ON THE SOUND FIELD DUE TO A MOVING SOURCE IN A GAS-FILLED GENERAL TWISTED TUBE WITH A SLOWLY VARYING CIRCULAR SECTION*

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الخلاصة :

لقد استعملنا النظام الاحداثي المنحني الخطي غير المتعامد لتكوين مسألة قيمة الحدود المرتبطه بحقل الصوت الناتج عن حركة نقطة كروية في أنبوب ملتوى ذو قاطع دائرى بطيء التغيير مملؤ بالغاز . وقد طورنا حلا لحالتين من الهندسة الانبوبية .

ABSTRACT

A non-orthogonal curvilinear coordinate system is used to formulate the boundary-value problem associated with the sound field induced by the motion of a monopole point source in a gas-filled general twisted tube with a slowly varying circular section. A solution scheme is presented for two cases of tube geometry.

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1. INTRODUCTION

The sound field induced by a monopole point source moving in an infinite inviscid compressible fluid is well known [1] in the literature of linear acoustics. Problems associated with sound sources moving in the vicinity of solid boundaries have also been investigated. However, no formulation to date includes the effects of curvature, torsion, and section variation on the motion induced by a sound source in a gas-filled general twisted tube.

It is the purpose of this paper to use the non-orthogonal curvilinear coordinate system [2] to formulate the boundary-value problem associated with the sound field induced by a monopole point source moving in an infinitely long general twisted tube with slowly varying circular section which is filled with an inviscid compressible fluid. A solution scheme is presented which involves expansions in terms of two and three small parameters for the two cases of tube geometry considered.

2. FORMULATION OF THE PROBLEM

We denote the interior and boundary of an infinite tube in R_3 by D_3 and ∂D_3, respectively. The tube orientation is specified by a curve L (Figure 1) which has a prescribed unit tangent vector $\mathbf{t}_1(\xi^1)$ where ξ^1 measures arc length along L from the origin O to the point O'. The point O' is the center of the circular section denoted by $D_2 \cup \partial D_2$ which is normal to L and has radius $a(\xi^1)$. The unit tangent vector $\mathbf{t}_1$ is given by

$$\mathbf{t}_1 \equiv \begin{pmatrix} \cos\theta \\ \sin\theta \sin\phi \\ \sin\theta \cos\phi \end{pmatrix}, \tag{2.1}$$

where the angles θ and ϕ are prescribed twice differentiable functions of ξ^1. We will also need the two unit vectors $\mathbf{t}_2$ and $\mathbf{t}_3$ where

$$\mathbf{t}_2 \equiv \begin{pmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{pmatrix}, \tag{2.2}$$

and

$$\mathbf{t}_3 \equiv \begin{pmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{pmatrix}, \tag{2.3}$$

respectively. The vectors $\mathbf{t}_i$, $i=1, 2, 3$, are then mutually orthogonal.

It has been shown [1] that a non-orthogonal curvilinear coordinate system can be constructed for the tube. The coordinates are denoted by ξ^i, $i=1, 2, 3$, where $\xi^2=0$ on ∂D_2 and $\xi^2=-\infty$ at O' for all values of ξ^1 and ξ^3. The transformation from Cartesian coordinates x^i, $i=1, 2, 3$, to the curvilinear coordinates ξ^i, $i=1, \bar{2}, 3$, is given by

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = T_1(\phi)^{-1}\, T_2(\theta)^{-1} \begin{pmatrix} 0 \\ v \\ u \end{pmatrix} + \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}^{(0)}, \tag{2.4}$$

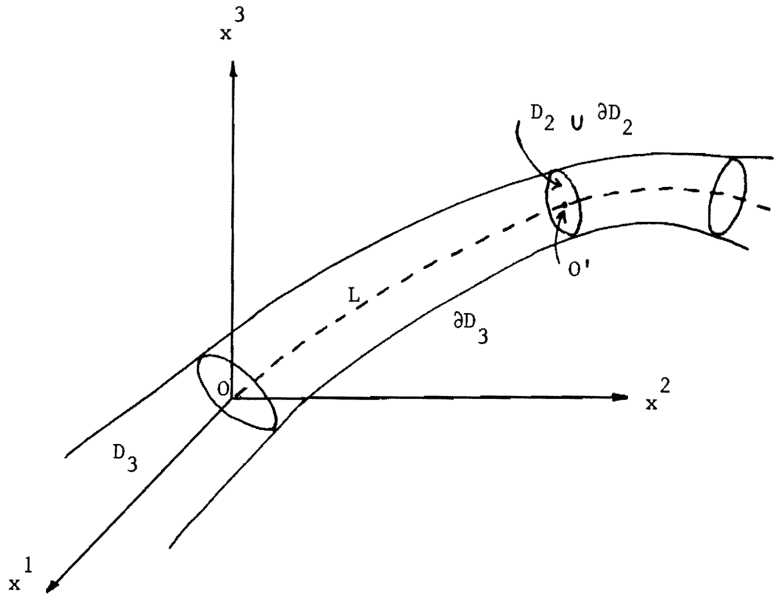

Figure 1. Geometry of the Tube

where

$$T_1(\phi) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \tag{2.5}$$

$$T_2(\theta) \equiv \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{2.6}$$

and

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}^{(0)} \equiv \int_0^{\xi_1} \mathbf{t}_1(\bar{\xi}^1)\mathrm{d}\bar{\xi}^1. \tag{2.7}$$

The point O' is represented by the vector in (2.7) and

$$v + iu = a(\xi^1)\mathrm{e}^{\xi^2 + i\xi^3}. \tag{2.8}$$

Also, $O'v$, $O'u$ are the axes of a Cartesian frame of reference which coincide with the unit vectors $\mathbf{t}_2$ and $\mathbf{t}_3$, respectively, and the coordinate ξ^3 measures the angle between $O'v$ and $O'P$ (Figure 2).

In what follows it is convenient to employ the transformation

$$\xi'^1 = \xi^1,\ \xi'^2 = \xi^2,\ \xi'^3 = \xi^3 - \chi, \tag{2.9}$$

where $\chi \equiv \int_0^{\xi^1} \omega(\bar{\xi}^1)\mathrm{d}\bar{\xi}^1$, $\omega(\equiv \dot{\phi}\cos\theta)$ is the rate at which the frame $O'v$, $O'u$ rotates about L as ξ^1 varies, and the dot notation represents differentiation with respect to ξ^1(or ξ'^1). The angle ξ'^3 measures the angle between $O'P$ and $\mathbf{i}(\equiv \cos\chi \mathbf{t}_2 + \sin\chi \mathbf{t}_3)$ where the vector $\mathbf{i}$ does not rotate about L as ξ'^1 varies [2]. It can readily be shown that the components of the covariant and contravariant metric tensors associated with the transformation given by

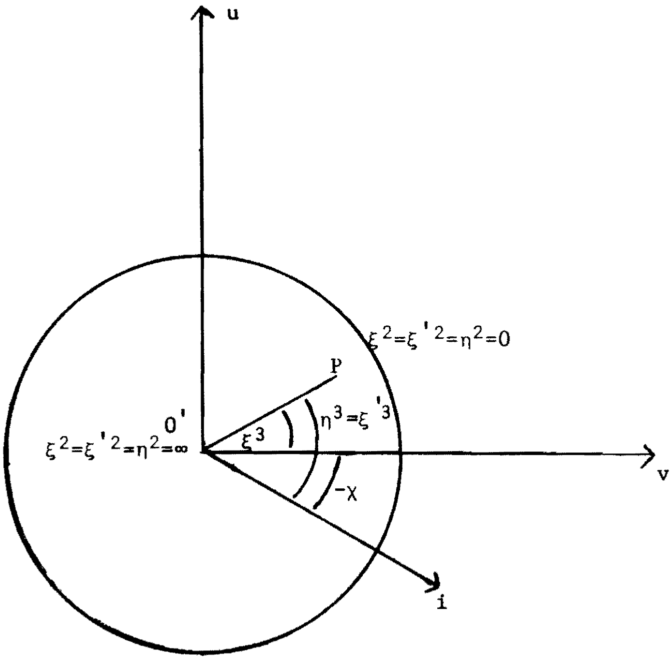

Figure 2. Circular Section of the Tube

Equations (2.4)–(2.9) are given by

$$g_{11}=\dot{a}^2 e^{2\xi'^2}+(1-\kappa a e^{\xi'^2}\sin(\chi+\xi'^3+\gamma))^2, \tag{2.10}$$

$$g_{12}=g_{21}=a\dot{a}e^{2\xi'^2}, \tag{2.11}$$

$$g_{13}=g_{31}=0, \tag{2.12}$$

$$g_{23}=g_{32}=0, \tag{2.13}$$

$$g_{22}=g_{33}=a^2 e^{2\xi'^2}, \tag{2.14}$$

and

$$g^{11}=J^{-2}a^4 e^{4\xi'^2}, \tag{2.15}$$

$$g^{12}=g^{21}=-J^{-2}a^3\dot{a}e^{4\xi'^2}, \tag{2.16}$$

$$g^{13}=g^{31}=0, \tag{2.17}$$

$$g^{23}=g^{32}=0, \tag{2.18}$$

$$g^{22}=J^{-2}a^2 e^{2\xi'^2}\{\dot{a}^2 e^{2\xi'^2}+[1-\kappa a e^{\xi'^2}\sin(\chi+\xi'^3+\gamma)]^2\}, \tag{2.19}$$

$$g^{33}=J^{-2}a^2 e^{2\xi'^2}[1-\kappa a e^{\xi'^2}\sin(\chi+\xi'^3+\gamma)]^2, \tag{2.20}$$

where $J\equiv a^2 e^{2\xi'^2}[1-\kappa a e^{\xi'^2}\sin(\chi+\xi'^3+\gamma)]$ is the Jacobian of the transformation given by Equations (2.4) through (2.9), $\kappa(\xi'^1)$ is the curvature of L, and the angle γ is defined by $\dot{\theta}=\kappa\cos\gamma$ and $\dot{\phi}\sin\theta=\kappa\sin\gamma$.

For a monopole point source moving parallel to L along the curve $\xi'^2=\xi_0'^2$, $\xi'^3=\xi_0'^3$ with uniform velocity U, the velocity potential Ψ associated with the fluid motion induced in the tube is governed by the equation

$$\frac{1}{J}\frac{\partial}{\partial\xi'^i}\left(Jg^{ij}\frac{\partial\Psi}{\partial\xi'^j}\right)=\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}+\frac{q(t)\delta(\xi'^1-Ut)\delta(\xi'^2-\xi_0'^2)\delta(\xi'^3-\xi_0'^3)}{J}. \tag{2.21}$$

In Equation (2.21), $q(t)$ is a prescribed function of time t, δ represents the Dirac delta function, and the constant c is the speed of sound in the gas.

The boundary condition associated with Equation (2.21) is

$$g^{2i}\frac{\partial\Psi}{\partial\xi'^i}=0 \text{ on } D_3. \tag{2.22}$$

Using the expressions for J and g^{ij}, $i, j=1, 2, 3$, together with the relation $\dot{\gamma}=-\tau-\omega^3$, where $\tau(\xi'^1)$ is the torsion of L, we find that Equations (2.21) and (2.22) can be written in the form

$$\begin{aligned}
&b_1a^2\frac{\partial^2\Psi}{\partial\xi'^{12}}+b_1(b_1^2\mathrm{e}^{-2\xi'^2}+\dot{a}^2)\frac{\partial^2\Psi}{\partial\xi'^{22}}+b_1^3\mathrm{e}^{-2\xi'^2}\frac{\partial^2\Psi}{\partial\xi'^{32}}\\
&-2a\dot{a}b_1\frac{\partial^2\Psi}{\partial\xi'^1\partial\xi'^2}+a^3b_2\mathrm{e}^{\xi'^2}\frac{\partial\Psi}{\partial\xi'^1}\\
&+[b_1(\dot{a}^2-a\ddot{a})-a^2\dot{a}b_2\mathrm{e}^{\xi'^2}-b_1^2a\mathrm{e}^{-\xi'^2}\kappa\sin\beta]\frac{\partial\Psi}{\partial\xi'^2}\\
&-b_1^2a\mathrm{e}^{-\xi'^2}\kappa\cos\beta\frac{\partial\Psi}{\partial\xi'^3}=\frac{b_1^3a^2}{c^2}\frac{\partial^2\Psi}{\partial t^2}\\
&+b_1^2\mathrm{e}^{-2\xi'^2}q(t)\delta(\xi'^1-Ut)\delta(\xi'^2-\xi_0'^2)\delta(\xi'^3-\xi_0'^3),
\end{aligned} \tag{2.23}$$

and

$$(\dot{a}^2\mathrm{e}^{2\xi'^2}+b_1^2)\frac{\partial\Psi}{\partial\xi'^2}-a\dot{a}\mathrm{e}^{2\xi'^2}\frac{\partial\Psi}{\partial\xi'^1}=0 \text{ on } \partial D_3, \tag{2.24}$$

where $b_1=1-\kappa a\mathrm{e}^{\xi'^2}\sin\beta$, $b_2=\dot{\kappa}\sin\beta-\kappa\tau\cos\beta$, and $\beta=\chi+\gamma+\xi'^3$. This completes the formulation of the problem.

3. SOLUTION SCHEME

Let $Q\equiv\max\limits_{-\infty<\xi'^1<\infty} a$, $\varepsilon_1(<1)\equiv\max\limits_{-\infty<\xi'^1<\infty}\kappa Q$, and $\eta^1=\xi'^1/Q$. We can write $\kappa Q=\varepsilon_1 f_1(\eta^1)$, where $f_1(\eta^1)$ is $O(1)$. Also, if τQ is $O(1)$ for $-\infty<\xi'^1<\infty$, we can write $\tau Q=f_2(\eta^1)$, where $f_2(\eta^1)$ is $O(1)$. Since the tube section is slowly varying we can write $a=Q(1+\varepsilon_3 f_3(\eta^1))$ where $0<\varepsilon_3<1$ and $f_3(\eta^1)$ is $O(1)$. If we set η^2, $\eta_0^2=\xi'^2, \xi_0'^2; \eta^3, \eta_0^3=\xi'^3, \xi_0'^3; \bar{t}=Ut/Q$, $\bar{q}(\bar{t})=q(t)/UQ^2$, $\Psi=UQ\bar{\Psi}$, and $\bar{c}=c/U$ then Equations (2.23) and (2.24) can be written (after conveniently dropping the bars on $\bar{t}$, $\bar{q}$, $\bar{\Psi}$, and $\bar{c}$) in the form

$$\begin{aligned}
&\Psi_{,11}+\nabla^2\Psi-\frac{1}{c^2}\Psi_{,tt}-q(t)\mathrm{e}^{-2\eta^2}\delta(\eta^1-t)\delta(\eta^2-\eta_0^2)\delta(\eta^3-\eta_0^3)\\
&=\varepsilon_1 f_1\mathrm{e}^{\eta^2}[s_\beta\Psi_{,11}+3s_\beta\nabla^2\Psi+[f_2c_\beta-M_1(|f_1|)s_\beta]\Psi_{,1}\\
&\quad+M_2(\Psi)-\frac{3s_\beta}{c^2}\Psi_{,tt}-2q(t)s_\beta\mathrm{e}^{-2\eta^2}\delta(\eta^1-t)\delta(\eta^2-\eta_0^2)\delta(\eta^3-\eta_0^3)]\\
&\quad-\varepsilon_3 f_3\Big[2\Psi_{,11}-2M_1(|f_3|)\mathrm{e}^{\eta^2}\Psi_{,12}-M_1(|f_3|)M_1(|f_{3,1}|)\mathrm{e}^{\eta^2}\Psi_{,2}\\
&\quad-\frac{2}{c^2}\Psi_{,tt}\Big]-\varepsilon_1^2 f_1^2\mathrm{e}^{2\eta^2}s_\beta\Big[3s_\beta\nabla^2\Psi+2M_2(\Psi)\\
&\quad-\frac{3s_\beta}{c^2}\Psi_{,tt}-q(t)s_\beta\mathrm{e}^{-2\eta^2}\delta(\eta^1-t)\,\delta(\eta^2-\eta_0^2)\,\delta(\eta^3-\eta_0^3)\Big]-
\end{aligned}$$

$$-\varepsilon_3^2 f_3^2\Big[\Psi_{,11}+M_1^2(|f_3|)e^{2\eta^2}\Psi_{,22}-2M_1(|f_3|)e^{\eta^2}\Psi_{,12}$$

$$+M_1(|f_3|)[2M_1(|f_3|)-M_1(|f_{3,1}|)]e^{\eta^2}\Psi_{,2}-\frac{1}{c^2}\Psi_{,tt}\Big]$$

$$+\varepsilon_1\varepsilon_3 f_1 f_3 e^{\eta^2}\Big[\Big[3s_\beta\Psi_{,11}+3s_\beta\nabla^2\Psi-2M_1(|f_3|)s_\beta e^{\eta^2}\Psi_{,12}$$
$$+3[f_2c_\beta-M_1(|f_1|)s_\beta]\Psi_{,1}-M_1(|f_3|)\{[M_1(|f_{3,1}|)-M_1(|f_1|)]s_\beta$$
$$+f_2c_\beta\}e^{\eta^2}\Psi_{,2}+M_2(\Psi)-\frac{9s_\beta}{c^2}\Psi_{,tt}$$

$$-2q(t)s_\beta e^{-2\eta^2}\delta(\eta^1-t)\,\delta(\eta^2-\eta_0^2)\,\delta(\eta^3-\eta_0^3)\Big]\Big]$$

$$+\varepsilon_1^3 f_1^3 e^{3\eta^2}s_\beta^2\Big[s_\beta\nabla^2\Psi+M_2(\Psi)-\frac{s_\beta}{c^2}\Psi_{,tt}\Big]$$

$$-2\varepsilon_1^2\varepsilon_3 f_1^2 f_3 e^{2\eta^2}s_\beta\Big[3s_\beta\nabla^2\Psi+2M_2(\Psi)-\frac{6s_\beta}{c^2}\Psi_{,tt}$$

$$-q(t)s_\beta e^{-2\eta^2}\delta(\eta^1-t)\,\delta(\eta^2-\eta_0^2)\,\delta(\eta^3-\eta_0^3)\Big]$$

$$+\varepsilon_1\varepsilon_3^2 f_1 f_3^2 e^{\eta^2}\Big[\Big[3s_\beta\Psi_{,11}+M_1^2(|f_3|)s_\beta e^{2\eta^2}\Psi_{,22}-4M_1(|f_3|)s_\beta e^{\eta^2}\Psi_{,12}$$
$$+3[f_2c_\beta-M_1(|f_1|)s_\beta]\Psi_{,1}+2M_1(|f_3|)\{s_\beta[M_1(|f_3|)-M_1(|f_{3,1}|)$$
$$+M_1(|f_1|)]-f_2c_\beta\}e^{\eta^2}\Psi_{,2}-\frac{9s_\beta}{c^2}\Psi_{,tt}\Big]\Big]$$

$$+\varepsilon_1^3\varepsilon_3 f_1^3 f_3 e^{3\eta^2}s_\beta^2\Big[3s_\beta\nabla^2\Psi+3M_2(\Psi)-\frac{5s_\beta}{c^2}\Psi_{,tt}\Big]$$

$$+\varepsilon_1\varepsilon_3^3 f_1 f_3^3 e^{\eta^2}\Big[s_\beta[\Psi_{,11}+M_1^2(|f_3|)e^{2\eta^2}\Psi_{,22}$$
$$-2M_1(|f_3|)e^{\eta^2}\Psi_{,12}]+[f_2c_\beta-M_1(|f_1|)s_\beta]\Psi_{,1}$$
$$-M_1(|f_3|)\{s_\beta[M_1(|f_{3,1}|)-2M_1(|f_3|)-M_1(|f_1|)]+f_2c_\beta\}e^{\eta^2}\Psi_{,2}$$
$$-\frac{3s_\beta}{c^2}\Psi_{,tt}\Big]\Big]-\varepsilon_1^2\varepsilon_3^2 f_1^2 f_3^2 e^{2\eta^2}s_\beta\Big[3s_\beta\nabla^2\Psi+2M_2(\Psi)$$
$$-\frac{18s_\beta}{c^2}\Psi_{,tt}-q(t)s_\beta e^{-2\eta^2}\delta(\eta^1-t)\,\delta(\eta^2-\eta_0^2)\,\delta(\eta^3-\eta_0^3)\Big]$$

$$+\varepsilon_1^3\varepsilon_3^2 f_1^3 f_3^2 e^{3\eta^2}s_\beta^2\Big[3s_\beta\nabla^2\Psi+3M_2(\Psi)-\frac{10s_\beta}{c^2}\Psi_{,tt}\Big]+$$

$$+12\varepsilon_1^2\varepsilon_3^3 f_1^2 f_3^3 e^{2\eta^2}\frac{s_\beta^2}{c^2}\Psi_{,tt}+\varepsilon_1^3\varepsilon_3^3 f_1^3 f_3^3 e^{3\eta^2}s_\beta^2\left[s_\beta\nabla^2\Psi+M_2(\Psi)-\frac{10s_\beta}{c^2}\Psi_{,tt}\right]$$

$$+3\varepsilon_1^2\varepsilon_3^4 f_1^2 f_3^4 e^{2\eta^2}\frac{s_\beta^2}{c^2}\Psi_{,tt}-5\varepsilon_1^3\varepsilon_3^4 f_1^3 f_3^4 e^{3\eta^2}\frac{s_\beta^3}{c^2}\Psi_{,tt}$$

$$-\varepsilon_1^3\varepsilon_3^5 f_1^3 f_3^5 e^{3\eta^2}\frac{s_\beta^3}{c^2}\Psi_{,tt} \tag{3.1}$$

and

$$\begin{aligned}\Psi_{,2}&=2\varepsilon_1 f_1 e^{\eta^2}s_\beta\Psi_{,2}+\varepsilon_3 f_3 M_1(|f_3|)e^{\eta^2}\Psi_{,1}\\&-\varepsilon_1^2 f_1^2 e^{2\eta^2}s_\beta^2\Psi_{,2}+2\varepsilon_1\varepsilon_3 f_1 f_3 e^{\eta^2}s_\beta\Psi_{,2}+\varepsilon_3^2 f_3^2 M_1(|f_3|)e^{\eta^2}[\Psi_{,1}\\&-M_1(|f_3|)e^{\eta^2}\Psi_{,2}]-2\varepsilon_1^2\varepsilon_3 f_1^2 f_3 e^{2\eta^2}s_\beta^2\Psi_{,2}\\&-\varepsilon_1^2\varepsilon_3^2 f_1^2 f_3^2 e^{2\eta^2}s_\beta^2\Psi_{,2}\ \text{on}\ \partial D_3,\end{aligned} \tag{3.2}$$

where $c_\beta, s_\beta \equiv \cos\beta$; $\sin\beta$, $(\ )_{,1}\equiv\partial(\ )/\partial\eta^1$; $(\ )_{,j}=e^{-\eta^2}\partial(\ )/\partial\eta^j$, $j=2,3$, $\nabla^2\equiv e^{-2\eta^2}(\partial^2/\partial\eta^{22}+\partial^2/\partial\eta^{32})$, $M_1(\ )=(\ln(\ ))_{,1}$, $M_2(\ )=e^{-\eta^2}(s_\beta(\ )_{,2}+c_\beta(\ )_{,3})$, and $(\ )_{,t}=\partial(\ )/\partial t$.

For sufficiently small values of ε_1, ε_3, we will seek the solution of (3.1) and (3.2) in the form

$$\Psi=\sum_{n=0}^{\infty}\varepsilon_1^i\varepsilon_3^k\Psi_{ik}^{(n)},\quad i+k=n. \tag{3.3}$$

The system of boundary-value problems for $\Psi_{ik}^{(n)}$, $n\geq 0$ is

$$\Psi_{00,11}^{(0)}+\nabla^2\Psi_{00}^{(0)}-\frac{1}{c^2}\Psi_{00,tt}^{(0)}=q(t)e^{-2\eta^2}\delta(\eta^1-t)\delta(\eta^2-\eta_0^2)\delta(\eta^3-\eta_0^3), \tag{3.4}$$

$$\Psi_{00,2}^{(0)}=0\ \text{on}\ \partial D_3, \tag{3.5}$$

together with

$$\Psi_{ik,11}^{(n)}+\nabla^2\Psi_{ik}^{(n)}-\frac{1}{c^2}\Psi_{ik,tt}^{(n)}=\Phi_{ik}^{(n-1)},\quad i+k=n,\quad n\geq 1, \tag{3.6}$$

$$\Psi_{ik,2}^{(n)}=\chi_{ik}^{(n-1)}\ \text{on}\ \partial D_3,\quad i+k=n\quad n\geq 1, \tag{3.7}$$

where $\Phi_{ik}^{(n-1)}$ and $\chi_{ik}^{(n-1)}$ are given by

$$\begin{aligned}\Phi_{ik}^{(n-1)}=f_1 e^{\eta^2}\Big[&s_\beta\Psi_{i-1\,k,11}^{(n-1)}+3s_\beta\nabla^2\Psi_{i-1\,k}^{(n-1)}+[f_2c_\beta-M_1(|f_1|)s_\beta]\Psi_{i-1k,1}^{(n-1)}\\&+M_2(\Psi_{i-1\,k}^{(n-1)})-\frac{3s_\beta}{c^2}\Psi_{i-1\,k,tt}^{(n-1)}-2q(t)s_\beta e^{-2\eta^2}\delta(\eta^1-t)\delta(\eta^2-\eta_0^2)\delta(\eta^3-\eta_0^3)\Big]\\&-f_3\Big[2\Psi_{ik-1,11}^{(n-1)}-2M_1(|f_3|)e^{\eta^2}\Psi_{ik-1,12}^{(n-1)}-M_1(|f_3|)M_1(|f_{3,1}|)e^{\eta^2}\Psi_{ik-1,2}^{(n-1)}\\&-\frac{2}{c^2}\Psi_{ik-1,tt}^{(n-1)}\Big]-f_1^2e^{2\eta^2}s_\beta\Big[3s_\beta\nabla^2\Psi_{i-2k}^{(n-2)}+2M_2(\Psi_{i-2k}^{(n-2)})\\&-\frac{3}{c^2}s_\beta\Psi_{i-2\,k,tt}^{(n-2)}-q(t)s_\beta e^{-2\eta^2}\delta(\eta^1-t)\delta(\eta^2-\eta_0^2)\delta(\eta^3-\eta_0^3)\Big]-\end{aligned}$$

$$-f_3^2\Bigg[\Psi^{(n-2)}_{ik-2,11}+M_1^2(|f_3|)e^{2\eta^2}\Psi^{(n-2)}_{ik-2,22}-2M_1(|f_3|)e^{\eta^2}\Psi^{(n-2)}_{ik-2,12}$$

$$+M_1(|f_3|)[2M_1(|f_3|)-M_1(|f_{3,1}|)]e^{\eta^2}\Psi^{(n-2)}_{ik-2,2}-\frac{1}{c^2}\Psi^{(n-2)}_{ik-2,tt}\Bigg]$$

$$+f_1f_3e^{\eta^2}\Bigg[\!\!\Bigg[3s_\beta\Psi^{(n-2)}_{i-1\,k-1,11}+3s_\beta\nabla^2\Psi^{(n-2)}_{i-1\,k-1}-2M_1(|f_3|)s_\beta e^{\eta^2}\Psi^{(n-2)}_{i-1\,k-1,12}$$

$$+3[f_2c_\beta-M_1(|f_1|)s_\beta]\Psi^{(n-2)}_{i-1\,k-1,1}-M_1(|f_3|)\{[M_1(|f_{3,1}|)-M_1(|f_1|)]s_\beta$$

$$+f_2c_\beta\}e^{\eta^2}\Psi^{(n-2)}_{i-1\,k-1,2}+M_2(\Psi^{(n-2)}_{i-1\,k-1})-\frac{9s_\beta}{c^2}\Psi^{(n-2)}_{i-1\,k-1,tt}$$

$$-2q(t)s_\beta e^{-2\eta^2}\delta(\eta^1-t)\delta(\eta^2-\eta_0^2)\delta(\eta^3-\eta_0^3)\Bigg]\!\!\Bigg]$$

$$+f_1^3e^{3\eta^2}s_\beta^2\Bigg[s_\beta\nabla^2\Psi^{(n-3)}_{i-3\,k}+M_2(\Psi^{(n-3)}_{i-3\,k})-\frac{s_\beta}{c^2}\Psi^{(n-3)}_{i-3\,k,tt}\Bigg]$$

$$-2f_1^2f_3e^{2\eta^2}s_\beta\Bigg[3s_\beta\nabla^2\Psi^{(n-3)}_{i-2\,k-1}+2M_2(\Psi^{(n-3)}_{i-2\,k-1})-\frac{6s_\beta}{c^2}\Psi^{(n-3)}_{i-2\,k-1,tt}$$

$$-q(t)s_\beta e^{-2\eta^2}\delta(\eta^1-t)\delta(\eta^2-\eta_0^2)\delta(\eta^3-\eta_0^3)\Bigg]$$

$$+f_1f_3^2e^{\eta^2}\Bigg[\!\!\Bigg[3s_\beta\Psi^{(n-3)}_{i-1\,k-2,11}+M_1^2(|f_3|)s_\beta e^{2\eta^2}\Psi^{(n-3)}_{i-1\,k-2,22}$$

$$-4M_1(|f_3|)s_\beta e^{\eta^2}\Psi^{(n-3)}_{i-1\,k-2,12}+3[f_2c_\beta-M_1(|f_1|)s_\beta]\Psi^{(n-3)}_{i-1\,k-2,1}$$

$$+2M_1(|f_3|)\{s_\beta[M_1(|f_3|)-M_1(|f_{3,1}|)+M_1(|f_1|)]-f_2c_\beta\}e^{\eta^2}\Psi^{(n-3)}_{i-1\,k-2,2}$$

$$-\frac{9s_\beta}{c^2}\Psi^{(n-3)}_{i-1\,k-2,tt}\Bigg]\!\!\Bigg]$$

$$+f_1^3f_3e^{3\eta^2}s_\beta^2\Bigg[3s_\beta\nabla^2\Psi^{(n-4)}_{i-3\,k-1}+3M_2(\Psi^{(n-4)}_{i-3\,k-1})-\frac{5s_\beta}{c^2}\Psi^{(n-4)}_{i-3\,k-1,tt}\Bigg]$$

$$+f_1f_3^3e^{\eta^2}\Bigg[\!\!\Bigg[s_\beta[\Psi^{(n-4)}_{i-1\,k-3,11}+M_1^2(|f_3|)e^{2\eta^2}\Psi^{(n-4)}_{i-1\,k-3,22}$$

$$-2M_1(|f_3|)e^{\eta^2}\Psi^{(n-4)}_{i-1\,k-3,12}]+[f_2c_\beta-M_1(|f_1|)s_\beta]\Psi^{(n-4)}_{i-1\,k-3,1}$$

$$-M_1(|f_3|)\{s_\beta[M_1(|f_{3,1}|)-2M_1(|f_3|)-M_1(|f_1|)]+f_2c_\beta\}e^{\eta^2}\Psi^{(n-4)}_{i-1\,k-3,2}$$

$$-\frac{3s_\beta}{c^2}\Psi^{(n-4)}_{i-1\,k-3,tt}\Bigg]\!\!\Bigg]-f_1^2f_3^2e^{2\eta^2}s_\beta\Bigg[3s_\beta\nabla^2\Psi^{(n-4)}_{i-2\,k-2}$$

$$+2M_2(\Psi^{(n-4)}_{i-2k-2})-\frac{18s_\beta}{c^2}\Psi^{(n-4)}_{i-2k-2,tt}-$$

$$-q(t)s_\beta e^{-2\eta^2}\delta(\eta^1-t)\delta(\eta^2-\eta_0^2)\delta(\eta^3-\eta_0^3)\Big]$$

$$+f_1^3f_3^2e^{3\eta^2}s_\beta^2\left[3s_\beta\nabla^2\Psi^{(n-5)}_{i-3\,k-2}+3M_2(\Psi^{(n-5)}_{i-3\,k-2})-\frac{10s_\beta}{c^2}\Psi^{(n-5)}_{i-3\,k-2,tt}\right]$$

$$+12f_1^2f_3^3e^{2\eta^2}\frac{s_\beta^2}{c^2}\Psi^{(n-5)}_{i-2\,k-3,tt}+f_1^3f_3^3e^{3\eta^2}s_\beta^2\Big[s_\beta\nabla^2\Psi^{(n-6)}_{i-3\,k-3}+M_2(\Psi^{(n-6)}_{i-3\,k-3})$$

$$-\frac{10s_\beta}{c^2}\Psi^{(n-6)}_{i-3\,k-3,tt}\Big]+3f_1^2f_3^4e^{2\eta^2}\frac{s_\beta^2}{c^2}\Psi^{(n-6)}_{i-2\,k-4,tt}$$

$$-5f_1^3f_3^4e^{3\eta^2}\frac{s_\beta^3}{c^2}\Psi^{(n-7)}_{i-3\,k-4,tt}-f_1^3f_3^5e^{3\eta^2}\frac{s_\beta^3}{c^2}\Psi^{(n-8)}_{i-3\,k-5,tt}, \tag{3.8}$$

and

$$\begin{aligned}\chi_{ik}^{(n-1)}=\;&2f_1e^{\eta^2}s_\beta\Psi^{(n-1)}_{i-1\,k,2}+f_3M_1(|f_3|)e^{\eta^2}\Psi^{(n-1)}_{i\,k-1,1}\\&-f_1^2e^{2\eta^2}s_\beta^2\Psi^{(n-2)}_{i-2\,k,2}+2f_1f_3e^{\eta^2}s_\beta\Psi^{(n-2)}_{i-1\,k-1,2}\\&+f_3^2M_1(|f_3|)e^{\eta^2}[\Psi^{(n-2)}_{i\,k-2,1}-M_1(|f_3|)e^{\eta^2}\Psi^{(n-2)}_{i\,k-2,2}]\\&-2f_1^2f_3e^{2\eta^2}s_\beta^2\Psi^{(n-3)}_{i-2\,k-1,2}-f_1^2f_3^2e^{2\eta^2}s_\beta^2\Psi^{(n-4)}_{i-2\,k-2,2}\end{aligned} \tag{3.9}$$

with $\Psi^{(m)}_{i_1i_2}$, $i_1+i_2=m$, and all derivatives of $\Psi^{(m)}_{i_1i_2}$ identically zero when $i_1<0$ or $i_2<0$.

Equations (3.4) and (3.5) can be considered as the equations in the *cylindrical* coordinate system (η^1, η^2, η^3) with scaling factors $1, e^{\eta^2}, e^{\eta^2}$ which govern the motion due to a monopole point source moving in a gas-filled infinitely long *straight* tube with uniform circular section. The solution of this boundary-value problem can be obtained by standard methods when the function $q(t)$ is prescribed. The effects of tube curvature, torsion, and section variation are exhibited in the system of boundary-value problems given by (3.6) and (3.7). For $n\geq 1$, each of these boundary-value problems governs the motion induced by a source function $\Phi_{ik}^{(n-1)}$ and a prescribed boundary velocity $\chi_{ik}^{(n-1)}$ in a gas-filled infinitely long *straight* tube with uniform circular section. Again, these can be solved by standard methods.

If in addition the torsion of the tube is small, $|\tau|Q\ll 1$ for $-\infty<\xi'^1<\infty$, we can write $\tau Q=\varepsilon_2 f_2(\eta^1)$, where $\varepsilon_2=\max|\tau|Q\ll 1$ and $f_2(\eta^1)$ is again $O(1)$. Equations (2.23) and (2.24) now take a form which can be obtained from (3.1) and (3.2) by replacing f_2 throughout by $\varepsilon_2 f_2$.

In this case we seek a solution of the form

$$\Psi=\sum_{n=0}^{\infty}\varepsilon_1^i\varepsilon_2^j\varepsilon_3^k\Psi^{(n)}_{ijk},\quad i+j+k=n. \tag{3.10}$$

The system of boundary-value problems for $\Psi^{(n)}_{ijk}$, $n\geq 0$, is

$$\Psi^{(0)}_{000,11}+\nabla^2\Psi^{(0)}_{000}-\frac{1}{c^2}\Psi^{(0)}_{000,tt}=q(t)e^{-2\eta^2}\delta(\eta^1-t)\delta(\eta^2-\eta_0^2)\delta(\eta^3-\eta_0^3), \tag{3.11}$$

$$\Psi^{(0)}_{000,2}=0 \text{ on } \partial D_3, \tag{3.12}$$

together with

$$\Psi^{(n)}_{ijk,11}+\nabla^2\Psi^{(n)}_{ijk}-\frac{1}{c^2}\Psi^{(n)}_{ijk,tt}=\Phi^{(n-1)}_{ijk},\quad i+j+k=n,\quad n\geq 1, \tag{3.13}$$

$$\Psi^{(n)}_{ijk,2}=\chi^{(n-1)}_{ijk} \text{ on } \partial D_3,\quad i+j+k=n,\quad n\geq 1. \tag{3.14}$$

We obtain $\Phi_{ijk}^{(n-1)}$ in terms of Ψs by the following prescription: in (3.8) replace $\Phi_{ik}^{(n-1)}$ by $\Phi_{ijk}^{(n-1)}$ and replace each $\Psi_{\beta\gamma}^{(\alpha)}$ by $\Psi_{\beta j\gamma}^{(\alpha)}$, except that if f_2 occurs in the coefficient of $\Psi_{\beta\gamma}^{(\alpha)}$ replace it by $\Psi_{\beta j-1\gamma}^{(\alpha-1)}$. To obtain $\chi_{ijk}^{(n-1)}$ replace $\chi_{ik}^{(n-1)}$ by $\chi_{ijk}^{(n-1)}$ and each $\Psi_{\beta\gamma}^{(\alpha)}$ by $\Psi_{\beta j\gamma}^{(\alpha)}$ in (3.9).

The method of solution of the resulting boundary-value problems is formally the same as in the case $\tau Q = O(1)$.

In [3] we have discussed the Dirichlet boundary value problems associated with a general twisted tube with uniform cross-section. There we obtained the Green functions associated with the following four problems:

(i) tube of finite length,
(ii) closed tube of finite length,
(iii) semi-infinite tube,
(iv) infinite tube.

Perhaps the ideas contained there could be extended in our present case. Furthermore, special forms of $q(t)$ in Equation (2.21) such as that for the harmonic motion of a pulsating sphere ($q = q_0 \cos \omega t$) could be treated (see, for example, [1]).

REFERENCES

[1] P. M. Morse and K. O. Ingard, *Theoretical Acoustics.* New York: McGraw-Hill, 1968, pp. 717–732.

[2] J. C. Murray, 'Curvilinear Coordinate Systems Associated with a Class of Boundary-Value Problems', *Journal of the Institute of Mathematics and its Applications*, **25** (1980), pp. 397–411.

[3] J. C. Murray, 'On the Boundary-Value Problem Associated with a General Twisted Tube with a Uniform Non-Rotating Section', *Quarterly Journal of Applied Mathematics*, **39** (1981), pp. 341–349.

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