HAMILTONIAN STRUCTURES OF SOME NON-LINEAR EVOLUTION EQUATIONS

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الخلاصـــة :

نناقش في هذا البحث التركيب الهاملتوني لنموذج سيجما الغير خطي ، و(٢ ، ١) ، الذي عمم معادلات أ ك ن س . وباختزال نموذج سيجما الغير خطي ، و(٢ ، ١) ، لصورته الهاملتونية ، أمكننا أن نشتق بعض قوانين المحافظة . كذلك عرضنا سلسلة جديدة من معادلات التجذير الغير خطية وبينا أنها معادلات هاملتونية معممة لها عدد لا نهائي من قوانين المحافظة .

ABSTRACT

The Hamiltonian structures of the O(2, 1) non-linear sigma model, generalized AKNS equations, are discussed. By reducing the O(2, 1) non-linear sigma model to its Hamiltonian form, some new conservation laws are derived. A new hierarchy of non-linear evolution equations is proposed and shown to be generalized Hamiltonian equations with an infinite number conservation laws.

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1. INTRODUCTION

Both from mathematical and from physical points of view the theory of Hamiltonian systems is beautiful and elegant. The great importance of the Hamiltonian theory comes from its deep physical insight. As is well known, the early development of quantum mechanics and statistical mechanics is based entirely on Hamiltonian equations. Moreover, the great significance of the Hamiltonian formalism also comes from the fact that many branches of mathematics are involved in the recent development of the theory, such as the theory of representation of Lie groups, the theory of canonical operators, and symplectic manifolds of finite or infinite dimension. In the past 15 years the emergence of the soliton theory gives a new impulsion to the study of the Hamiltonian theory. Since the pioneer work [1–5], many important non-linear equations, arising from various branches of physics, such as the Yang–Mills equations, Ernst equations, isotropic Heisenberg equations, Thirring model, and non-linear sigma models, are found to be completely integrable Hamiltonian equations which can be solved by the powerful IST technique [6].

In the soliton theory, the non-linear evolution equations (NLEEs) are usually derived as an isospectral deformation equation. To be more precise, let

$$\psi_x = U\psi, \qquad \psi_t = V\psi \tag{1.1}$$

be a couple of linear equations with $\psi \equiv (\psi_1, \ldots, \psi_N)^{\mathrm{T}}$ being an N-vector, and U and V being two $N \times N$ matrices whose entries are dependent on some potential $u = u(x, t) = (u^1, \ldots, u^M)$ and a spectral parameter λ. In most cases the matrices U and V are taken from some Lie algebra such as $\mathrm{sl}(N)$. Equation (1.1) can be written as

$$\mathrm{d}\psi = \Omega\psi, \tag{1.2}$$

where $\Omega = U\mathrm{d}x + V\mathrm{d}t$ is a one-form related to the spectral problem (1.1). Taking the exterior derivatives on both sides of (1.2), we obtain the following important 'zero-curvature' condition

$$\mathrm{d}\Omega = \Omega \wedge \Omega, \tag{1.3}$$

where $\wedge$ represents the wedge product. In terms of matrices U and V, this integrability condition (1.3) can be written equivalently as

$$U_t - V_x + [U,V] = 0, \tag{1.4}$$

where $[U,V] = UV - VU$ denotes the usual commutator of matrices U and V. In most cases studied thus far, for a properly chosen matrix U there always exist different forms of V, usually taking the form of $V = \Sigma\lambda^{-k}V_k$, such that Equation (1.4) reduces to a hierarchy of λ-independent non-linear evolution equations (see, for example, [7]),

$$u_t = JL^n f(u) \tag{1.5a}$$

or

$$Ku_t = L^n f(u), \tag{1.5b}$$

where J, K, and L are some linear differential operators. Following the terminology in soliton literature we call these equations the soliton equations, since the first such equation—the celebrated KdV equation—exhibits soliton solutions. The extensive investigation undergone in the past decade reveals many intriguing features of soliton equations. One feature among others is the fact that there always exists an infinite set $\{h_n\}$ such that

$$L^n f(u) = (\delta/\delta u)h_n(u), \tag{1.6}$$

where $\delta/\delta u$ stands for the variational derivative. In terms of these h_n, the above hierarchy (1.5) can be written in the form of generalized Hamiltonian equations:

$$u_t = J\delta H/\delta u \tag{1.7a}$$

$$Ku_t = \delta H/\delta u. \tag{1.7b}$$

To transform an evolution equation $u_t = F(u)$ into its Hamiltonian form is not only a matter of beauty, but other rewards can also be received from doing so. In fact, as we show in Section 3, by reducing the O(2, 1) non-linear sigma model to its Hamiltonian form we are able to find three new conservation laws.

The organization of this paper is as follows. In Section 2 some basic notions on generalized Hamiltonian equations are briefly sketched; in Section 3 the non-linear O(2,1) sigma model is shown to possess three new conservation laws by reducing the model to its Hamiltonian form. In Section 4, by following a simple approach presented in [7–10], the Hamiltonian form of the hierarchy of NLEEs relating to the generalized $N \times N$ Zakharov–Shabat spectral problem is derived. Finally, in Section 5 a new hierarchy of equations with the form (1.5b) is proposed.

The first couple of equations in this new hierarchy read:

$$\begin{aligned} u_t &= u_x + 2v \\ v_t &= \pm 2uv. \end{aligned} \tag{1.8}$$

By setting $v = \exp(w)$, the above couple of equations can be reduced to a single equation

$$w_{tt} - w_{tx} \pm 4 \exp w = 0,$$

and it is shown that Equation (1.6) holds in this case; thus the whole hierarchy of equations takes the form of generalized Hamiltonian equations.

2. PRELIMINARIES

As is well known, the ordinary Hamiltonian equation for continuous media reads

$$\frac{\partial p_i}{\partial t} = \frac{-\delta H}{\delta q_i}, \quad \frac{\partial q_i}{\partial t} = \frac{\delta H}{\delta p_i}, \quad i = 1, \ldots, n, \tag{2.1}$$

where $\delta/\delta p$ and $\delta/\delta q$ denote the variational derivatives. Setting $u = (p_1, \ldots, p_n, q_1, \ldots, q_n)$ and

$$\frac{\delta}{\delta u} = \left(\frac{\delta}{\delta p_1}, \ldots, \frac{\delta}{\delta q_n}\right)^{\mathrm{T}}, \quad J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where T represents the transpose and I_n is the identity matrix of order n, we may rewrite Equation (2.1) in the concise form

$$u_t = J_n \frac{\delta H}{\delta u}. \tag{2.2}$$

In this ordinary case the Poisson bracket of Equation (2.1) is defined by

$$\{F, G\} = \sum_i \left(\frac{\delta F}{\delta q_i}\frac{\delta G}{\delta p_i} - \frac{\delta F}{\delta p_i}\frac{\delta G}{\delta q_i}\right),$$

which can be written as

$$\{F, G\} = \left(\frac{\delta F}{\delta u}\right)^{\mathrm{T}} J_n \left(\frac{\delta G}{\delta u}\right). \tag{2.3}$$

To give the definition of generalized Hamiltonian equations let $u = (u^1, \ldots, u^M)^{\mathrm{T}}$, $u^i = u^i(x_1, \ldots, x_N, t)$ be an M-vector, and

$$u^r_{i_1 \ldots i_k} = \frac{\partial^k u^r}{\partial x_{i_1} \ldots \partial x_{i_k}}, \quad u^{(k)} = \{u^r_{i_1 \ldots i_k}\}$$

be its x-derivatives of order k. Set

$$D_i = \frac{\partial}{\partial x_i} + \sum_{rk_1, \ldots, i_k} u^r_{i_1 \ldots i_k i} \frac{\partial}{\partial u^r_{i_1 \ldots i_k}}$$

as the operator of total differentiation with respect to x_i. The variational derivatives are defined by $\delta/\delta u = (\delta/\delta u^1, \ldots, \delta/\delta u^M)^{\mathrm{T}}$, with

$$\frac{\delta}{\delta u^r} = \sum_{ki_1 \ldots i_k} (-1)^k D_{i_1} \ldots D_{i_k} \frac{\partial}{\partial u^r_{i_1 \ldots i_k}}.$$

An operator $K = K(u)$, which depends on the function u, is called *symplectic* if: (i) it is linear and skew symmetric, that is $K^* = -K$, where* stands for the formal conjugation

$$(A_{ij})^* = (A^*_{ji}), \quad \left(\sum a_i D^i\right)^* = \sum (-D)^i (a_i \cdot);$$

(ii) the bracket defined by $\{f, g, h\}_1 = f \cdot K'[g]h$ satisfies the Jacobi identity

$$\{f, g, h\} + \{g, h, f\} + \{h, f, g\} \overset{D}{=} 0$$

where $f \cdot g = \sum_{i=1}^{M} f^i g^i$, $f \overset{D}{=} g$ means $f - g = \Sigma D_i h_i$ for some vector $h = (h_1, \ldots, h_N)$ and $K'[g]h$ refers to the Gateaux derivative

$$K'[u]f = \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right) K(u + \varepsilon f) \quad {}_{\varepsilon = 0}$$

An operator $J = J(u)$ is called *cosymplectic* if: (i) it is linear and skew symmetric; (ii) the bracket defined by $\{f, g, h\}_2 = f . J'[Jg]h$ satisfies the Jacobi identity. The equation

$$u_t = \frac{J \delta H}{\delta u} \tag{2.2'}$$

or

$$K u_t = \frac{\delta H}{\delta u} \tag{2.2''}$$

is called a *generalized Hamiltonian equation* if the operator J is cosymplectic or K is symplectic. The *Poisson bracket* of the generalized Hamiltonian equation (2.2′) is defined by an equation similar to Equation (2.3)

$$\{F, G\} = \left(\frac{\delta F}{\delta u}\right)^{\mathrm{T}} J \left(\frac{\delta G}{\delta u}\right). \tag{2.3'}$$

A scalar function $f = f(u, u^{(1)}, \ldots, u^{(k)})$, which depends on $u(x, t)$ and its space derivatives $u^{(k)}$, is called a conserved density of Equation (2.2) if $f_t \overset{D}{=} 0$ holds when $u(x,t)$ is taken to be solutions of Equations (2.2′) and (2.2″).

Some typical examples of the generalized Hamiltonian equations are as follows:

Korteweg–de Vries (KdV) equation $u_t = uu_x + u_{xxx}$,

$$J = D = \frac{d}{dx}, \quad H = \frac{u^3}{6} - \frac{u_x^2}{2}.$$

Modified KdV equation $u_t = u^2 u_x + u_{xxx}$,

$$J = D, \quad H = \frac{u^4}{12} - \frac{u_x^2}{2}.$$

Non-linear Schrödinger equation $q_t = \mathrm{i}(q_{xx} + |q|^2 q_x)$,

$$J = J_1, \quad H = \mathrm{i}\left(\frac{q^2 p^2}{2} - q_x p_x\right), \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

($p = \bar{q}$, the complex conjugate of q, $u = (p, q)^{\mathrm{T}}$.)

Boussinesq equation $q_{tt} = q_{xx} + 3(q_x^2)_x + q_{xxxx}$,

$$J = J_1, \quad H = \frac{(p^2 + q_x^2 + 2q_x^3 - q_{xx}^2)}{2}, \quad p = q_t, \quad u = (q, p)^{\mathrm{T}}.$$

Sine–Gordon equation $q_{tt} - q_{xx} = \sin q$,

$$J = J_1, \quad H = \frac{(q_x^2 + p^2)}{2} - \cos q, \quad p = q_t, \quad u = (q, p)^{\mathrm{T}}.$$

Benjamin–Bona–Mahony (BBM) equation $u_t = u_x + uu_x + u_{txx}$,

$$J = D(1 - D^2)^{-1}, \quad H = \frac{u^2}{2} + \frac{u_x^2}{2}.$$

For a given non-linear equation

$$G(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0 \tag{2.4}$$

a function $\eta = \eta(u) = (\eta^1, \ldots, \eta^M)$ is called a generalized symmetry of Equation (2.4) if the equation remains form invariant under the infinitesimal transformation $u' = u + \varepsilon\eta$, where ε is an infinitesimal parameter. The condition for invariance can be written as

$$\left(\frac{d}{d\varepsilon}\right) G(u + \varepsilon\eta)|_{\varepsilon=0} = 0. \tag{2.5}$$

In particular, we call $\eta = \eta(u)$ a symmetry of the evolution equation $u_t = F(u, u^{(1)}, u^{(2)}, \ldots)$ if

$$\eta_t = \left(\frac{d}{d\varepsilon}\right) F(u + \varepsilon\eta)|_{\varepsilon=0}. \tag{2.5'}$$

To write this condition (2.5′) more precisely we introduce the operator

$$V(f) = \begin{bmatrix} V_1(f^1) \ldots V_M(f^1) \\ \ldots\ldots\ldots\ldots\ldots \\ V_1(f^M) \ldots V_M(f^M) \end{bmatrix},$$

where

$$V_A(f^B) = \sum_{j \geq 0} \left(\frac{\partial f^B}{\partial u_j^A}\right) D^j \text{ (when } u \text{ is real).}$$

and

$$V_A(f^B) = \sum_j \left(\frac{\partial f^B}{\partial u_j^A}\right) D^j + \left(\frac{\partial f^B}{\partial \bar{u}_j^A}\right) CD^j \text{ (when } u \text{ is complex),}$$

where $\bar{f}$ stands for the complex conjugation of f and $Cf = \bar{f}$. With this operator the condition (2.5′) can be written as

$$[\eta, F] \equiv V(\eta)F - V(F)\eta = 0.$$

We need also the following transformation formula (chain rule) of variational derivatives:

$$\begin{matrix} u = u(v, v_1, \ldots, v_q) \\ v_i = D^i v \end{matrix} \Rightarrow \frac{\delta}{\delta v} = V^*(u(v))\frac{\delta}{\delta u}. \tag{2.6}$$

The transformation formula (2.6) plays a key role in the recurrent method [7–10] which we shall follow in Sections 4 and 5.

Note that the two brackets $\{,\}$ and $[,]$ introduced above are related by the following important formula (see, for example, [11–14]):

$$\left[\frac{J\delta F}{\delta u}, \frac{J\delta G}{\delta u}\right] = \frac{J\delta}{\delta u}\{F, G\},$$

from which we deduce the following Noether type correspondence between conserved densities and symmetries of generalized Hamiltonian equations in the case that the operator J is invertible [15–17]:

G is a conserved density of Equation (1.7a) $\Leftrightarrow \eta = \dfrac{J\delta G}{\delta u}$ is a symmetry of Equation (1.7a) (2.7).

3. NEW CONSERVATION LAWS OF O(2,1) NON-LINEAR SIGMA MODELS

The O(2,1) non-linear sigma model [18,19] is defined by the Lagrangian $L = \frac{1}{2} w_u w_v$, subjected to the constraint $w^2 \equiv (w^1)^2 + (w^2)^2 - (w^3)^2 = -1$, where $w = (w^1, w^2, w^3)$ and the subscripts u and v refer to the differentiation. From the Euler–Lagrange equation and the constraint, it follows that

$$w_{uv} - (w_u \cdot w_v)w = 0. \tag{3.1}$$

By taking the parametrization [18]

$$w=\frac{1}{r}(\mathrm{i}(z-\bar{z}),\ 1-z\bar{z},\ 1+z\bar{z}),$$

where $r=z+\bar{z}$, the Lagrangian and Equation (3.1) reduce, respectively, to

$$L=\frac{(z_u\bar{z}_v+\bar{z}_u z_v)}{r^2}$$
$$z_{uv}=\frac{2z_u z_v}{r}. \tag{3.2}$$

In [18] it is shown that this equation exhibits an infinite number of conservation laws,

$$\partial_v \chi_0=0, \tag{3.3a}$$

$$\partial_v \chi_1+\partial_u(\mathrm{i}ba^{-1}|a|)=0 \quad (\mathrm{i}=\sqrt{-1}), \tag{3.3b}$$

$$\partial_v \chi_2+\partial_u[\tfrac{1}{2}ba^{-1}(2\chi_0-(a-\bar{a})+a^{-1}a_u)]=0, \tag{3.3c}$$

$$\partial_v \chi_{k+3}+\partial_u(ba^{-1}\chi_{k+1})=0, \quad k=0,1,\ldots, \tag{3.3d}$$

where

$$a=\frac{z_u}{r}, \quad b=\frac{z_v}{r},$$

and χ_r can be calculated in the following recurrent way:

$$\chi_0=-\tfrac{1}{2}|a|^{-1}|a|_u, \quad \chi_1=-\tfrac{1}{2}\mathrm{i}|a|^{-1}(l-\chi_{0u}-\chi_0^2),$$

$$\chi_{k+2}=\tfrac{1}{2}\mathrm{i}|a|^{-1}\Big(\chi_{k+1,u}+\sum_{s=0}^{k+1}\chi_{k+1-s}\chi_s\Big), \quad k=0,1,\ldots,$$

with

$$l=\tfrac{1}{2}[(a-\bar{a})-a^{-1}a_u]_u+\tfrac{1}{2}[(a-\bar{a})-a^{-1}a_u]^2.$$

The calculation of χ_r, even the direct verification of Equation (3.3b), is rather tedious. We show that there are some simple conservation laws which do not involve the modulus $|a|$ and $|b|$:

$$\left[\frac{(\bar{z}_u-z_u)}{r^2}\right]_v+\left[\frac{(\bar{z}_v-z_v)}{r^2}\right]_u=0, \tag{3.4a}$$

$$\left[\frac{(z\bar{z}_u+\bar{z}z_u)}{r^2}\right]_v+\left[\frac{(z\bar{z}_v+\bar{z}z_v)}{r^2}\right]_u=0, \tag{3.4b}$$

$$\left[\frac{(z^2\bar{z}_u-\bar{z}^2z_u)}{r^2}\right]_v+\left[\frac{(z^2\bar{z}_v-\bar{z}^2z_v)}{r^2}\right]_u=0. \tag{3.4c}$$

Although the direct verification of (3.4a)–(3.4c) by using (3.2) is a simple matter, we consider it worth giving the derivation of these conservation laws. To do this, we first make the substitution of variables u and v:

$$x=u+v, \quad t=u-v$$

which reduces the Lagrangian and Equation (3.2), respectively, to

$$L=\frac{(z_x\bar{z}_x-z_t\bar{z}_t)}{r^2}, \tag{3.5}$$

and

$$z_{xx}-z_{tt}=\frac{2(z_x^2-z_t^2)}{r}. \tag{3.6}$$

Following the usual way we introduce the momentum

$$p=\frac{\partial L}{\partial z_t}=\frac{-\bar{z}_t}{r^2} \tag{3.7}$$

and the Hamiltonian

$$H=z_t\frac{\partial L}{\partial z_t}+\bar{z}_t\frac{\partial L}{\partial \bar{z}_t}-L=\frac{-(z_t\bar{z}_t+z_x\bar{z}_x)}{r^2}.$$

With the new variables, z and p Equation (3.6) can be written as

$$\begin{bmatrix} z_t \\ p_t \end{bmatrix}=\begin{bmatrix} -2r^2\bar{p} \\ \dfrac{2rp\bar{p}-\bar{z}_{xx}}{r^2}+\dfrac{2\bar{z}_x^2}{r^3} \end{bmatrix}\equiv F \tag{3.8}$$

or equivalently written in the following Hamiltonian form:

$$\begin{bmatrix} z \\ p \end{bmatrix}_t=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dfrac{\delta H}{\delta z} \\ \dfrac{\delta H}{\delta p} \end{bmatrix}. \tag{3.9}$$

According to the definition (2.5), a symmetry $f=f(z)$ of Equation (3.2) satisfies $(\mathrm{d}/\mathrm{d}\varepsilon)G(z+\varepsilon f)|_{\varepsilon=0}=0$, where $G(z)=(z+\bar{z})z_{uv}-2z_u z_v$, or equivalently

$$(z+\bar{z})f_{uv}+(f+\bar{f})\frac{2z_u z_v}{r}=2f_u z_v+2f_v z_u \tag{3.10}$$

and a vector $\eta=(\eta^1,\ \eta^2)$ would be a symmetry of Equation (3.8) if

$$V(\eta)F-V(F)\eta=0.$$

From Equation (3.7) we see that there exists a one-to-one correspondence between symmetries f and symmetries η:

$$f \Leftrightarrow \eta=\begin{bmatrix} \eta^1 \\ \eta^2 \end{bmatrix}$$

with

$$\eta^1=f,\quad \eta^2=\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right)p(z+\varepsilon f)_{\varepsilon=0}=\frac{-\bar{f}_t}{r^2}+2\bar{z}_t\frac{(f+\bar{f})}{r^3}. \quad (3.11)$$

Now we proceed to search for the symmetries f of Equation (3.2). We consider only the symmetries of the form $f=f(z,\bar{z})$ and suppose that f is smooth with respect to its variables z and $\bar{z}$. Substituting $f=f(z,\bar{z})$ into (3.10) and comparing the coefficients of $\bar{z}_u\bar{z}_v$, $z_u z_v$ and $\bar{z}_v z_u+z_v\bar{z}_u$ on both sides of the resulting equation, we find that $f_{\bar{z}}=0$, that is, f is analytic in z, and

$$f_{zz}(\bar{z}+z)^2-2(z+\bar{z})f_z+2(f+\bar{f})=0.$$

Suppose that $f=\Sigma a_k z^k$: we then have for $k\geq 3$ that $k(k-1)a_k-2ka_k+2a_k=0$ or $(k-1)(k-2)a_k=0$, from which we deduce that $a_k=0$ for $k\geq 3$. Furthermore, it is easy to verify that the expression

$$f=\mathrm{i}\alpha_0+\alpha_1 z+\mathrm{i}\alpha_2 z^2,\quad \mathrm{i}=\sqrt{-1}, \qquad (3.12)$$

where α_k are arbitrary real constants, satisfies the above equation for f and thus gives a symmetry of (3.2). From (3.11) we see that the corresponding symmetry of (3.8) reads

$$\eta^1=\mathrm{i}\alpha_0+\alpha_1 z+\mathrm{i}\alpha_2 z^2,\quad \eta^2=-p(\alpha_1+2\mathrm{i}\alpha_2 z).$$

It is easy to see that

$$\begin{bmatrix}\mathrm{i}\alpha_0+\alpha_1 z+\mathrm{i}\alpha_2 z^2\\ \\ -p(\alpha_1+2\mathrm{i}\alpha_2 z)\end{bmatrix}=\begin{bmatrix}0 & 1\\ \\ -1 & 0\end{bmatrix}\begin{bmatrix}\dfrac{\delta H}{\delta z}\\ \\ \dfrac{\delta H}{\delta p}\end{bmatrix},$$

where $H=h+\bar{h}$ and $h=(\mathrm{i}\alpha_0+\alpha_1 z+\mathrm{i}\alpha_2 z^2)p$. According to the relation (2.7) between symmetries and conserved densities, we obtain the following three conserved densities:

$$H_1=p-\bar{p},\quad H_2=zp+\bar{z}\bar{p},\quad H_3=z^2p-\bar{z}^2\bar{p}.$$

In fact, one can easily verify that

$$H_{1t}=\left[\frac{(z_x-\bar{z}_x)}{r^2}\right]_x,$$

$$H_{2t}=\left[\frac{-(z\bar{z}_x+\bar{z}z_x)}{r^2}\right]_x,$$

$$H_{3t}=\left[\frac{(\bar{z}^2z_x-z^2\bar{z}_x)}{r^2}\right]_x.$$

Noting that $H_t=G_x \Leftrightarrow (H-G)_u=(H+G)_v$ we are finally led to the conservation laws (3.4a)–(3.4c).

If we continue to search for the symmetry with the form $f=f(z,\bar{z},z_u,\bar{z}_u,z_v,\bar{z}_v)$, then the similar procedure leads to the symmetry

$$\eta^1=f=\tfrac{1}{2}(z_u+z_v)=z_x$$

and the corresponding $\eta^2=p_x$. Since

$$\begin{bmatrix}z_x\\ \\ p_x\end{bmatrix}=\begin{bmatrix}\dfrac{\delta H}{\delta p}\\ \\ \dfrac{-\delta H}{\delta z}\end{bmatrix},\quad \begin{bmatrix}\bar{z}_x\\ \\ \bar{p}_x\end{bmatrix}=\begin{bmatrix}\dfrac{\delta H}{\delta \bar{p}}\\ \\ \dfrac{-\delta H}{\delta \bar{z}}\end{bmatrix},$$

where $H=pz_x+\bar{p}\bar{z}_x$, we accordingly obtain the following conservation laws:

$$(pz_x+\bar{p}\bar{z}_x)_t=-\left(\frac{(z_x\bar{z}_x)}{r^2}+r^2p\bar{p}\right)_x,$$

which is equivalent to

$$\left(\frac{(z_v\bar{z}_v)}{r^2}\right)_u+\left(\frac{(z_u\bar{z}_u)}{r^2}\right)_v=0.$$

This conserved density $(z_u\bar{z}_u)/r^2$ is just $|a|^2$ as mentioned in Equation (3.3a). One may naturally expect to find more general symmetries with the form $f=f(z,\bar{z},z_u,\bar{z}_u,z_v,\bar{z}_v,z_{uu},\bar{z}_{uu},z_{vv},\bar{z}_{vv})$. Note that since $z_{uv}=2z_uz_v/r$, the two terms z_{uv} and $\bar{z}_{uv}$ can be excluded from the above expression. Unfortunately, if we suppose that f is smooth with respect to its variables, it seems that no such symmetry exists. And it seems also that the infinite number of conservation laws found by Chinea and Guil [18] should be related to some kinds of symmetry which are not smooth with respect to their variables. Any further results in this respect would be very helpful.

As another remark, we examine the algebra spanned by the four symmetries of (3.8):

$$\eta_1=\begin{pmatrix}\mathrm{i}\\0\end{pmatrix},\quad \eta_2=\begin{pmatrix}z\\-p\end{pmatrix},\quad \eta_3=\begin{pmatrix}\mathrm{i}z^2\\-2\mathrm{i}zp\end{pmatrix},\quad \eta_4=\begin{pmatrix}z_x\\p_x\end{pmatrix}.$$

An easy calculation shows that

$$[\eta_i,\eta_4]=0\quad (i=1,2,3)$$

and

$$[\eta_1,\eta_2]=-\eta_1,\quad [\eta_1,\eta_3]=2\eta_2,\quad [\eta_2,\eta_3]=-\eta_3.$$

For example,

$$[\eta_2,\eta_3]=V(\eta_2)\eta_3-V(\eta_3)\eta_2$$

$$=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\mathrm{i}z^2\\-2\mathrm{i}zp\end{pmatrix}-\begin{pmatrix}2\mathrm{i}z & 0\\-2\mathrm{i}p & -2\mathrm{i}z\end{pmatrix}\begin{pmatrix}z\\-p\end{pmatrix}=$$

$$=\begin{pmatrix} iz^2 \\ 2izp \end{pmatrix}-\begin{pmatrix} 2iz^2 \\ 0 \end{pmatrix}=\begin{pmatrix} -iz^2 \\ 2izp \end{pmatrix}=-\eta_3.$$

If we set

$$\zeta_1=\frac{(\eta_3+\eta_1)}{2},\quad \zeta_2=\frac{(\eta_3-\eta_1)}{2},\quad \zeta_3=\eta_2,$$

then

$$[\zeta_1,\zeta_3]=\zeta_2,\quad [\zeta_1,\zeta_2]=\zeta_3,\quad [\zeta_2,\zeta_3]=\zeta_1.$$

Comparing these relations with the Lie algebra so(2,1) whose basis matrices (see, for example, [22], p. 265) are

$$x_1=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ x_2=\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ x_3=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

with

$$[x_1,x_3]=x_2,\quad [x_1,x_2]=x_3,\quad [x_2,x_3]=x_1,$$

we see that the subalgebra spanned by η_1, η_2, and η_3 is isomorphic to so(2, 1).

4. HAMILTONIAN STRUCTURES OF N × N AKNS EQUATIONS

Consider the spectral problem

$$\psi_x=U\psi,\quad U=\lambda A+P \tag{4.1}$$

with

$$\psi=(\psi_1,\ldots,\psi_N)$$

$$A=\text{diag }(a_1,\ldots,a_N),\quad a_i\neq a_j,\quad a_i=\text{const.}$$

and

$$P=(P_{ij}),\quad P_{ii}=0.$$

Choosing $V=\Sigma\lambda^{n-j}V_j$, $V_j=V_j(P)$ and substituting U and V into the integrability condition (1.4), we can solve for V_j and obtain the following hierarchy of equations [20, 21]:

$$P_t=JL^n[C,P^{\mathrm{T}}]_A, \tag{4.2}$$

where

$$C=\text{diag }(c_1,c_2,\ldots,c_N),\quad c_i=\text{const.}$$

$$JQ=[A,Q^{\mathrm{T}}],\ LQ=(-Q_x-[P^{\mathrm{T}},Q]_F+[P^{\mathrm{T}},I[P^{\mathrm{T}},Q]_D])_A,$$

where $S_D=\text{diag }(s_{11},s_{22},\ldots,s_{NN})$, $S_F=S-S_D$, and for a matrix Q_F the matrix Q_{FA} is defined by the equation $[A,Q_{FA}]=Q_F$.

We shall prove that there exists a series $\{H_n\}$ such that

$$L^n[C,P^{\mathrm{T}}]_A=\frac{\delta H_n}{\delta P}. \tag{4.3}$$

To this end we introduce first

$$Y^{(i,j)}\equiv\frac{\psi_i}{\psi_j} \tag{4.4}$$

and set

$$H^{(i)}=\sum_{j=1}^{N}P_{ij}Y^{(j,i)}, \tag{4.5}$$

then expand $H^{(i)}$ into a series of λ^{-1}:

$$H^{(i)}=\sum_{n=1}^{\infty}\lambda^{-n}H_n^{(i)}. \tag{4.6}$$

We show that (4.3) holds with

$$H_n\equiv\sum_{i=1}^{N}c_iH_n^{(i)}. \tag{4.7}$$

It is easy to see that Equations (4.3), (4.5), and (4.6) are equivalent to a single equation

$$(L-\lambda)\frac{\delta H^{(i)}}{\delta P}=[P^{\mathrm{T}},E_i]_A, \tag{4.8}$$

where $E_i=(\delta_{ik}\delta_{il})$. Since the discussions for different i are the same we shall, for simplicity of notation, prove (4.8) in the case $i=1$, that is,

$$(L-\lambda)\frac{\delta H}{\delta P}=[P^{\mathrm{T}},E_1]_A,$$

$$H\equiv H^{(1)}=\sum_{j=1}^{N}P_{1j}y_j,\quad y_j\equiv Y^{(j,1)}.$$

Note first that from (4.1) and (4.4) we have

$$y_{jx}=P_{j1}+\sum_{k=2}^{N}(P_{jk}y_k-P_{1k}y_jy_k)+(a_j-a_1)\lambda y_j. \tag{4.9}$$

In order to calculate the variational derivative $\delta H/\delta P$ we make the transformation

$$P=(P_{ij})\rightarrow\bar{P}=(\bar{P}_{ij})$$

with

$$\bar{P}_{12}=H=\sum_{j=2}^{N}P_{1j}y_j,$$

$$\bar{P}_{j1}=y_j,\quad j=2,\ldots,N,$$

$$\bar{P}_{ij}=P_{ij}y_j\quad (i=1,j\geq 3\text{ or }i,j\geq 2).$$

We can also solve for P_{ij} in terms of $\bar{P}_{ij}$ as follows:

$$P_{12}=\frac{\left(H-\sum_{i=3}^{N}\bar{P}_{1i}\right)}{y_2},$$

$$P_{j1}=y_{jx}+Hy_j-\sum_{k=2}^{N}\bar{P}_{jk}+(a_1-a_j)\lambda y_j,$$

$$P_{ij}=\frac{\bar{P}_{ij}}{y_j},\quad i=1,\quad j\geq 3 \text{ or } i,j\geq 2,$$

from which we obtain

$$V(P_{12})=\frac{1}{y_2},\quad V_{\bar{P}_{1i}}(P_{12})=-\frac{1}{y_2},$$

$$V_{y_i}(P_{12})=-\left(\frac{P_{12}}{y_2}\right)\delta_{i2},\quad V_{\bar{P}_{ij}}(P_{12})=0\quad (i,j\geq 2);$$

$$V_H(P_{j1})=y_j,\quad V_{\bar{P}_{1i}}(P_{j1})=0,$$

$$V_{y_i}(P_{j1})=\delta_{ij}(D+H+(a_1-a_j)\lambda),$$

$$V_{\bar{P}_{kl}}(P_{j1})=-\delta_{jk}\quad (k,l\geq 2);$$

$$V_H(P_{1j})=0,\quad V_{\bar{P}_{1k}}(P_{1j})=\frac{\delta_{jk}}{y_j},$$

$$V_{y_k}(P_{1j})=\delta_{kj}\left(\frac{-P_{1j}}{y_j}\right),\quad V_{\bar{P}_{kl}}(P_{1j})=0\quad (k,l>2);$$

$$V_H(P_{ij})=0,\quad V_{y_k}(P_{ij})=\delta_{jk}\left(\frac{-P_{ij}}{y_j}\right),$$

$$V_{\bar{P}_{1j}}(P_{ij})=0,\quad V_{\bar{P}_{kl}}(P_{ij})=\frac{\delta_{ik}\delta_{jl}}{y_j}\quad (k,l\geq 2).$$

By the chain rule (2.6) we have

$$\begin{aligned}\frac{\delta}{\delta H}&=V_H^*(P_{12})\frac{\delta}{\delta P_{12}}+\sum_{i=2}^{N}V_H^*(P_{i1})\frac{\delta}{\delta P_{i1}}\\&+\sum_{j=3}^{N}V_H^*(P_{1j})\frac{\delta}{\delta P_{1j}}\\&+\sum_{i,j=2}^{N}V_H^*(P_{ij})\frac{\delta}{\delta P_{ij}},\end{aligned}$$

and similar expressions for $\delta/\delta Y_k$, $\delta/\delta P_{ij}$, and $\delta/\delta\bar{P}_{ij}(i,j\geq 2)$. Setting $K_{ij}=\delta H/\delta P_{ij}$ and applying both sides of the above operator identities to H, we then find

$$K_{1l}=y_l\left(1-\sum_{i=2}^{N}y_iK_{i1}\right),$$

$$K_{lk}=y_kK_{l1},\quad l,k\geq 2,\quad l\neq k,$$

and

$$\begin{aligned}K_{l1x}&=(a_1-a_l)\lambda K_{l1}-P_{1l}+\sum_{i=2}^{N}(P_{1i}K_{li}-P_{il}K_{il})\\&+P_{1l}\left(\sum_{i=2}^{N}y_iK_{il}+y_lK_{l1}\right),\end{aligned}\tag{4.10}$$

from which we obtain

$$\begin{aligned}(K_{lk})_x&=(a_k-a_l)\lambda K_{lk}+P_{kl}(y_lK_{l1}-y_kK_{k1})\\&+\sum_{j=1}^{N}(P_{kj}K_{lj}-P_{jl}K_{jk})\quad (l,k\geq 2;\ l\neq k)\end{aligned}$$

and

$$\begin{aligned}(K_{1l})_x&=(a_l-a_1)\lambda K_{1l}+\sum_{i=2}^{N}(P_{li}K_{1i}-K_{il}P_{i1})+P_{l1}\\&-P_{l1}\sum_{i=2}^{N}(1+\delta_{il})y_iK_{i1}.\end{aligned}$$

It is easy to see that the above three equations are equivalent to the following single matrix equation:

$$(K^{\mathrm{T}})_x=\lambda[A,K^{\mathrm{T}}]-[K^{\mathrm{T}},P]_F+[B,P]-[E_1,P],\tag{4.11}$$

where

$$B=\operatorname{diag}\left(\sum_{i=2}^{N}y_iK_{i1},\ -y_2K_{21},\ -y_3K_{31},\ldots,\ -y_NK_{N1}\right).$$

Using (4.9) and (4.10), we can easily verify that

$$\begin{aligned}(y_lK_{l1})_x&=\sum_{k=1}^{N}(p_{lk}K_{lk}-p_{kl}K_{kl})\\&=([P,K^{\mathrm{T}}]_D)_u,\quad (1\geq 2).\end{aligned}\tag{4.12}$$

Since tr $[B,A]=0$ for any pair of matrices A and B, we see that

$$([K^{\mathrm{T}},P]_D)_{11}=-\sum_{l=2}^{N}([K^{\mathrm{T}},P]_D)_u=\sum_{l=2}^{N}(y_lK_{l1})_x,\tag{4.13}$$

which together with (4.10) imply that

$$B_x=[K^{\mathrm{T}},P]_D.\tag{4.14}$$

From (4.9), (4.10), and the definition of L we finally obtain

$$(L-\lambda)K=[P^{\mathrm{T}},E_1]_A$$

as desired.

5. A NEW HIERARCHY OF HAMILTONIAN EQUATIONS†

Consider the spectral problem $\psi_x=U\psi$ with

$$U=\begin{bmatrix}-\lambda-\lambda^{-1}\varepsilon v & u-v\lambda^{-1}\\ u+v\lambda^{-1} & \lambda+\lambda^{-1}\varepsilon v\end{bmatrix},\quad \varepsilon=\pm 1,\tag{5.1}$$

†A short summary of the results in this section has been announced without proof in [23].

which is a reduction of the spectral problem discussed in [8]. To derive the corresponding hierarchy of equations we introduce the auxiliary problem

$$\psi_t = V\psi, \quad V = \sum_{j=0}^{n} V_j \lambda^{n-j}, \quad n = 2m+1,$$

$$V_j = \begin{bmatrix} d_j & (e_j+f_j)/2 \\ (e_j-f_j)/2 & -d_j \end{bmatrix}. \tag{5.2}$$

Substituting U and V into the integrability condition (1.4) and comparing the coefficients of λ^j on both sides of the resulting equation, we obtain

$$\begin{aligned} &v_t \varepsilon \delta_{jn+1} + d_{jx} + uf_j + ve_{j-1} = 0, \\ &2u_t \delta_{jn} - e_{jx} - 2f_{j+1} + 4d_{j-1}v - 2\varepsilon vf_{j-1} = 0, \\ &2v_t \delta_{jn+1} + f_{jx} + 4ud_j + 2e_{j+1} + 2\varepsilon ve_{j-1} = 0, \end{aligned} \tag{5.3}$$

from which we can calculate successively e_j, f_j as follows:

$$\begin{aligned} &e_0 = f_0 = 0; \\ &e_1 = 2u, \quad f_1 = 0; \quad e_2 = 0, \quad f_2 = -(u_x + 2v); \\ &e_{j+1} = -\tfrac{1}{2}f_{jx} + 2uI(uf_j + ve_{j-1}) \\ &\qquad -\varepsilon ve_{j-1}, \left(I \equiv \tfrac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right)dx'\right), \\ &f_{j+1} = -(\tfrac{1}{2}e_{jx} + 2vI(uf_{j-1} + ve_{j-2}) + \varepsilon vf_{j-1}); \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} u_t &= -f_{n+1}, \\ v_t &= -\varepsilon ve_n. \end{aligned} \tag{5.5}$$

From (5.4) we deduce that

$$\begin{bmatrix} -f_{n+1} \\ -\varepsilon ve_n \end{bmatrix} = L^* \begin{bmatrix} -f_{n-1} \\ -\varepsilon ve_{n-2} \end{bmatrix},$$

where

$$L^* = \begin{bmatrix} \tfrac{1}{4}D^2 - (u^2 + \varepsilon v) - (u_x + 2v)Iu, & \tfrac{1}{2}D - \varepsilon u - \varepsilon(u_x + 2v)I \\ -\tfrac{1}{2}\varepsilon vD + 2\varepsilon uvIu, & 2uvI - \varepsilon v \end{bmatrix}.$$

Therefore the corresponding hierarchy of equations reads

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} -f_{2m+2} \\ -\varepsilon ve_{2m+1} \end{bmatrix} = L^{*m} \begin{bmatrix} -f_2 \\ -\varepsilon ve_1 \end{bmatrix}$$

or

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = L^{*m} \begin{bmatrix} u_x + 2v \\ -2\varepsilon uv \end{bmatrix}. \tag{5.6}$$

In particular, setting $m=0$ we obtain the first couple of equations in this hierarchy

$$u_t = u_x + 2v, \quad v_t = \pm 2uv,$$

which can be reduced to a single equation

$$w_{tt} - w_{tx} \pm 4\exp w = 0.\dagger$$

Since this hierarchy of equations is derived from the isospectral problems (5.1) and (5.2), we can calculate the common conserved densities via the standard procedure [20]. To this end, we set $Z = \psi_2/\psi_1$ then from (5.1) we have

$$Z_x = (u + v\lambda^{-1}) + (2\lambda + 2\lambda^{-1}\varepsilon v)Z - (u - v\lambda^{-1})Z^2. \tag{5.7}$$

Expanding Z into a series of λ^{-1}: $Z = \Sigma_{m=1}^{\infty} Z_n \lambda^{-n}$ and then comparing the coefficients of λ^{-n} we obtain

$$Z_1 = -\frac{u}{2}, \quad Z_2 = -(\tfrac{1}{4})(u_x + 2v),$$

$$Z_3 = -\frac{(u_{xx} + 2v_x - 4\varepsilon uv - u^3)}{8},$$

$$Z_{n+1} = \frac{(Z_{nx} - 2\varepsilon vZ_{n-1} + u\sum_{j+k=n} Z_k Z_j - v\sum_{j+k=n-1} Z_k Z_j)}{2},$$

$$n \geq 3.$$

The generating function H of the conserved densities $\{H_n\}$ is

$$H = \sum_{n=-1}^{\infty} H_n \lambda^{-n} = (-\lambda - \lambda^{-1}\varepsilon v) + (u - v\lambda^{-1})Z, \tag{5.8a}$$

which gives

$$H_{-1} = -1, \quad H_0 = 0, \quad H_1 = -\left(\varepsilon v + \frac{u^2}{2}\right),$$

$$H_2 = \frac{-(u^2)_x}{8}, \; H_n = uZ_n - vZ_{n-1} \; (n \geq 2). \tag{5.8b}$$

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†At this point we would like to quote the following comment from one of the reviewers of this paper: 'The equation $w_{tt} - w_{tx} + 4\exp w = 0$ can be transformed into the Liouville equation $W_{\xi\eta} = k\exp w$ (under the change of coordinates $\xi = t + x, \eta = x$) whose exact solutions, conservation laws, and Bäcklund transformations have been known explicitly since the last century for arbitrary constant k. So perhaps there are more underlying points in this paper than investigated by the author. Especially in this way it may be possible to embed the Liouville equation into a hierarchy of completely integrable Hamiltonian systems. This would be a significant contribution.' We are very grateful to the reviewer for this valuable comment.

Note that only H_{2k-1} give non-trivial conserved densities for the whole hierarchy of equations. The variational derivatives of the first two non-trivial conserved densities are

$$\frac{\delta}{\delta q}H_1=\begin{pmatrix}-u\\-\varepsilon\end{pmatrix},\quad \left(q=(u,v)^T,\ \frac{\delta}{\delta q}\equiv\left(\frac{\delta}{\delta u},\frac{\delta}{\delta v}\right)^{\mathrm{T}}\right),$$

$$\frac{\delta}{\delta q}H_3=\begin{bmatrix}\dfrac{-u_{xx}}{4}+\dfrac{u^3}{2}+\varepsilon uv-\dfrac{v_x}{2}\\[2mm] \dfrac{u_x}{2}+\dfrac{\varepsilon u^2}{2}+v\end{bmatrix}.$$

It is easy to verify that

$$K\begin{bmatrix}u_x+2v\\ \\ -2\varepsilon v\end{bmatrix}=\begin{bmatrix}\dfrac{\delta H_3}{\delta u}\\[2mm] \dfrac{\delta H_3}{\delta v}\end{bmatrix},\tag{5.9}$$

where

$$K=\begin{bmatrix}uIu-\dfrac{D}{4} & \varepsilon uI-\dfrac{1}{2}\\[2mm] \varepsilon Iu+\dfrac{1}{2} & I\end{bmatrix}.$$

Note that

$$K^*=-K,\quad KL^*=LK.\tag{5.10}$$

It should be noted that the operator K is singular; in fact,

$$K\begin{bmatrix}f\\ \\ -\dfrac{f_x}{2}-\varepsilon uf\end{bmatrix}=0$$

holds for any f.

Now we proceed to prove that

$$L\left(\frac{\delta H_{2k-1}}{\delta q}\right)=\frac{\delta H_{2k+1}}{\delta q},\quad \frac{\delta H_{2k}}{\delta q}=0,\tag{5.11}$$

or equivalently

$$(L-\lambda^2)\frac{\delta H}{\delta q}=\lambda\begin{pmatrix}u\\ \varepsilon\end{pmatrix}.\tag{5.12}$$

To calculate $\delta H/\delta q$, we deduce first from (5.7) and (5.8a) that

$$1=-\lambda^{-1}\varepsilon V_H(v)+Z(V_H(u)-\lambda^{-1}V_H(v)),$$
$$0=V_H(u)+V_H(v)\lambda^{-1}+\lambda^{-1}\varepsilon ZV_H(v)-Z,$$

from which it results that

$$V_H(u)=\varepsilon,\tag{5.13a}$$

$$V_H(v)=-\frac{\lambda\varepsilon(1-\varepsilon Z)}{(1+\varepsilon Z)}.\tag{5.13b}$$

In the same manner we deduce that

$$V_Z(u)=\frac{1}{(1+\varepsilon Z)}D+\frac{(H-\lambda-\varepsilon u)}{(1+\varepsilon Z)},\tag{5.13c}$$

$$V_Z(\mathrm{v})=\frac{\varepsilon\lambda Z}{(1+\varepsilon Z)^2}D+\frac{\varepsilon\lambda}{(1+\varepsilon Z)^2}(HZ-\lambda Z$$
$$-\lambda^{-1}v(1+\varepsilon Z)+u).\tag{5.13d}$$

Now by the chain rule (2.6) we have

$$\begin{bmatrix}\dfrac{\delta}{\delta H}\\[2mm] \dfrac{\delta}{\delta z}\end{bmatrix}=\begin{bmatrix}V_H^*(u) & V_H^*(v)\\ \\ V_Z^*(u) & V_Z^*(v)\end{bmatrix}\begin{bmatrix}\dfrac{\delta}{\delta u}\\[2mm] \dfrac{\delta}{\delta v}\end{bmatrix}.$$

Applying the operators on both sides of the equation to H, we find

$$1=V_H^*(u)R+V_H^*(v)S,$$
$$0=V_Z^*(u)R+V_Z^*(v)S,$$

where $R=\delta H/\delta u$, $S=\delta H/\delta v$. Using Equations (5.13a)–(5.13d), we then obtain

$$R=\varepsilon+YS,\tag{5.14a}$$

$$S_x=2\varepsilon uS-2R,\tag{5.14b}$$

where

$$Y=\frac{\lambda(1-\varepsilon Z)}{(1+\varepsilon Z)}.$$

Since Z satisfies Equation (5.7) we see that

$$Y_x=-(\lambda^2+2v\varepsilon)-2\varepsilon vY+Y^2.\tag{5.14c}$$

The above three equations, (5.14a)–(5.14c), can be used to calculate R_x, R_{xx} in terms of R, S, and Y:

$$R_x=-(2\varepsilon Y+(\lambda^2+2\varepsilon v)S+SY^2)$$
$$R_{xx}=(8v+4\varepsilon\lambda^2)+4uY+S((-2\varepsilon v_x-2\varepsilon\lambda^2u-4uv)$$
$$+(4\lambda^2+8\varepsilon v)Y+(2\varepsilon u)Y^2).$$

Now it is straightforward to verify that

$$(\varepsilon u_x + 2v\varepsilon)R - 2uvS = \left(\frac{R_x}{2} + \varepsilon uR + \varepsilon vS + \lambda^2 S\right)_x \qquad (5.15\text{a})$$

$$\frac{R_{xx}}{4} - (\varepsilon v + \lambda^2)R + \frac{\varepsilon u R_x}{2} + \left(uv + \varepsilon\lambda^2 u + \frac{\varepsilon Dv}{2}\right)S = 0. \qquad (5.15\text{b})$$

In the following, we suppose that u, v along with all their x-derivatives tend to zero when x goes to $\pm\infty$. Assuming this, we have from (5.8b) that

$$S = \frac{\delta H}{\delta v} \to \frac{\delta H_1}{\delta v}\lambda^{-1} = -\varepsilon\lambda^{-1},$$

$$R \to 0 \quad (x \to \pm\infty).$$

Thus the integration of (5.15a) gives

$$I((\varepsilon u_x + 2v\varepsilon)R - 2uvS) = \lambda\varepsilon + \frac{R_x}{2} + \varepsilon uR + \varepsilon vS + \lambda^2 S, \qquad (5.16\text{a})$$

which together with (5.15b) yields

$$\frac{R_{xx}}{4} - (u^2 + \varepsilon v)R - \lambda^2 R + uI((u_x + 2v)R - 2uv\varepsilon S) = \lambda u. \qquad (5.16\text{b})$$

The combination of (5.16a) with (5.16b) gives the desired equation, (5.12). The proof of Equation (5.11) is thus completed. From (5.11), (5.9), and (5.10), we see that

$$K\begin{pmatrix} u \\ v \end{pmatrix}_t = KL^{*m}\begin{pmatrix} u_x + 2v \\ -2\varepsilon uv \end{pmatrix} = L^m K\begin{pmatrix} u_x + 2v \\ -2\varepsilon uv \end{pmatrix}$$

$$= L^m \frac{\delta H_3}{\delta q} = \frac{\delta H_{2m+3}}{\delta q}, \qquad (5.17)$$

which shows that all equations in this new hierarchy take the form of generalized Hamiltonian equations (1.5b).

As a final remark, let us introduce the Poisson bracket $\{F, H\}$ when $\delta F/\delta q$, $\delta H/\delta q$ lie in the image of K, $\delta F/\delta q = Kf$, $\delta H/\delta q = Kh$:

$$\{F, H\} = (Kf)^{\mathrm{T}} h.$$

Set $\hat{H}_m = H_{2m}$; then from (5.17)

$$\frac{\delta \hat{H}_{m+1}}{\delta q} = KS_m \quad \left(S_m \equiv L^{*m} f, \quad f = \begin{pmatrix} u_x + 2v \\ -2\varepsilon uv \end{pmatrix}\right),$$

and accordingly

$$\{\hat{H}_{m+1}, \hat{H}_{n+1}\} = KS_m \cdot S_n = KL^{*m} f \cdot L^{*n} f$$
$$= LKL^{*m-1} f \cdot L^{*n} f \simeq KL^{*m-1} f \cdot L^{*n+1} f$$
$$= KS_{m-1} \cdot S_{n+1} = \{\hat{H}_m, \hat{H}_{n+2}\},$$

from which and the fact that $\{F,H\} \simeq -\{H,F\}$ we deduce that

$$\{\hat{H}_m, \hat{H}_n\} \simeq 0.$$

It is not clear at present whether this new hierarchy of equations shares some other common properties possessed by many other completely integrable equations, such as exhibiting Bäcklund transformations and soliton solutions. We hope that these problems could be solved in the near future by the interested reader.

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