation (2.4).

$$V(\theta*|\mathbf{x}) = \frac{\gamma*(np+a-3,S_n)\ \gamma*(np+a-1,S_n)-\{\gamma*(np+a-2,S_n)\}^2}{\{\gamma*(np+a-1,S_n)\}^2} \times S_n^2 \qquad (2.4)$$

provided $np+a > 3$.

The reliability function under (1.1) is given by:

$$R(t) = P(X > t) = \frac{\gamma\left(p, \frac{B(d)}{\theta}\right) - \gamma\left(p, \frac{B(t)}{\theta}\right)}{\Gamma(p)} \qquad (2.5)$$

Using the convergent expansion:

$$\gamma(a,x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{a+m}}{m!\,(a+m)} \qquad (2.6)$$

(Erdelyi *et al.* [3] p. 135) we get

$$R(t) = \frac{1}{\Gamma(p)} \sum_{m=0}^{\infty} \frac{(-1)^m (1/\theta)^{p+m}}{m!\ (p+m)} [B^{p+m}(d) - B^{p+m}(t)] \qquad (2.7)$$

The RHS of (2.7) is convergent by (2.6) and hence, using (2.2), term by term integration gives the Bayes' estimator of $R(t)$ under a squared error loss as:

$$R*(t) = \frac{1}{\Gamma(p)} \times \sum_{m=0}^{\infty} \frac{(-1)^m \gamma*(p(n+1)+m+a-1,S_n)[B^{p+m}_{(d)} - B^{p+m}_{(t)}]}{m!(p+m)\gamma*(np+a-1,S_n)S_n^{p+m}} \qquad (2.8)$$

In case p is a positive integer, $R(t)$ reduces to

$$R(t) = \sum_{m=0}^{p-1} \frac{1}{m!} \left[\left\{\exp\left(-\frac{B(t)}{\theta}\right)\right\}\left(\frac{B(t)^m}{\theta}\right) - \left\{\exp\left(-\frac{B(d)}{\theta}\right)\right\}\left(\frac{B(d)^m}{\theta}\right)\right]. \qquad (2.9)$$

The corresponding Bayes' estimator under squared error loss (1.2) is

$R**(t) = E[R(t)|x] = \int_\alpha^\beta R(t)\,h_1(\theta|x)\,d\theta$. Noting that

$$\int_\alpha^\beta \{\exp(-B(t)/\theta)\}(B(t)/\theta)^m h_1(\theta|x)\,d\theta = \frac{S_n^{np+a-1}}{\gamma*(np+a-1,S_n)} \times \frac{B^m(t)\gamma*(np+a+m-1,S_n+B(t))}{\{S_n+B(t)\}^{np+a+m-1}},$$

we easily get

$$R**(t) = \sum_{m=0}^{p-1} \frac{1}{m!} \times \frac{S_n^{np+a-1}}{\gamma*(np+a-1,S_n)} \times \left[\frac{B^m(t)\gamma*(np+a+m-1,S_n+B(t))}{\{S_n+B(t)\}^{np+a+m-1}} - \frac{B^m(d)\gamma*(np+a+m-1,S_n+B(d))}{\{S_n+B(d)\}^{np+a+m-1}}\right]. \qquad (2.10)$$

Next we consider the problem under the exponential prior (2). In this case, the posterior density is given by Equation (2.11):

$$h_2(\theta|\mathbf{x}) = \begin{cases} \dfrac{\dfrac{\theta^{-np}}{\Gamma^n(p)} \prod_{i=1}^{n} |B'(x_i)|[B(x_i)]^{p-1}\exp(-S_n/\theta)\ \lambda\exp(-\theta/\lambda)}{\int_0^\infty \dfrac{\theta^{-np}}{\Gamma^n(p)} \prod_{i=1}^{n} |B'(x_i)|[B(x_i)]^{p-1}\exp(-S_n/\theta)\ \lambda\exp(-\theta/\lambda)}, & \alpha < \theta < \infty \\ 0, \text{ otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{\theta^{-np}\exp(-\theta/\lambda - S_n/\theta)}{\int_0^\infty \theta^{-np}\exp(-\theta/\lambda - S_n/\theta)}, & 0 < \theta < \infty \\ 0, \text{ otherwise} \end{cases} \qquad (2.11)$$

To evaluate the denominator of (2.11), we use the relation:

$$K_\nu(az) = \tfrac{1}{2}a^\nu\int_0^\infty \exp(-\tfrac{1}{2}zu - a^2z/2u) \times \frac{1}{u^{\nu+1}}\, du,$$

$$a, z > 0;\ \nu \geq 0 \qquad (2.12)$$

$K_\nu(az)$ is called the Modified Bessel function of the third kind of order ν. (Erdelyi et al. [8]).

Let $S_n = \dfrac{a^2z}{2}$, $\dfrac{1}{\lambda} = \dfrac{z}{2}$ and $\nu = np-1$. Substituting in the denominator of (2.11) and using (2.12) we get:

$$\int_0^\infty \theta^{-np}\exp(-\theta/\lambda - S_n/\theta)\, d\theta = \frac{2K_{np-1}\{2\sqrt{(S_n/\lambda)}\}}{\{\sqrt{(\lambda S_n)}\}^{np-1}}.$$

Hence (2.11) reduces to:

$$h_2(\theta|x) = C(1/\theta)^{np}\exp(-\theta/\lambda - S_n/\theta) \qquad (2.13)$$

where

$$C^{-1} = 2(\lambda S_n)^{-\frac{1}{2}(np-1)}\ K_{np-1}\{2\sqrt{(S_n/\lambda)}\}.$$

Under squared error loss function (1.2) the Bayes' estimator of θ is

$$\hat{\theta} = \int_0^\infty \theta h_2(\theta|x)\, d\theta$$

$$= \int_0^\infty C(1/\theta)^{np-1}\exp(-\theta/\lambda - S_n/\theta)\ d\theta \qquad (2.14)$$

Using (2.12) we easily get:

$$\hat{\theta} = (\lambda S_n)^{\frac{1}{2}}\,\frac{K_{np-2}\{2\sqrt{(S_n/\lambda)}\}}{K_{np-1}\{2\sqrt{(S_n/\lambda)}\}},\ np \geq 2 \qquad (2.15)$$

and its variance is

$$V(\hat{\theta}|\mathbf{x}) = \lambda S_n$$

$$\times\frac{K_{np-3}\{2\sqrt{(S_n/\lambda)}\}K_{np-1}\{2\sqrt{(S_n/\lambda)}\}K^2_{np-2}-\{2\sqrt{(S_n/\lambda)}\}}{K^2_{np-1}\{2\sqrt{(S_n/\lambda)}\}},$$

$$np \geq 3 \qquad (2.16)$$

Using (2.12) and (2.13), term-by-term integration of (2.7) gives the estimator of the reliability function by:

$$\hat{R}(t) = \frac{1}{\Gamma(p)}$$

$$\times \sum_{m=0}^{\infty}\frac{(-1)^m[B^{p+m}_{(d)} - B^{p+m}_{(t)}]K_{p(n+1)+m-1}\{2\sqrt{(S_n/\lambda)}\}}{m!(p+m)\ (\lambda S_n)^{+\frac{1}{2}(p+m)}K_{np-1}\ \{2\sqrt{(S_n/\lambda)}\}}$$

$$(2.17)$$

Using (2.12) and (2.13) we get:

$$\int_0^\infty e^{-B(t)/\theta}\left(\frac{B(t)}{\theta}\right)^m.\ h_2(\theta|x)\, d\theta =$$

$$\frac{B^m(t)\ K_{np+m-1}\ 2\sqrt{\{S_n+B(t)\}/\lambda}}{\{\lambda(S_n+B(t))\}^{\frac{np+m-1}{2}}}$$

$$\times\frac{(\lambda S_n)^{\frac{np-1}{2}}}{K_{np-1}\ \{2\sqrt{(S_n/\lambda)}\}}$$

similarly, we evaluate $\int_0^\infty e^{-B(d)/\theta}\ \{B(d)\}\ \dfrac{m}{\theta}h_2(\theta|x)d\theta$

and hence, the estimator of the reliability function in (2.9) is given by:

$$\hat{R}(t) = \sum_{m=0}^{p-1}\frac{1}{m!}$$

$$\times\left[\frac{B^m(t)\times K_{np+m-1}\left\{2\sqrt{\left(\frac{S_n+B(t)}{\lambda}\right)}\right\}}{\{\lambda(S_n+B(t))\}^{\frac{np+m-1}{2}}} - \frac{B^m(d)\times K_{np+m-1}\left\{2\sqrt{\left(\frac{S_n+B(d)}{\lambda}\right)}\right\}}{\{\lambda(S_n+B(d))\}^{\frac{np+m-1}{2}}}\right]$$

$$\times\frac{(\lambda S_n)^{\frac{np-1}{2}}}{K_{np-1}\ (2\sqrt{S_n/\lambda})} \qquad (2.18)$$

The third prior we consider is the inverted gamma (3). As before, the posterior density of θ is easily found to be:

$$h_3(\theta|\mathbf{x}) = \frac{(1/\theta)^{np+b+1}(S_n+\mu)^{np+b}\exp(-S_n-\mu)/\theta}{\Gamma(np+b)},$$

$$0 < \theta < \infty \qquad (2.19)$$

Under squared error loss function given in (1.2), the Bayes' estimator of θ is:

$$\tilde{\theta} = E[\Theta|\mathbf{x}]$$

$$= \int_0^\infty \theta\ h_3(\theta|\mathbf{x})\ d\theta$$

Carrying out the integration, we easily get:

$$\tilde{\theta} = \frac{S_n+\mu}{np+b-1},\quad np+b > 1 \qquad (2.20)$$

The variance of $\tilde{\theta}$ is the posterior variance of θ given by

$$V(\tilde{\theta}|\mathbf{x}) = E[\Theta^2|\mathbf{x}] - (E[\Theta|\mathbf{x}])^2$$

Using (2.19) we get:

$$V(\tilde{\theta}|\mathbf{x}) = \frac{(S_n+\mu)^2}{(np+b-1)^2(np+b-2)}, \quad np+b>2 \quad (2.21)$$

Again, using (2.19), term by term integration of (2.7) and (2.9) gives the Bayes' estimators of the reliability functions of (2.7) and (2.9), respectively, by:

$$\tilde{R}(t) = \frac{1}{\Gamma(p)} \sum_{m=0}^{\infty} \frac{(-1)^m[B^{p+m}_{(d)} - B^{p+m}_{(t)}]}{m!\,(p+m)} \times \frac{\Gamma(p(n+1)+b+m)}{\Gamma(np+b)(S_n+\mu)^{p+m}} \quad (2.22)$$

$$\tilde{\tilde{R}}(t) = \sum_{m=0}^{p-1} \frac{(S_n+\mu)^{np+b}}{m!\,\Gamma(np+b)} \Gamma(np+b+m) \times \left[\frac{B^m(t)}{(S_n+B(d)+\mu)^{np+b+m}} - \frac{B^m(d)}{(S_n+B(d)+\mu)^{np+b+m}}\right] \quad (2.23)$$

3. PRIOR IGNORANCE AND QUASI-DENSITIES

The case of lack of knowledge on the prior density of θ is treated by letting $a=0$ and $(\alpha,\beta)\rightarrow(0,\infty)$ in (1), or more generally, by using the quasi-density

$$g(\theta) = \frac{1}{\theta^a}, \quad 0<\theta<\infty \quad (3.1)$$

(Justification and detailed discussion can be found in [6], [7], and [9].)

Under (3.1), (2.2) reduces to

$$h(\theta|\mathbf{x}) = \frac{(1/\theta)^{np+a}\, S_n^{np+a-1}\exp(-S_n/\theta)}{\Gamma(np+a-1)}, \quad 0<\theta<\infty,\ np+a>1. \quad (3.2)$$

It is evident that the results in this case are obtained by letting $\alpha\rightarrow 0$ and $\beta\rightarrow\infty$ and the estimators (2.3), (2.4), (2.8), and (2.10), respectively reduce to:

$$\theta_1^* = \frac{S_n}{np+a-2}; \quad np+a>2 \quad (3.3)$$

$$V(\theta_1^*|\mathbf{x}) = \frac{S_n^2}{(np+a-2)^2(np+a-3)}, \quad np+a>3 \quad (3.4)$$

$$R_1^*(t) = \frac{\Gamma(p(n+1)+a-1)}{\Gamma(p+1)\,\Gamma(np+a-1)} \left[\left(\frac{B(d)}{S_n}\right)^p {}_2F_1\left(p(n+1)+a-1, p; p+1; -\frac{B(d)}{S_n}\right) - \left(\frac{B(t)}{S_n}\right)^p {}_2F_1\left(p(n+1)+a-1, p; p+1; -\frac{B(t)}{S_n}\right)\right] \quad (3.5)$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function (Erdelyi *et al.* [8]).

$$R_1^{**}(t) = \sum_{m=0}^{p-1} \frac{1}{m!} S_n^{np+a-1} \frac{\Gamma(np+a+m-1)}{\Gamma(np+a-1)} \times \left[\frac{1}{(S_n+B(t))^{np+a+m-1}} - \frac{1}{(S_n+B(d))^{np+a+m-1}}\right]. \quad (3.6)$$

If $\alpha\rightarrow\beta$, then by applying L'Hopital's rule to the RHS of (2.3), (2.4), (2.8), and (2.10) they, respectively, reduce to:

$$\theta_2^* = \beta \quad ; \quad V(\theta*|\mathbf{x}) = 0$$

$$R_2^*(t) = \frac{\gamma(p, B(d)/\beta) - \gamma(p, B(t)/\beta)}{\Gamma(p)}, \text{ and}$$

$$R_2^{**}(t) = \sum_{m=0}^{p-1} \frac{1}{m!}\left[\left(\frac{B(t)}{\beta}\right)^m \exp(-B(t)/\beta) - \left(\frac{B(d)}{\beta}\right)^m \exp(-B(d)/\beta)\right]$$

These results come out as expected of the case of degenerate prior density of θ namely, $g(\theta) = \begin{cases} 1, & \theta=\beta \\ 0, & \theta\neq\beta \end{cases}$

In many instances, Bayes confidence intervals for θ are of interest.

It is seen from (2.3) that $\frac{S_n}{\theta}$ is distributed as Gamma $(np+a-1,1)$ and hence $100(1-\alpha)\%$ Bayes' confidence intervals for θ can easily be constructed. If $2(np+a-1)$ is a positive integer, $\frac{2S_n}{\theta}$ has a χ^2 distribution with $2(np+a-1)$ degrees of freedom, and $100(1-\alpha)\%$ Bayes' confidence interval for θ is given by $\left(\frac{2S_n}{\chi^2_{1-\alpha/2}}, \frac{2S_n}{\chi^2_{\alpha/2}}\right)$ where $\chi^2_\alpha = 100\,\alpha\%$ point of χ^2 distribution with $2(np+a-1)$ d.f. In case of lack of knowledge about the prior of θ we use $a=0$, but if the experimenter's knowledge about θ is only vague, he can use $a=1$ which corresponds to Jeffery's invariant prior.

4. SPECIAL CASES

In this section we summarize the previous results for two special forms of $B(\mathbf{x})$ which contain many members.

(a) Let $p=1$ and $B(\mathbf{x})$ be monotone increasing such that $B(\mathbf{x}) \to \infty$ as $\mathbf{x} \to d$, d being finite or infinite.

The special cases of such $B(\mathbf{x})$ include: the exponential, Rayleigh, and Weibull with known shape parameter. The reliability function is

$$R(t) = e^{-B(t)/\theta}, \quad t \geq c$$

(b) Let $p=1$ and $B(\mathbf{x})$ be monotone decreasing such that $B(\mathbf{x}) \to 0$ as $\mathbf{x} \to d$. Such family has inverse Rayleigh and inverse Weibull distributions [10] as special cases. The reliability function is

$$R(t) = 1 - e^{-B(t)/\theta}, \quad t \geq c$$

The Bayes estimators of $R(t)$ for cases (a) and (b) under the different prior densities of θ are listed in Table 1.

Table 1.

Prior of θ	Bayes' estimator of $R(t)$	
	Case (a)	Case (b)
Uniform (1)	$R_U^{*} = \left(\frac{S_n}{S_n+B(t)}\right)^{n+a-1} \times \frac{\gamma*(n+a-1, S_n+B(t))}{\gamma*(n+a-1, S_n)}$	$1-R_U^{*}$
Exponential (2)	$R_E^{*} = \left(\frac{S_n}{S_n+B(t)}\right)^{\frac{1}{2}(n-1)} \times \frac{K_{n-1}\left\{2\sqrt{\left(\frac{S_n+B(t)}{\lambda}\right)}\right\}}{K_{n-1}\{2\sqrt{S_n/\lambda}\}}$	$1-R_E^{*}$
Inverted Gamma (3)	$R_{IG}^{*} = \left(\frac{S_n+\mu}{S_n+\mu+B(t)}\right)^{n+b}$	$1-R_{IG}^{*}$
Quasi-Prior $\frac{1}{\theta^a}, 0<\theta<\infty$	$R_Q^{*} = \left(\frac{S_n}{S_n+B(t)}\right)^{n+a-1}$	$1-R_Q^{*}$

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