STABILITY THEOREMS FOR AN INTEGRO-DIFFERENTIAL EQUATION

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الخلاصــة :

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ABSTRACT

We consider the scalar integro-differential equation

$$x' = A(t)x + \int_0^t C(t, s)x(s)\mathrm{d}s,$$

where A and C are continuous, and give necessary and sufficient conditions for the stability of the equation without $A(t)$ being necessarily bounded.

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STABILITY THEOREMS FOR AN INTEGRO-DIFFERENTIAL EQUATION

1. INTRODUCTION

We consider the integro-differential equation

$$x' = A(t)x + \int_0^t C(t, s)x(s)\mathrm{d}s, \tag{1.1}$$

where $A(t)$ and $C(t,s)$ are real-valued functions continuous for $0 \le t < \infty$ and $0 \le s \le t < \infty$, respectively.

Many stability results in differential and integral equations have been obtained by constructing Liapunov functionals. Such functionals for (1.1) require that $A(t)$ be negative. In order to avoid this restrictive condition, one chooses a continuous function $G(t,s)$ with

$$\partial G(t,s)/\partial t = C(t,s) \tag{1.2}$$

so that (1.1) takes the form

$$x' = Q(t)x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t G(t, s)x(s)\mathrm{d}s, \tag{1.3}$$

where

$$Q(t) = A(t) - G(t, t). \tag{1.4}$$

We intend to study stability of solutions of (1.1) via the construction of Liapunov functionals for (1.3). This approach turns out to be very fruitful and has led to several interesting results. In [1], we obtained the following characterization of stability of the zero solution of (1.1).

Theorem 1. Let $G(t,s)$ and $Q(t)$ be defined by (1.2) and (1.4). Suppose there are constants Q_1, Q_2, J, and R with $R < 2$ such that

(i) $0 < Q_1 \le |Q(t)| \le Q_2$,

(ii) $\int_0^t |G(t, s)|\mathrm{d}s \le J < 1,$

and

(iii) $\int_0^t |G(t, s)|\mathrm{d}s + \int_t^\infty |G(u, t)|\mathrm{d}u \le RQ_1/Q_2$

for $0 \le t < \infty$. Furthermore, suppose there is a continuous function h: $[0, \infty) \to [0, \infty)$ with $|G(t,s)| \le h(t-s)$ and $h(u) \to 0$ as $u \to \infty$. Then the zero solution of (1.1) is stable if and only if $Q(t) < 0$.

It was shown in [1] that Theorem 1 includes a result of Brauer [2] when $A(t) = A =$constant and $C(t,s) = C(t-s)$. However, we quickly observe that when $C(t,s) = C(t-s)$, $G(t,t) = G(0)$ and hence, by (1.4), condition (i) of Theorem 1 requires that $A(t)$ be bounded. It is therefore desirable to relax the boundedness condition on $Q(t)$ so that (1.1) may be stable while $A(t)$ is unbounded. The purpose of this paper is then to give a new characterization of stability of (1.1) under milder conditions on $A(t)$ and $C(t,s)$ than that of Theorem 1.

To define a solution of (1.1) we require a $t_0 \ge 0$ and a continuous initial function ϕ: $[0, t_0] \to R$. Then a solution of (1.1) is a continuous function x: $[0,\infty) \to R$, denoted by $x(t, t_0, \phi)$ or simply $x(t)$, which satisfies (1.1) for $t \ge t_0$ and such that $x(t, t_0, \phi) = \phi(t)$ for $0 \le t \le t_0$. Under the above hypothesis, (1.1) has a unique solution. Details on existence, uniqueness, and continuation of solutions are to be found in Driver [3].

The following definitions are natural extensions of stability definitions for ordinary differential equations. They have been used in integro-differential equations as well as in delay-differential equations; see Driver [3] and Miller [4].

Definition 1. The solution $x = 0$ of (1.1) is stable if for every $\varepsilon > 0$ and every $t_0 \ge 0$, there is a $\delta = \delta(\varepsilon, t_0) > 0$ such that $|\phi(t)| < \delta$ on $[0, t_0]$ implies $|x(t, t_0, \phi)| < \varepsilon$ for $t \ge t_0$.

Definition 2. The solution $x = 0$ of (1.1) is uniformly stable if it is stable and the δ in the definition of stability is independent of t_0.

Definition 3. The solution $x = 0$ of (1.1) is asymptotically stable if it is stable and for every $t_0 \ge 0$ there is a $b = b(t_0) > 0$ such that $|\phi(t)| < b$ on $[0, t_0]$ implies $x(t, t_0, \phi) \to 0$ as $t \to \infty$.

Definition 4. The solution $x = 0$ of (1.1) is uniformly asymptotically stable if it is uniformly stable, the b in the definition of asymptotic stability is independent of t_0, and for each $\eta > 0$ there is a $T = T(\eta) > 0$ such that $|\phi(t)| < b$ on $[0, t_0]$ implies $|x(t, t_0, \phi)| < \eta$ for $t \ge t_0 + T$.

When a function is written without its argument, it is understood that the argument is t.

2. STABILITY

Let $\alpha(t)$ be a real-valued function continuous on $[0,\infty)$. For each $s\in[0,\infty)$, we let $H(t,s)$ denote the solution of the differential equation

$$\partial H(t,s)/\partial t-\alpha(t)H(t,s)=C(t,s) \tag{2.1}$$

satisfying

$$H(s,s)=A(s)-\alpha(s) \tag{2.2}$$

with $0\le s\le t$. We then write (1.1) as

$$x'=\alpha(t)x-\alpha(t)\int_0^t H(t,s)x(s)\mathrm{d}s+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t H(t,s)x(s)\mathrm{d}s \tag{2.3}$$

and consider the functional equation

$$V(t,x(\cdot))=\left|x-\int_0^t H(t,s)x(s)\mathrm{d}s\right|+v\int_0^t\left[\mu|\alpha(s)|-\int_s^t|\alpha(u)||H(u,s)|\mathrm{d}u\right]|x(s)|\mathrm{d}s, \tag{2.4}$$

where v and μ are arbitrary constants.

The derivative of $V(t,x(\cdot))$ along a solution $x(t)=x(t,t_0,\phi)$ of (2.3) satisfies

$$V'_{(2.3)}(t,x(\cdot))=\alpha(t)\left|x-\int_0^t H(t,s)x(s)\mathrm{d}s\right|+v\mu|\alpha(t)||x|-v\int_0^t|\alpha(t)||H(t,s)||x(s)|\mathrm{d}s. \tag{2.5}$$

Theorem 2. Let $H(t,s)$ be defined by (2.1) and (2.2), and suppose that for some $\alpha(t)\le 0$ there is a constant J such that

(i) $\displaystyle\int_0^t|H(t,s)|\mathrm{d}s\le J<1$

and

(ii) $\displaystyle\int_t^\infty|\alpha(u)||H(u,t)|\mathrm{d}u\le|\alpha(t)|$.

Then the zero solution of (1.1) is stable.

Proof. Since $\alpha(t)\le 0$, then it follows from (2.5) that

$$V'_{(2.3)}(t,x(\cdot))\le-|\alpha(t)|\left[|x|-\int_0^t|H(t,s)||x(s)|\mathrm{d}s\right]+v\mu|\alpha(t)||x|-v|\alpha(t)|\int_0^t|H(t,s)||x(s)|\mathrm{d}s.$$

Thus,

$$V'_{(2.3)}(t,x(\cdot))\le(v\mu-1)|\alpha(t)||x|+(1-v)|\alpha(t)|\int_0^t|H(t,s)||x(s)|\mathrm{d}s. \tag{2.6}$$

By taking $v=\mu=1$, we have $V'_{(2.3)}(t,x(\cdot))\le 0$ and hence

$$\begin{aligned}V(t,x(\cdot))\le V(t_0,\phi(\cdot))&=\left|\phi(t_0)-\int_0^{t_0}H(t_0,s)\,\phi(s)\mathrm{d}s\right|+\int_0^{t_0}\left[|\alpha(s)|-\int_s^{t_0}|\alpha(u)||H(u,s)|\mathrm{d}u\right]|\phi(s)|\mathrm{d}s\\&\le|\phi(t_0)|+\int_0^{t_0}|H(t_0,s)||\phi(s)|\mathrm{d}s+\int_0^{t_0}\left[|\alpha(s)|-\int_s^{t_0}|\alpha(u)||H(u,s)|\mathrm{d}u\right]|\phi(s)|\mathrm{d}s\end{aligned}$$

for all $t\ge t_0$.

If $|\phi(t)|<\delta$ for $0\le t\le t_0$, then by (i) and the above inequality we have

$$V(t,x(\cdot))\le\delta[1+J+R(t_0)]\overset{\text{def}}{=}\delta N$$

where

$$R(t_0)=\int_0^{t_0}\left[|\alpha(s)|-\int_s^{t_0}|\alpha(u)||H(u,s)|\mathrm{d}u\right]\mathrm{d}s. \tag{2.7}$$

On the other hand, (2.4) and (ii) imply that

$$V(t,x(\cdot))\ge\left|x(t)-\int_0^t H(t,s)x(s)\mathrm{d}s\right|\ge|x|-\int_0^t|H(t,s)||x(s)|\mathrm{d}s.$$

Thus

$$|x(t)|\le\delta N+\int_0^t|H(t,s)||x(s)|\mathrm{d}s \tag{2.8}$$

for all $t\ge t_0$. Now, for any $\varepsilon>0$ and $t_0\ge 0$, we choose δ so that $0<\delta<\min\{\varepsilon,\varepsilon(1-J)/N\}$. As $|\phi(\mathrm{t})|<\delta<\varepsilon$, then either $|x(t)|<\varepsilon$ for all $t\ge t_0$ or there is a $t_1>t_0$ such that $|x(t_1)|=\max_{0\le t\le t_1}|x(t)|$. In the latter case we have by (2.8),

$$|x(t_1)|\le\delta N+\int_0^{t_1}|H(t_1,s)||x(s)|\mathrm{d}s\le\delta N+\varepsilon J<\varepsilon,$$

a contradiction. Thus, $x=0$ is stable and this completes the proof.

If we take $\alpha(t) \equiv 0$ in Theorem 2, we obtain the following result.

Corollary 1. Let $H(t,s)$ satisfy $\partial H(t,s)/\partial t = C(t,s)$ and $H(s,s) = A(s)$. If there is a constant J such that

$$\int_0^t |H(t,s)|ds \leq J < 1,$$

then the zero solution of (1.1) is stable.

Remark. By the use of the function $H(t,s)$ as defined by (2.1) and (2.2) the conditions of Theorem 2 on the kernel $C(t,s)$ become much weaker than the conditions of Theorem 1. This can be seen from the following example.

Example. For $A(t) = -t$ and $C(t,s) = -3/(t-s+1)^4 + (t+1)/(t-s+1)^3$, 'we choose $\alpha(t) = -t-1$ so that $H(t,s) = 1/(t-s+1)^3$ satisfies (2.1) and (2.2). Furthermore.

$$\int_0^t |H(t,s)|ds = \tfrac{1}{2}[1-(t+1)^{-2}] < \tfrac{1}{2}$$

and

$$\int_t^\infty |\alpha(u)||H(u,t)|du = \int_t^\infty (u+1)(u-t+1)^{-3}\,du = 1 + \frac{t}{2} \leq |\alpha(t)|.$$

Thus, all the conditions of Theorem 2 are satisfied and hence the zero solution of (1.1) is stable. However, Theorem 1 fails to apply as $G(t,s)$, defined by (1.2), is of the form

$$G(t,s) = (t-s+1)^{-3} - (t-s+1)^{-1} - \frac{s}{2}(t-s+1)^{-2} + h(s)$$

which is not integrable for any choice of $h(s)$.

Theorem 3. Let $H(t,s)$ be defined by (2.1) and (2.2) and suppose there is a continuous function $h:[0,\infty] \to [0,\infty]$ such that $|H(t,s)| \leq h(t-s)$ with $h(u) \to 0$ as $u \to \infty$. Furthermore, suppose there are positive constants a, J, and $\mu < 1$ and a function $\alpha(t)$ such that

(i) $|\alpha(t)| \geq a$,

(ii) $\int_0^t |H(t,s)|ds \leq J < 1$,

and

(iii) $\int_t^\infty |\alpha(u)||H(u,t)|du \leq \mu |\alpha(t)|$.

Then the zero solution of (1.1) is stable if and only if $\alpha(t) < 0$.

Proof. We need only prove the converse. Suppose that $\alpha(t) > 0$. Then, by (2.5), we have

$$V'_{(2.3)}(t,x(\cdot)) \geq \alpha(t)\left[|x| - \int_0^t |H(t,s)||x(s)|ds\right] + \nu\mu\alpha(t)|x| - \nu\int_0^t \alpha(t)|H(t,s)||x(s)|ds.$$

Thus

$$V'_{(2.3)}(t,x(\cdot)) \geq (\nu\mu+1)\,\alpha(t)|x| - (\nu+1)\alpha(t)\int_0^t |H(t,s)||x(s)|\,ds.$$

Choose $\nu = -1$ to obtain

$V'_{(2.3)}(t,x(\cdot)) \geq (1-\mu)\,a|x|$ for all $t \geq t_0$.

From this and (2.4), we have

$$\left|x(t) - \int_0^t H(t,s)x(s)ds\right| \geq V(t,x(\cdot)) \geq V(t_0,\phi(\cdot)) + (1-\mu)a\int_{t_0}^t |x(s)|ds. \tag{2.9}$$

Now, for any $t_0 \geq 0$ and $\delta > 0$ there is a continuous function $\phi:[0,t_0] \to R$ such that $V(t_0,\phi(\cdot)) > 0$. If $x(t) = x(t,t_0,\phi)$ is a bounded solution of (1.1), then it follows from (2.9) and (ii) that $x(t)$ is in $L^1[0,\infty)$. Since

$$\int_0^t |H(t,s)||x(s)|ds \leq \int_0^t h(t-s)|x(s)|ds \tag{2.10}$$

and the right-hand side is the convolution of an L^1-function with a function tending to zero, then the right-hand side tends to zero and hence

$$\int_0^t |H(t,s)||x(s)|ds \to 0 \text{ as } t \to \infty.$$

Thus, by (2.9), $x(t)$ is bounded away from zero for all sufficiently large values of t. This contradicts $x(t)$ being in L^1. Thus, $x(t)$ is unbounded and the proof is complete.

Theorem 4. Let the conditions of Theorem 2 hold and suppose, in addition, that

$$\int_0^\infty \left[|\alpha(s)| - \int_t^\infty |\alpha(u)||H(u,t)|du\right]dt < \infty. \tag{2.11}$$

Then the zero solution of (1.1) is uniformly stable.

Proof. It is enough to observe that $R(t_0)$ in (2.7) is an increasing function of t_0 so that, by (2.11), $R(t_0) \leq M$ for some positive constant M and all $t_0 \geq 0$.

Consequently, the choice of δ in the proof of Theorem 2 is independent of t_0. This completes the proof.

Theorem 5. Let the conditions of Theorem 3 hold. Then solutions of (1.1) are in $L^1[0,\infty)$ if and only if $\alpha(t)<0$.

Proof. Suppose $\alpha(t)<0$. Then, for $v=1$, it follows from (i) of Theorem 3 and (2.6) that $V'_{(2.3)}(t,x(\cdot))\leq(\mu-1)a|x(t)|$.

Integration of both sides from t_0 to t yields

$$0\leq V(t,x(\cdot))\leq V(t,\phi(\cdot))+(\mu-1)a\int_{t_0}^{t}|x(s)|\mathrm{d}s$$

and hence $x(t)$ is in $L^1[0,\infty)$.

Suppose now that $\alpha(t)>0$ and show that $x(t)$ is not in L^1. If $x(t)$ is in L^1, then it follows as in the proof of Theorem 3 that (2.9) and (2.10) imply that $x(t)$ is bounded away from zero for all sufficiently large t. This contradicts $x(t)$ being in L^1. The proof is now complete.

Miller [4] showed that in the convolution equation

$$x'=Ax+\int_0^t C(t-s)x(s)\mathrm{d}s \tag{2.12}$$

solutions are in L^1 if and only if the zero solution is uniformly asymptotically stable. Obviously, when $C(t,s)=C(t-s)$, the function $H(t,s)$ defined by (2.1) and (2.2) reduces to $H(t-s)$. Thus, if $\alpha(t)=\alpha=$constant and $H(t)$ is the solution of the initial-value problem

$$H'(t)-\alpha H(t)=C(t) \tag{2.13}$$

$$H(0)=A-\alpha, \tag{2.14}$$

we obtain the following result.

We may combine Theorem 5 and Miller's result to obtain the following theorem.

Theorem 6. Let $H(t)$ be defined by (2.13) and (2.14) and suppose that $H(t)\to 0$ as $t\to\infty$ and there is a non-zero constant α such that

$$\int_0^\infty |H(t)|\mathrm{d}t<1.$$

Then the zero solution of (2.12) is uniformly asymptotically stable if and only if $\alpha<0$.

REFERENCES

[1] T. A. Burton and W. E. Mahfoud, 'Stability Criteria for Volterra Equations', *Transactions of the American Mathematical Society*, **279** (1983), pp. 143–174.

[2] F. Brauer, 'Asymptotic Stability of a Class of Integro-Differential Equations', *Journal of Differential Equations*, **28** (1978), pp. 180–188.

[3] R. D. Driver, Existence and Stability of Solutions of a Delay-Differential System', *Archives of Rational Mechanics and Analysis*, **10** (1962), pp. 401–426.

[4] R. K. Miller, 'Asymptotic Stability Properties of Linear Volterra Integro-Differential Equations', *Journal of Differential Equations*, **10** (1971), pp. 485–506.

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