COMPLETE COLLINEATION GROUP OF THE NEW FLAG TRANSITIVE PLANE OF ORDER 27

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الخلاصـة :

لقد حددنا في هذا البحث زمرة المسافة الكاملة للمستوى المتعدي ذي العلم الجديد التي اكتشفها راو راو (مداولات الجمعية الرياضية الأمريكية ٥٩ و (١٩٧٦) و ص ٣٣٧) . ولقد وجدنا أن مكملة النقل لهذا المستوى هي زمرة فوق دورية عدد عناصرها ١٦٨ .

ABSTRACT

The complete collineation group of the new flag transitive plane of order 27 of Rao and Rao [see *Proceedings of the American Mathematical Society*, **59** (1976), p. 337] is determined. The translation complement of this plane is a metacyclic group of order 168.

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0377–9211/85/020183–05$01.00

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1. INTRODUCTION

Rao and Rao [1] have constructed a flag transitive plane of order 27 and have proved that their new plane is not isomorphic to Hering's plane of order 27 [2]. The aim of this paper is to determine the complete collineation group of the flag transitive plane π_N constructed by Rao and Rao [1]. It is established that the translation complement G of π_N is of order 168, a very small number in comparison with the order of the translation complement of Hering's plane [3], which is 2,184. The group G is actually a metacyclic group obtained by extending a cyclic group of order 56 by a group of order 3.

2. NOTATION

Throughout this paper we use the notation of Rao and Rao [1]. They defined their new flag transitive plane π_N through a 2-spread set $\mathscr{B}$ (See Table 2.1 of [1]) and exhibited a flag transitive collineation T of π_N. Since the 2-spread set is not amenable to easy manipulation, we define a new 2-spread set $\mathscr{B}_1$ using M_8, $M_{15} \in \mathscr{B}$ (see Table 2.1 of [1]).

Let

$$\mathscr{B}_1 = \{N_i | N_i = (M_8 - M_{15})^{-1}(M_i - M_{15}),\ M_i \in \mathscr{B}\}.$$

Obviously $\mathscr{B}_1$ is a 2-spread set over $GF(3)$ and defines a plane π isomorphic to π_N [4]. The 2-spread set $\mathscr{B}_1$ is exhibited in Table 1. The entry abc in Table 1 under the heading 'C.P of N_i' indicates that N_i has $-\lambda^3 + a\lambda^2 + b\lambda + c$ as its characteristic polynomial. Throughout this paper every matrix is a matrix over $GF(3)$ and for convenience, we denote a 3×3 matrix

$$\begin{pmatrix} x & y & z \\ p & q & r \\ a & b & c \end{pmatrix}$$

by (xyz, pqr, abc). We now determine the translation complement of the plane π associated with the 2-spread set $\mathscr{B}_1$. In what follows, by a collineation we mean a collineation belonging to the translation complement of π.

Table 1. The 2-Spread Set $\mathscr{B}_1$

i	N_i	C.P of N_i	i	N_i	C.P of N_i
0	(001, 021, 222)	102	15	(000, 000, 000)	000
2	(010, 001, 102)	201	16	(022, 200, 020)	012
3	(111, 121, 021)	111	17	(021, 120, 122)	122
4	(122, 220, 201)	102	18	(212, 101, 211)	011
5	(112, 012, 202)	122	19	(210, 111, 121)	111
6	(201, 011, 120)	011	20	(211, 212, 111)	102
7	(220, 222, 011)	212	21	(011, 122, 210)	221
8	(100, 010, 001)	001	22	(200, 020, 002)	002
9	(101, 112, 212)	122	23	(012, 202, 221)	111
10	(002, 201, 112)	201	24	(110, 022, 100)	012
11	(120, 211, 010)	212	25	(202, 002, 220)	221
12	(102, 221, 110)	012	26	(222, 110, 022)	201
13	(121, 100, 101)	221	27	(020, 102, 012)	212
14	(221, 210, 200)	011			

Since I, $- \in \mathscr{B}_1$ and $\mathscr{B}_1$ is a 2-spread set, we may conclude that the characteristic polynomial of every non-scalar matrix in $\mathscr{B}_1$ is irreducible.

3. SOME COLLINEATIONS OF π

The lines of π that contain the zero vector are denoted by V_i, where $V_i = \{(x, y, z, p, q, r) | x, y, z \in GF(3), (p, q, r) = (x, y, z)N_i,\ N_i \in \mathscr{B}_1\}$ and $V_1 = \{(0, 0, 0, x, y, z) | x, y, z \in GF(3)\}$. Any nonsingular linear transformation of $V(6, 3)$ induces a collineation of π if and only if the linear transformation permutes the subspaces $V_i, 0 \leqslant i \leqslant 27$, among themselves. Rao and Rao have exhibited a flag transitive collineation T of π_N [1]. By applying an appropriate transformation on T using the mapping

$$M \to (M_8 - M_{15})^{-1}(M - M_{15})$$

we obtain the corresponding flag transitive collineation α of π given by

$$\alpha = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

where $A = (222, 012, 110)$, $B = (121, 210, 222)$ and its action restricted to the set of ideal points is

$$\alpha: (0, 1, 2, \ldots, 26, 27).$$

It is easily verified that

$$\alpha^7=\begin{pmatrix} I & 2I \\ I & I \end{pmatrix},\ \alpha^{14}=\begin{pmatrix} 0 & I \\ 2I & 0 \end{pmatrix},\ \text{and } \alpha^{21}=\begin{pmatrix} I & I \\ 2I & I \end{pmatrix},$$

where I is the 3×3 identity matrix and 0 is the 3×3 zero matrix. We now exhibit another collineation β of π. Let

$$\beta=\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}$$

where H = (001, 022, 201). It is easily verified that $V_0\beta=V_{20}$ and $\beta^{-1}\alpha\beta=\alpha^9$. Using these relations we get that

$$V_i\beta=V_0\alpha^i\beta=V_0\beta\ \beta^{-1}\alpha^i\beta=V_{20}\alpha^{9i}=V_j$$

where $j\equiv 20+9i \pmod{28}$. This implies that β permutes the subspaces V_i, $0\leqslant i\leqslant 27$ among themselves and therefore β induces a collineation of π and its action restricted to the ideal points is

$$\begin{aligned}\beta\colon &(1)(8)(15)(22)(0,\ 20,\ 4)(2,\ 10,\ 26)\\ &(3,\ 19,\ 23)(5,\ 9,\ 17)(6,\ 18,\ 14)(7,\ 27,\ 11)\\ &(12,\ 16,\ 24)(13,\ 25,\ 21).\end{aligned}$$

Since

$$\beta^3=\begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix},$$

we find that β is of order 6. The rest of this paper is devoted to showing that the group generated by α and β is the translation complement of π and that its order is 168.

4. THE GROUP GENERATED BY α AND β

Let G be the translation complement of π. Since G contains α, G is transitive on the set of lines V_i, $0\leqslant i\leqslant 27$. Let G_1 be the subgroup of G consisting of all collineations of π that fix the line V_1. Then

$$G=\bigcup_{i=0}^{27} G_1\alpha^i,$$

and

$$|G|=28\cdot|G_1|.$$

We wish to show that G_1 is generated by β so that $G=\langle\alpha,\beta\rangle$ and $|G|=168$. We need some preliminary results in this connection.

Lemma 4.1 The matrices $-N_i$, $i=2$, 3, 5, and 6 cannot be expressed in the form $(L+M)$ where L, $M\in\mathscr{B}_1$, $L\neq M$, $L\neq N_{15}\neq M$.

Proof. The first row of $-N_2$ is 020 and the following are the only pairs (i,j), $i\neq j$, $i\neq 15\neq j$ such that N_i+N_j has 020 as its first row:

(0, 16), (3, 18), (4, 6), (5, 20), (7, 8),
(9, 26), (10, 17), (11, 22), (12, 14), (13, 25),
(19, 24), (21, 23).

But none of these pairs satisfies the condition $-N_2=(N_i+N_j)$.

The other cases may be proved similarly.

For convenience we denote line V_i by line i.

Theorem 4.2 Every collineation that fixes line 1 also fixes line 15.

Proof. Suppose μ is a collineation of π that fixes line 1 and moves lines r and 15 onto lines 15 and s respectively. Then μ is of the form

$$\begin{pmatrix} B & BN_s \\ 0 & -N_r^{-1}BN_s \end{pmatrix}$$

for some $B\in GL(3,3)$ satisfying the condition that for each $M\in\mathscr{B}_1$, there exists $N\in\mathscr{B}_1$ (and *vice versa*) such that the following relation holds.

$$B^{-1}(I-MN_r^{-1})BN_s=N. \tag{4.1}$$

Adding the two relations obtained from Relation (4.1) by taking $M=N_8$ and N_{22} and using the fact that $N_8+N_{22}=(000,\ 000,\ 000)$, we get the existence of two matrices L, $K\in\mathscr{B}_1$ such that $L\neq K$ and $-N_s=L+K$. This is not possible for $s=2$, 3, 5, and 6 in view of Lemma 4.1. Hence there does not exist a collineation that fixes line 1 and moves line 15 onto line s, $s=2$, 3, 5, and 6. If μ is a collineation that fixes line 1 and moves line 15 onto line s, then μ, $\beta^{-1}\mu\beta$, $\beta^{-2}\mu\beta^2$, $\alpha^{14}\mu\alpha^{1-s}$, $\beta^{-1}\alpha^{14}\mu\alpha^{-s}\beta$, and $\beta^{-2}\alpha^{14}\mu\alpha^{1-s}\beta^2$ are collineations that fix line 1 and move line 15 onto line t where $t=s$, $t\equiv 9s+20 \pmod{28}$, $t\equiv 25s+4 \pmod{28}$, $t\equiv 2-s \pmod{28}$, $t\equiv 10-9s \pmod{28}$ and $t\equiv 26-25s \pmod{28}$, respectively. From this we may now conclude that there does not exist a collineation that fixes line 1 and moves line 15 onto line s, where $0\leqslant s\leqslant 27$, $s\neq 8$, 15, and 22. Suppose that μ is a collineation that fixes line 1 and moves lines r and 15 onto lines 15 and 8 respectively. It follows from what we have just proved above that r must be 8 or 22. If $r=8$, then μ is of the form

$$\begin{pmatrix} B & B \\ O & 2B \end{pmatrix}$$

and

$$\mu^{-1}\alpha^7\mu\alpha^{21}=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

which is not a collineation of π since N_2^{-1} does not belong to $\mathscr{b}_1$. This implies that μ is not a collineation. If $r=22$, then μ is of the form

$$\begin{pmatrix}B & B\\ 0 & B\end{pmatrix} \quad \text{and} \quad \mu^{-1}\alpha^{14}\mu\alpha^7=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

and therefore μ is not a collineation. Thus there does not exist a collineation that fixes line 1 and moves line 15 onto line 8. Taking $s=8$ in the congruence $t\equiv 2-s \pmod{28}$ we obtain that $t=22$, implying that there does not exist a collineation that fixes line 1 and moves line 15 onto line 22. Hence if a collineation fixes line 1, then it also fixes line 15.

We now state a lemma without proof which follows from Schur's lemma and Wedderburn's theorem regarding irreducible rings of endomorphisms of vector spaces.

Lemma 4.2 Let $M\in F_3$, the ring of all 3×3 matrices over $GF(3)$ and let the characteristic polynomial of M be irreducible over $GF(3)$. Let $Z(M)=\{P\in F_3 \mid MP=PM\}$. Then $Z(M)$ is the subfield of F_3 generated by M.

Lemma 4.3 Let $M, N\in F_3$ and let there exist $L\in F_3$ such that $L^{-1}ML=N$. Let $T(M,N)=\{P\in F_3 \mid MP=PN\}$. Then $T(M,N)=Z(M)L$ where $Z(M)$ is the centralizer of M in F_3.

Proof. Follows easily.

Theorem 4.3 There does not exist a collineation that fixes lines 1 and 15 and interchanges lines 8 and 22.

Proof. If μ is a collineation that fixes lines 1 and 15 and interchanges lines 8 and 22, then it is of the form $\begin{pmatrix}B & 0\\ 0 & -B\end{pmatrix}$ satisfying the condition that for each $M\in \mathscr{b}_1$, there exist $N\in \mathscr{b}_1$ (and *vice versa*) such that

$$-B^{-1}MB=N.$$

The matrices N_2, N_{10}, and N_{26} are the only matrices in $\mathscr{b}_1$ having the characteristic polynomial $-\lambda^3+2\lambda^2+1$. The matrices $-N_0$, $-N_4$, and $-N_{20}$ have the characteristic polynomial $-\lambda^3+2\lambda^2+1$. This forces μ to map lines 2, 10, and 26 onto lines 0, 4, and 20 in some order, and *vice versa*. Further, μ moves all the lines 2, 10, 26, 0, 4 and 20. Then either μ or μ^3 is such that the line 2 is interchanged with one of the lines 0, 4, or 20. It is therefore enough if we consider the existence of collineations μ_i whose actions restricted to the ideal points are

$$\mu_1:(1)(15)(8,\,22)(2,\,0)(\ldots)\ldots$$
$$\mu_2:(1)(15)(8,\,22)(2,\,4)(\ldots)\ldots$$
$$\mu_3:(1)(15)(8,\,22)(2,\,20)(\ldots)\ldots$$

The collineation μ_1 is of the form

$$\begin{pmatrix}B & 0\\ 0 & -B\end{pmatrix}$$

where $B\in GL(3,\,3)$ satisfying the following relations.

$$N_2B=BN_0 \tag{4.2}$$
$$N_0B=BN_2. \tag{4.3}$$

The general form of B satisfying Relations (4.2) and (4.3) may be obtained using Lemmas 4.2 and 4.3 or by solving simultaneous equations obtained from (4.2) and (4.3). These forms are

$$T(N_2,\,-N_0)=\left\{P=\begin{pmatrix}a & b & c\\ c & b+c & 2a+2b+c\\ 2a+2b+c & 2a+2c & 2a+b+2c\end{pmatrix}\;\middle|\; a,\,b,\,c\in GF(3)\right\}$$

$$T(N_0,\,-N_2)=\left\{Q=\begin{pmatrix}a & b & c\\ 2a+2b+2c & a+2b+2c & 2a+2b\\ 2c & 2a & 2b+c\end{pmatrix}\;\middle|\; a,\,b,\,c\in GF(3)\right\}.$$

The Relations (4.2) and (4.3) are simultaneously satisfied if $B=P=Q$, which forces $B=(120,\,020,\,021)$ or $B=(210,\,010,\,012)$. But $B^{-1}(2N_{10})B=(222,\,201,\,112)$ does not belong to $\mathscr{b}_1$. This implies that B does not induce a collineation, implying that μ_1 does not exist. Similarly one may prove that μ_2 and μ_3 do not exist. Thus there does not exist a collineation that fixes lines 1 and 15 and interchanges lines 8 and 22.

Theorem 4.4 The group G_1 consisting of all collineations that fix line 1 is generated by β and is of order 6.

Proof. If σ is a collineation that fixes the line 1, then by Theorem 4.2 it fixes line 15 and therefore it is of the form

$$\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$$

satisfying the condition that

$$M \to A^{-1}MB$$

is a one-to-one mapping from ℓ_1 onto ℓ_1. Taking $M=N_8$ and N_{22} we find that $A^{-1}B$, $-A^{-1}B$ must both be in ℓ_1, which is possible only if $A=B$ or $2A=B$. $2A=B$ does not induce a collineation by Theorem 4.3. It therefore follows that G_1 consists of mappings of the form

$$M \to A^{-1}MA \tag{4.4}$$

from ℓ_1 onto ℓ_1. The collineation β given earlier is obtained by taking $A=(102, 220, 100)$. We observe that N_2, N_{10}, and N_{26} are similar, and no other matrices in ℓ_1 are similar to these matrices. The mapping σ, therefore, must map N_2, N_{10}, and N_{26} onto N_2, N_{10}, and N_{26} in some order. The collineation β acts transitively on the set consisting of lines 2, 10, and 26 and consequently G_1 acts transitively on the set consisting of lines 2, 10, and 26. It is therefore enough if we obtain all collineations σ that fix lines 1, 8, 15, 22, and 2 since any collineation that fixes the lines 1, 8, 15, and 22 is a combination of σ with an element of $\langle \beta \rangle$. Suppose σ fixes lines 1, 8, 15, 22, 2, 10, and 26. Then the matrix A in (4.4) must be in $Z(N_2) \cap Z(N_{10}) \cap Z(N_{26})$ (Lemma 4.2). Since N_2, N_{10}, and N_{26} do not commute with each other, $Z(N_2)$, $Z(N_{10})$, and $Z(N_{26})$ are distinct fields. Therefore

$$Z(N_2) \cap Z(N_{10}) \cap Z(N_{26}) = \{0, I, 2I\}.$$

From this we conclude that σ is a scalar collineation. Suppose σ is a collineation that fixes lines 1, 8, 15, 22, and 2 and interchanges lines 10 and 26. Then σ^2 is a scalar collineation because it fixes lines 1, 8, 15, 22, 2, 10, and 26. From this we obtain that $A \neq I$, $2I$ but $A^2 = I$ or $2I$ where $A \in Z(N_2)$, which is a contradiction, since $Z(N_2)$ contains no such A. Thus whenever a collineation fixes lines 1, 8, 15, 22, and 2, it must fix lines 10 and 26 and therefore it is a scalar collineation. The group G_1 of all collineations that fix the line 1 is generated by β, and G_1 is of order 6.

Theorem 4.5 The translation complement G of π is the metacyclic group generated by α and β and is of order 168.

Proof. Follows from Theorem 4.4 and the transitivity of G on the lines i, $0 \leqslant i \leqslant 27$. The subgroup $\langle \alpha \rangle$ is of order 56 containing a cyclic subgroup of order 8 which happens to be the Sylow subgroup of G corresponding to the prime 2. The Sylow subgroups of G corresponding to the other primes are cyclic since they are of prime orders. The group G is therefore metacyclic.

The complete collineation group of π is indeed generated by the translation complement and the translation group of π.

It is interesting to note that the two flag transitive planes of order 25 constructed by Foulser [5] have the property that the flag transitive collineation groups are themselves the translation complements [6, 7]. This is not the case with this plane π.

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Paper Received 20 December 1983; Revised 22 May 1984.