HIGHER ORDER SHOCK STRUCTURE FOR A CLASS OF GENERALIZED BURGERS' EQUATIONS

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الخلاصة :

نعالج في هذا البحث فئة من معادلات برجر العامة حيث تكون اللاخطيه بشكل اعتباطي بينما يكون التبديد خطأ وذو معامل صغير ، ك . نبين هنا أن الحل لهذه المعادلات يؤدى الى هزات وانه يمكن الحصول على الحل بدقه داخل مجال هذه الهزات وخارجه بواسطه تمديدات مقاربه متناسبه . كما نبين امكانيه تعيين موضع الهزه لادنى رتبه ممكنه بواسطة قاعدة ويثمان لتساوى المساحات باحدى وجوهها العامة كذلك نشتق تعبيرا عاما وصريحا برتبة ك للازاحة لا بتهيل الناتجه عن الانتشار .

ABSTRACT

A class of generalized Burgers' equations is considered in which the non-linearity is of arbitrary form whereas the dissipation is linear with small coefficient k. The solution is shown to develop shocks and the structure of the solution both within the shock and in the outer region is obtained accurately to order k by means of matched asymptotic expansions. It is shown that to lowest order the shock position can be determined by an extended version of Whitham's equal-areas rule whereas to order k a general explicit expression for Lighthill's 'displacement due to diffusion' is derived.

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1. INTRODUCTION

We consider the following generalization of Burgers' equation:

$$u_\tau - p(u)u_x = k(u_{xx} + cu_{x\tau}), \tag{1}$$

where $u = u(x, \tau)$. Burgers' equation itself corresponds to the case $p(u) = \lambda u$, $c = 0$. Equations of this type arise frequently in problems of wave propagation in the presence of non-linearity and dissipation. Examples of such problems are acoustic waves in fluids [1–4], waves in weakly viscoelastic solids [5–7], flood waves in rivers, and waves in traffic flow [8, 2].

It is well known (see, for example, [2, 9]) that when $k = 0$, the solution of Equation (1) for general initial values $u(x, 0)$ ceases to be uniquely defined for τ greater than some critical value τ_f. When $\tau > \tau_f$ the solution contains a discontinuity, or shock, whose position can be found from the usual jump conditions. When k is non-zero but sufficiently small, this discontinuity is replaced by a narrow shock layer in which the solution changes rapidly but smoothly from one value to another. When k is small, the position of the shock layer can again be calculated to lowest order in k from the jump conditions. However, there are higher order corrections to the shock position which require a more detailed investigation of the solution of (1).

In the case of Burgers' equation itself, the exact solution is known [10, 11] and it is possible to derive such properties as the detailed structure of the solution inside the shock layer and the higher order terms in the shock position from this complete solution [1]. With generalizations such as Equation (1), such properties can only be obtained with the aid of approximation techniques. Matched asymptotic approximations were first used for this purpose by Murray [12] with a class of equations which contains Equation (1), and this method has subsequently been used by a number of authors to investigate shock structure in different contexts [6, 7, 13–15]. In this paper, we shall show how matched asymptotic expansions can be used for Equation (1) to derive expressions accurate to order k for the solution inside the shock layer, and for the shock position (apart from an undetermined constant in the inner solution). Murray [12] derived an expression for this latter quantity by using a conserved integral, claiming that it could not be obtained by matching. However, this claim is not correct, as is clear from other particular calculations [13–15], and we shall show that higher order matching can be used successfully for the general class of Equations (1).

In Section 2 of the paper we derive the lowest order solution inside the shock layer. It is shown that in the special case $p(u) = \lambda u$, this solution is of hyperbolic tangent type for all values of c; this result is of course well known for $c = 0$. When $p(u)$ is non-linear, the structure of the shock layer is asymmetric in general. In Section 3 the lowest order solution outside the shock is obtained and matched to the inner solution. In the case $p(u) = \lambda u$, the matching conditions lead to Whitham's equal-areas rule for shock fitting [16, 1, 2]. We show that the equal-areas rule can be generalized to arbitrary $p(u)$ so as to provide a feasible method of calculating shock positions (though in general it is a less convenient method than Whitham's original rule).

In the remaining sections a similar calculation is carried out to first order in k. In Section 4 the inner solution is derived accurately to order k and its asymptotic form at the edges of the shock layer is obtained. In Section 5 the outer solution is obtained to order k and is matched to the inner solution. An explicit expression is derived for the order k displacement of the shock position, the 'displacement of the shock due to diffusion' [1]. In the case $p(u) = \lambda u$ this expression is obtained in simplified form and when $c = 0$ it agrees with Lighthill's classical result for Burgers' equation.

2. LOWEST ORDER SOLUTION IN THE SHOCK LAYER

In this section, we are interested in the form of the solution of Equation (1) in the vicinity of the shock. Therefore, if the position of the shock is $x = x_s(\tau)$, we introduce the stretched variable

$$\eta = k^{-1}[x - x_s(\tau)] \tag{2}$$

in place of x. Equation (1) then takes the form

$$(1 - cx_s')u_{\eta\eta} + [p(u) + x_s']u_\eta = k(u_\tau - cu_{\eta\tau}). \tag{3}$$

We seek the inner solution as an expansion in powers of k, retaining only the terms of order k:

$$u = u_0(\eta, \tau) + ku_1(\eta, \tau) + O(k^2). \tag{4}$$

Substituting (4) into (3) and comparing the coefficients of the two lowest powers of k, we obtain

$$(1-cx_s')u_{0_{\eta\eta}}+[p(u_0)+x_s']u_{0_\eta}=0 \tag{5}$$

$$(1-cx_s')u_{1_{\eta\eta}}+[p(u_0)+x_s']u_{1_\eta}+p'(u_0)u_{0_\eta}u_1 = u_{0_\tau}-cu_{0_{\eta\tau}}. \tag{6}$$

We shall return to Equation (6) later in the paper, and in this and the following sections we shall concentrate on the lowest approximation given by Equation (5). This equation may be integrated once immediately, to give

$$(1-cx_s')u_{0_\eta}+P(u_0)+x_s'u_0=A(\tau), \tag{7}$$

where $A(\tau)$ is the constant of integration and

$$P(u)=\int p(u)\mathrm{d}u. \tag{8}$$

As x increases from below to above the shock layer, the stretched variable η increases from $-\infty$ to $+\infty$. Denoting by $u_\pm$ the respective limits of u_0 as $\eta\to\pm\infty$, we have from (7) that

$$P(u_\pm)+x_s'u_\pm=A. \tag{9}$$

The limiting values $u_\pm$ must be matched with the limiting values of the outer solution above and below the shock, and then together with (9) we have a complete set of equations for x_s, A, and $u_\pm$ (see Section 3).

Integrating Equation (7), we obtain the following expression for the solution u_0 in the shock layer:

$$\eta+b(\tau)=(1-cx_s')\int_{u_L}^{u_0}\frac{\mathrm{d}u}{Q(u,\tau)}, \tag{10}$$

where

$$Q(u,\tau)=A-x_s'u-P(u) \tag{11}$$

and $b(\tau)$ is the constant of integration. It is convenient to choose u_L as some representative value of u_0, for example, the average value $(u_+ + u_-)/2$. Then we can regard $\eta=-b$ (or equivalently $x=x_s-kb$) as being the central position of the shock layer. The quantity (kb) therefore represents a displacement in space of order k in the position of the shock; Lighthill has termed it the 'displacement due to diffusion' [1]. It can only be determined from higher order matching (see Section 5).

Let us consider the shock structure given by (10) in the special case when p is a linear function of u,

$$p(u)=\lambda u. \tag{12}$$

We obtain from (11) that

$$Q(u,\tau)=\tfrac{1}{2}\lambda\,[a^2-(u+\lambda^{-1}x_s')^2], \tag{13}$$

where

$$a^2=\lambda^{-2}(2\lambda A+x_s'). \tag{14}$$

Therefore, from (10), the inner solution is found to be:

$$u_0=-\lambda^{-1}x_s'+a\tanh\xi, \tag{15}$$

where

$$\xi=\tfrac{1}{2}\lambda a(1-cx_s')^{-1}(\eta+b). \tag{16}$$

The hyperbolic tangent structure is a well-known property of the shock layers arising from Burgers' equation in the case of small damping (that is, when $p(u)=\lambda u$ and $c=0$). However, it is clear from (15), (16) that a hyperbolic tangent structure occurs for all c when $p(u)=\lambda u$.

The width of the shock layer does depend on c. From (15) we can take $|\xi|<1$ as a measure of the width, and hence from (2) and (16) we obtain the width in terms of physical variables to be:

$$|x-x_s+kb|<2k(1-cx_s')/\lambda a.$$

The shock structure when $p(u)$ is a quadratic function is given in [14] in the case $c=0$, and similar results apply in the present case. The notable feature is that the shock layer is no longer symmetrical about its center.

Certain general conclusions can be drawn from Equation (10) regarding possible shock structures. If $P(u)$ is an analytic function between u_- and u_+, then $Q(u,\tau)$ cannot vanish between these limits, otherwise the integral in (10) would diverge for some value of u_0 between $u_\pm$. Thus, $u_\pm$ are adjacent zeros of $Q(u,\tau)$ and, from (7), u_{0_η} has constant sign throughout the shock layer. Across the shock, u_0 changes monotonically from u_- to u_+, and *it is not possible to have oscillatory shock structures for the present class of equations.*

In general, Q will have simple zeros at $u_\pm$, and u always approaches $u_\pm$ exponentially as $\eta\to\pm\infty$. If we define

$$c_\pm=(1-cx_s')^{-1}q_\pm, \tag{17}$$

where

$$q_\pm=Q_u(u_\pm,\tau)=-x_s'-p(u_\pm), \tag{18}$$

then the asymptotic behavior of the solution is

$$u-u_\pm\sim\exp[-c_\pm\eta+\text{const.}]\ \text{as}\ \eta\to\pm\infty. \tag{19}$$

In general, $c_+\neq c_-$, so the two tails have different lengths.

Finally we observe from (10) that the sole effect of including the u_{xt} term in Equation (1) on the shock structure (in lowest order) is to rescale the spatial variable by a factor $(1-cx_s')$.

In the case of a steady shock (generalized Taylor shock) propagating with constant speed between uniform asymptotic states, the solution given by (10) is exact to all orders, with $b(\tau)=0$. With non-uniform outer states, this is not the case.

3. THE OUTER SOLUTION AND SHOCK FITTING

The solution outside the shock is obtained by seeking the solution of Equation (1) as a straightforward power series in k:

$$u=u_0(x,\tau)+ku_1(x,\tau)+\ldots.$$

Substituting and comparing powers of k, we obtain the equations

$$u_{0\tau}-p(u_0)u_{0x}=0$$
$$u_{1\tau}-p(u_0)u_{1x}-p'(u_0)u_{0x}u_1=u_{0xx}+cu_{0x\tau}. \quad (20)$$

Again, the second of these equations will be reserved for later discussion. The solution of the first is

$$u_0=\phi(\beta), \quad (21)$$

where β is a characteristic variable defined by:

$$\beta=x+\tau p[\phi(\beta)]\equiv x+\tau\psi(\beta) \quad (22)$$

in which $\psi(\beta)=p[\phi(\beta)]$.

The function ϕ is determined by the initial values of u since when $\tau=0$ we have $x=\beta$ and hence $\phi(\beta)=u_0(\beta,0)$. For a general point (x,τ), (22) then provides an implicit equation for β and hence allows u_0 to be found from (21).

In general, (22) determines a unique value of β for all x only when τ is less than some critical value. This critical value of τ is given by $\tau_f=[\psi'(\beta_m)]^{-1}$, where β_m is the value of β at which $\psi'(\beta)$ has its greatest positive value. For $\tau\geq\tau_f$, a shock must be included in the solution. The point at which the shock forms is given by $\tau=\tau_f$, $x=x_f\equiv\beta_m-\tau_f\psi(\beta_m)$.

A typical configuration involving the presence of a shock is illustrated in Figure 1. At a point $x=x_s(\tau)$ on the shock path there arrive two characteristics with characteristic coordinates $\beta^{\pm}$, one to the right and one to the left of the shock.

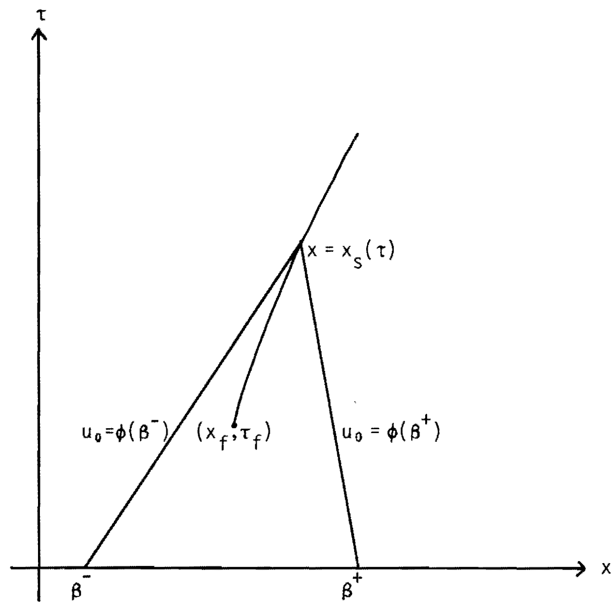

Figure 1. Two Characteristics Converging on the Shock Path

Then, from (22), we have

$$\beta^{\pm}=x_s+\tau\psi(\beta^{\pm}). \quad (23)$$

Furthermore, the signals $\phi(\beta^{\pm})$ arriving at the shock must agree with the limits $u_{\pm}$ of the inner solution obtained from (9):

$$u_{\pm}=\phi(\beta^{\pm}). \quad (24)$$

Eliminating A from the two equations (9), and using (24), we obtain

$$x_s'[\phi(\beta^+)-\phi(\beta^-)]=-\{P[\phi(\beta^+)]-P[\phi(\beta^-)]\}. \quad (25)$$

We note that (25) is equivalent to the usual jump condition which can be obtained directly from Equation (1) in the limit $k\to 0$. Equations (23) and (25) provide three equations from which $\beta^{\pm}$ and x_s can be determined.

In the special case of Burgers' equation, it is well known that these three equations lead to a geometrical construction, called the equal-areas rule, which allows the shock position to be found at any time [16, 1, 2]. It turns out that an equal-areas rule can also be given for an arbitrary function p, although when p is non-linear the rule becomes less convenient as a practical tool. In order to develop the generalized rule, we first consider the graphs of the functions $y=\psi(\beta)$ and $y=\tau^{-1}(\beta-x_s)$ in the $y\beta$-plane (see Figure 2).

From (23), the points A, B, where these graphs intersect, have $\beta=\beta^{\pm}$. The second graph is a straight

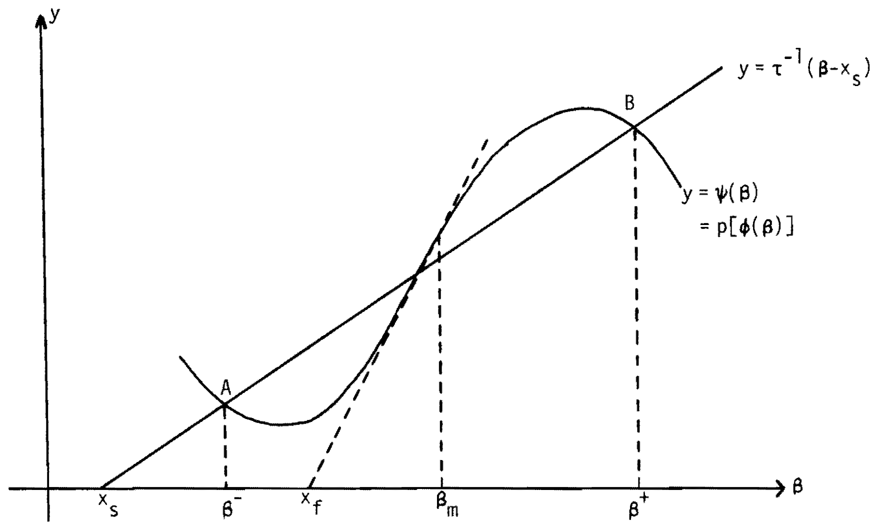

Figure 2. First Stage of the Equal-Areas Construction

line of slope τ^{-1} which meets the β-axis at x_s. At the instant of formation of the shock, $\tau=\tau_f$, the chord AB becomes the tangent to the curve at its point of inflection. The slope of the tangent is τ_f^{-1}, and it meets the β-axis at the position of formation x_f. For this limiting case, $\beta^-=\beta^+$.

When p is a linear function of u, it can be shown that the chord in Figure 2 cuts off equal areas above and below the curve $y=\psi(\beta)$. In the general case, this is not true. However, a similar result is found to hold if we map the graphs of Figure 2 onto the $z\beta$-plane where $y=p(z)$. The chord AB becomes a curve whose equation is $\beta=x_s+\tau p(z)$ while the curve in Figure 2 now has the equation $z=\phi(\beta)$ (see Figure 3).

The two curves again intersect at $\beta=\beta^\pm$ and the curve AB meets the β axis at x_s[provided that $p(0)=0$ which can always be arranged by suitable re-writing of the original equation].

Denoting by $A(\tau)$ the difference between the two shaded areas in Figure 3, we have that

$$A(\tau)=\int_{\beta^-}^{\beta^+}\left[\phi(\beta)-p^{-1}\left(\frac{\beta-x_s}{\tau}\right)\right]\mathrm{d}\beta,$$

where p^{-1} denotes the inverse function of p. Therefore,

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}\tau}=&\frac{\mathrm{d}\beta^+}{\mathrm{d}\tau}\left[\phi(\beta^+)-p^{-1}\left(\frac{\beta^+-x_s}{\tau}\right)\right]\\&-\frac{\mathrm{d}\beta^-}{\mathrm{d}\tau}\left[\phi(\beta^-)-p^{-1}\left(\frac{\beta^+-x_s}{\tau^-}\right)\right]\\&-\int_{\beta^-}^{\beta^+}\frac{\partial}{\partial\tau}\left[p^{-1}\left(\frac{\beta-x_s}{\tau}\right)\right]\mathrm{d}\beta=\end{aligned}$$

$$=-\int_{\beta^-}^{\beta^+}\frac{\partial z}{\partial\tau}\mathrm{d}\beta,$$

where $z=p^{-1}\left(\frac{\beta-x_s}{\tau}\right)$ and we have used (23). Hence,

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}\tau}&=\int_{\beta^-}^{\beta^+}\frac{1}{p'(z)}\left[\frac{\beta-x_s}{\tau^2}+\frac{x_s'}{\tau}\right]\mathrm{d}\beta\\&=\int_A^B[p(z)+x_s']\mathrm{d}z\\&=[P(z)+x_s'z]_A^B\end{aligned}$$

Since z changes from $\phi(\beta^-)$ to $\phi(\beta^+)$ between A and B, we conclude from (25) that $\mathrm{d}A/\mathrm{d}\tau=0$. Hence $A(\tau)=$const. But since $A(\tau_f)$ is clearly equal to zero, it follows that $A(\tau)=0$ for all $\tau\geq\tau_f$.

This equal-areas rule provides a quite feasible practical method of plotting the shock position at any time τ, although for a non-linear function p, the rule is obviously less convenient than the familiar one for Burgers' equation.

4. SECOND ORDER SOLUTION IN THE SHOCK LAYER

We shall now return to Equation (6) and investigate the second order inner solution u_1. Integrating this equation once, we get

$$(1-cx_s')u_{1\eta}+[p(u_0)+x_s']u_1=\int u_{0\tau}\mathrm{d}\eta-cu_{0\tau}+C(\tau).$$

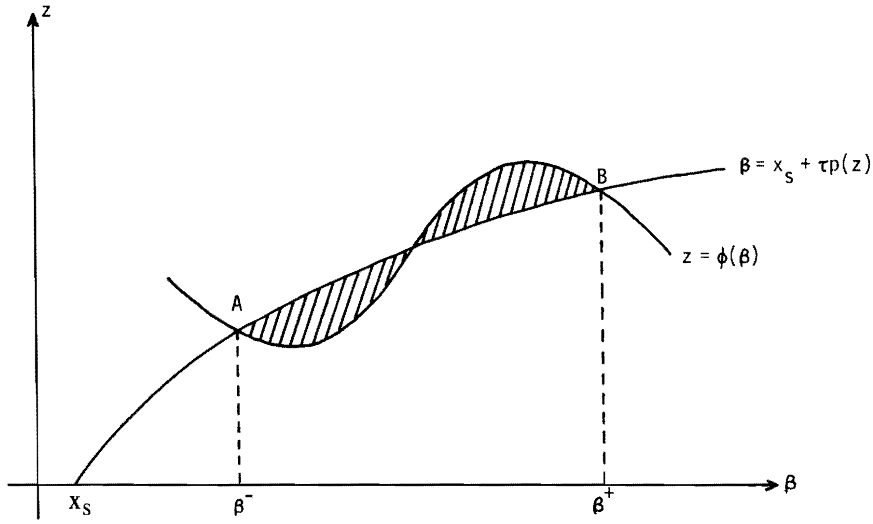

Figure 3. Final Stage of the Equal-Areas Construction

In view of Equation (5), the integrating factor for this equation is $u_{0\eta}^{-1}$, and it can be written in the form

$$(1-cx_s')u_{0\eta}\frac{\partial}{\partial\eta}\left(\frac{u_1}{u_{0\eta}}\right)=\int u_{0\tau}\mathrm{d}\eta-cu_{0\tau}+C(\tau). \quad (26)$$

In order to make use of the solution (10) it will be simpler to regard u_0 and τ as the independent variables and $\eta=\eta(u_0,\tau)$. Then, using standard transformation formulas, we can rewrite (26) in the following form:

$$(1-cx_s')\eta_{u0}^{-1}\frac{\partial}{\partial u_0}(\eta_{u0}u_1)=-\eta_{u_0}\int\eta_\tau\,\mathrm{d}u_0 +c\eta_\tau+C(\tau)\eta_{u0}. \quad (27)$$

From (10) and (11) we obtain that

$$\eta_{u0}=(1-cx_s')Q(u_0)^{-1},$$

$$\eta_\tau=-b_1-cx_s''\int_{u_L}^{u_0}Q(u)^{-1}\,\mathrm{d}u -(1-cx_s')\int_{u_L}^{u_0}(A'-x_s''u)Q(u)^{-2}\,\mathrm{d}u,$$

where the τ-dependence has been suppressed throughout and

$$b_1=b'(\tau)+(1-cx_s')u_L'(\tau)Q(u_L)^{-1}. \quad (28)$$

Substituting these expressions into (27), using the same lower limit u_L for all the integrals for convenience, we obtain after some manipulation the final solution

$$\begin{aligned}Q(u_0)^{-1}u_1&=b_1\int_{u_L}^{u_0}(u-u_L)Q(u)^{-2}\,\mathrm{d}u\\&+cx_s''\int_{u_L}^{u_0}Q(u)^{-2}\int_{u_L}^{u}(u-u')Q(u')^{-1}\,\mathrm{d}u'\,\mathrm{d}u\\&+(1-cx_s')\int_{u_L}^{u_0}Q(u)^{-2}\\&\int_{u_L}^{u}(u-u')(A'-x_s''u')Q(u')^{-2}\,\mathrm{d}u'\,\mathrm{d}u\\&-cb_1(1-cx_s')^{-1}\int_{u_L}^{u_0}Q(u)^{-1}\,\mathrm{d}u\\&-c^2x_s''(1-cx_s')^{-1}\int_{u_L}^{u_0}Q(u)^{-1}\int_{u_L}^{u}Q(u')^{-1}\,\mathrm{d}u'\,\mathrm{d}u\\&-c\int_{u_L}^{u_0}Q(u)^{-1}\int_{u_L}^{u}(A'-x_s''u')Q(u')^{-2}\,\mathrm{d}u'\,\mathrm{d}u\\&+C(\tau)\int_{u_L}^{u_0}Q(u)^{-2}\,\mathrm{d}u+E(\tau).\end{aligned} \quad (29)$$

For given $p(u)$, the solution u_1 can be explicitly evaluated from (29). When $p(u)=\lambda u$, u_0 is given by (15), (16) (this corresponds to the choice $u_L=-\lambda^{-1}x_s'$).

With the same notation, we obtain that

$$\lambda u_1 = b'[\tanh^2\xi - ca\lambda(1-cx_s')^{-1}\xi\,\mathrm{sech}^2\xi]$$
$$+(a\lambda)^{-2}x_s''(1-cx_s')[\tanh^2\xi - 2\xi\tanh\xi - \xi^2\,\mathrm{sech}^2\xi]$$
$$+c(a\lambda)^{-1}x_s''\Big[2\xi+\tanh\xi-\xi\,\mathrm{sech}^2\xi$$
$$-2(\tanh\xi+\xi\,\mathrm{sech}^2\xi)\ln\cosh\xi$$
$$+2\,\mathrm{sech}^2\xi\int_0^\xi \xi\tanh\xi\,\mathrm{d}\xi\Big]$$
$$-c^2x_s''(1-cx_s')^{-1}\xi^2\,\mathrm{sech}^2\xi$$
$$+\lambda a'[(a\lambda)^{-2}(1-cx_s')(2\xi-\xi\,\mathrm{sech}^2\xi-\tanh\xi)$$
$$-c(a\lambda)^{-1}(\tanh^2\xi+\xi^2\,\mathrm{sech}^2\xi)]$$
$$+a^{-1}C(\tanh\xi+\xi\,\mathrm{sech}^2\xi)+\tfrac{1}{2}(a\lambda)^2E\,\mathrm{sech}^2\xi. \tag{30}$$

The inner solution to order k is then given by

$$u=-\lambda^{-1}x_s'+a\tanh\xi+ku_1. \tag{31}$$

The solutions thus obtained involve three as yet undetermined constants b, C, and E. The first two of these can be determined by matching with the outer solution to order k. (Presumably E would be found by matching to order k^2.)* In order to perform this matching we need the asymptotic behavior of the inner solution as $\eta\to\pm\infty$. In the linear example, we find from (30) and (31) that

$$u\sim -\lambda^{-1}x_s'\pm a+\frac{k}{\lambda^2 a}(\lambda a'\pm x_s'')(\eta+b)$$
$$+k\left[\frac{b'}{\lambda}+\frac{1}{\lambda^3a^2}x_s''(1-x_s')-\frac{ca'}{\lambda a}\right]$$
$$\pm k\left[\frac{C}{\lambda a}+\frac{cx_s''}{\lambda^2 a}(1+2\ln 2)-\frac{(1-cx_s')a'}{\lambda^2a^2}\right]. \tag{32}$$

In the general case we must evaluate the asymptotic behavior of the various integrals in (29) as $u_0\to u_\pm$ in order to find the behavior of u_1 as $\eta\to\pm\infty$. Assuming that $Q(u_0)$ has only simple zeros at $u_0=u_\pm$, the non-zero asymptotic contributions to u_1 arise only from those terms on the right of (29) which are at least as singular as $(u_0-u_\pm)^{-1}$ in the limit. Using standard techniques to evaluate these leading singular terms, we find first of all from (10) that as $u_0\to u_\pm$

$$(1-cx_s')^{-1}(\eta+b)\sim\frac{1}{q_\pm}\ln\left|\frac{u_0-u_\pm}{u_\mathrm{L}-u_\pm}\right|$$
$$+\int_{u_\mathrm{L}}^{u_\pm}\left[\frac{1}{Q(u)}-\frac{1}{q_\pm(u-u_\pm)}\right]\mathrm{d}u,$$

where $q_\pm=-x_s'-p(u_\pm)$. Then from (29) we find that

$$u_1\sim\left[\frac{A'-x_s''u_\pm}{q_\pm^2}\right](\eta+b-c)$$
$$+\frac{1-cx_s'}{q_\pm^3}[2(A'-x_s''u_\pm)-x_s''(u_\pm-u_\mathrm{L})]$$
$$-\frac{1}{q_\pm}\Big[b'(u_\pm-u_\mathrm{L})+C-cx_s''\int_{u_\mathrm{L}}^{u_\pm}\frac{(u-u_\pm)}{Q(u)}\mathrm{d}u$$
$$+(1-cx_s')\frac{u_\mathrm{L}'(u_\pm-u_\mathrm{L})}{Q(u_\mathrm{L})}\Big]$$
$$+\frac{(1-cx_s')}{q_\pm^3}\int_{u_\mathrm{L}}^{u_\pm}\left[\frac{q_\pm(u-u_\pm)}{Q(u)}-1\right]\times$$
$$\left[q_\pm\frac{(A'-x_s''u)}{Q(u)}-x_s''\right]\mathrm{d}u, \tag{33}$$

after substituting from (28) for b_1. Making use of the fact that $A'-x_s''u_\pm=-q_\pm u_\pm'$, we can write this asymptotic formula in the following, much neater form:

$$q_\pm u_1\sim -u_\pm'(\eta+b-c)-b'(u_\pm-u_\mathrm{L})$$
$$-C-(1-cx_s')q_\pm^{-1}u_\pm'$$
$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{(1-cx_s')\int_{u_\mathrm{L}}^{u_\pm}\frac{(u-u_\pm)}{Q(u)}\mathrm{d}u\right\}. \tag{34}$$

5. SECOND ORDER SOLUTION OUTSIDE THE SHOCK LAYER

We shall now return to Equation (20) and calculate the corresponding second term in the outer solution. The first term u_0 in the outer region is given by (21) and (22). In solving for u_1 in this region, it is convenient to use the characteristic variable β in place of x. In terms of β and τ it is readily shown that Equation (20) takes the form

$$D^{-1}\frac{\partial}{\partial\tau}(Du_1)=(1+c\psi)\left\{\left[\phi''-\frac{\phi'\psi''}{\psi'}\right]D^{-2}\right.$$
$$\left.+\frac{\phi'\psi''}{\psi'}D^{-3}\right\}+c\phi'\psi'D^{-2}, \tag{35}$$

*This presumption is disputed by Kevorkian and Cole [15] on the grounds that the E term in (30) is exponentially small as $\xi\to\pm\infty$. However, we note that the same is true of the contribution of b to the lowest order solution ((15), for example) in the limit $\xi\to\pm\infty$.

where $\phi=\phi(\beta)$, $\psi=\psi(\beta)$, primes denote derivatives with respect to β, and $D=1-\tau\psi'(\beta)$. This equation can be immediately solved for u_1. The complementary function is $D^{-1}\phi_1(\beta)$, where $\phi_1(\beta)$ is determined by the initial conditions on the total solution u_0+ku_1. Generally it will be the case that the initial values of u are independent of k and the appropriate condition in such circumstances is that $u_1=0$ when $\tau=0$. With this condition, the solution for u_1 is

$$\begin{aligned} u_1 &= (1+c\psi)(\phi'\psi''/\psi'^2)(D^{-2}-D^{-1}) \\ &\quad -[(1+c\psi)(\phi'/\psi')'+c\phi']D^{-1}\ln|D|. \end{aligned} \tag{36}$$

This result agrees with Murray's equation (69) [12] except that the last term in this latter equation should be absent. (The error originates in the expression (65).)

In the special case $p(u)=\lambda u$, the solution given by (35) and (36) takes the form

$$u_1=[1+c\lambda\phi(\beta)]\frac{\phi''(\beta)}{\lambda\phi'(\beta)}[D^{-2}-D^{-1}]-c\phi'(\beta)\,D^{-1}\ln|D|,$$

where now $D=1-\lambda\tau\phi'(\beta)$.

For $\tau\geq\tau_f$, the outer solution $u=\phi(\beta)+ku_1$ must be matched on each side of the shock layer with the limiting values of the inner solution. To do this we set $x=x_s+k\eta$ in the outer solution and expand the result in powers of k. The corresponding value of β will be close to β^+ or β^-, depending on whether $x>x_s$ or $x<x_s$. From (22) and (23), to first order in k we get

$$\beta-\beta^{\pm}=D_{\pm}^{-1}(x-x_s)=D_{\pm}^{-1}k\eta,$$

where $D_{\pm}=1-\tau\psi'(\beta^{\pm})$. Therefore,

$$u_0=\phi(\beta)\simeq\phi(\beta^{\pm})+\phi'(\beta^{\pm})D^{-1}k\eta. \tag{37}$$

Combining (35)–(37) we get the following expression for the inner expansion of the outer solution:

$$u\sim\phi(\beta^{\pm})+\phi'(\beta^{\pm})D_{\pm}^{-1}k\eta+kK_{\pm},$$

where

$$\begin{aligned} K_{\pm} &= [1+c\psi(\beta^{\pm})]\frac{\phi'(\beta^{\pm})\psi''(\beta^{\pm})}{[\psi'(\beta^{\pm})]^2}[D_{\pm}^{-2}-D_{\pm}^{-1}] \\ &\quad -\left\{[1+c\psi(\beta^{\pm})]\left[\frac{\phi'(\beta^{\pm})}{\psi'(\beta^{\pm})}\right]' \right.\\ &\quad \left. +c\phi'(\beta^{\pm})\right\}D_{\pm}^{-1}\ln|D_{\pm}|. \end{aligned}$$

The corresponding outer limit of the inner solution is given by $u\sim u_{\pm}+ku_1$, where the limiting expression for u_1 is given by (34), or by (32) in the special case $p(u)=\lambda u$. Matching the leading terms of the two limits simply reproduces (24). The terms proportional to η give the conditions

$$D_{\pm}^{-1}\phi'(\beta^{\pm})=-q_{\pm}^{-1}u'_{\pm}.$$

It is not difficult to see that these conditions can be obtained by differentiating (24) with respect to τ. The only new conditions arise from matching the terms proportional to k, and they are:

$$\begin{aligned} & u'_{\pm}(b-c)+b'(u_{\pm}-u_L)+C+(1-cx'_s)q_{\pm}^{-1}u'_{\pm} \\ &\quad +\frac{d}{d\tau}\left\{(1-cx'_s)\int_{u_L}^{u_{\pm}}\frac{u-u_{\pm}}{Q(u)}du\right\}=-q_{\pm}K_{\pm}. \end{aligned} \tag{38}$$

These equations are sufficient to determine the unknowns $b(\tau)$ and $C(\tau)$. By using the conditions

$$u'_{\pm}=\phi'(\beta^{\pm})d\beta^{\pm}/d\tau,\ q_{\pm}=-x'_s-\psi(\beta^{\pm})=-D_{\pm}d\beta^{\pm}/d\tau,$$

we can show that

$$q_{\pm}K_{\pm}+(1-cx'_s)\frac{u'_{\pm}}{q_{\pm}}=\frac{d}{d\tau}\left\{\frac{1+cp(u_{\pm})}{p'(u_{\pm})}\ln|D_{\pm}|+cu_{\pm}\right\}$$

and therefore the matching conditions (38) take the simpler form

$$\begin{aligned} & \frac{d}{d\tau}\left\{u_{\pm}b+(1-cx'_s)\int_{u_L}^{u_{\pm}}\frac{u-u_{\pm}}{Q(u)}du\right. \\ &\quad \left.+\frac{1+cp(u_{\pm})}{p'(u_{\pm})}\ln|D_{\pm}|\right\}=u_L b'-C. \end{aligned} \tag{39}$$

The difference between these two conditions can be integrated immediately to give an explicit expression for $b(\tau)$. Since $\beta^+=\beta^-$ when $\tau=\tau_f$, the constant of integration is zero, and the result is:

$$\begin{aligned} (u_+-u_-)b &= -(1-cx'_s)\left\{\int_{u_L}^{u_+}\frac{u-u_+}{Q(u)}du-\int_{u_L}^{u_-}\frac{u-u_-}{Q(u)}du\right\} \\ &\quad -\left\{\frac{1+cp(u_+)}{p'(u_+)}\ln|D_+|\right. \\ &\quad \left.-\frac{1+cp(u_-)}{p'(u_-)}\ln|D_-|\right\}. \end{aligned} \tag{40}$$

We can verify from (40) that $b+(1-cx'_s)\int^{u_L}\frac{du}{Q(u)}$ is independent of the choice of u_L, as it should be from (10).

In the particular case $p(u)=\lambda u$, the expression for b becomes

$$b=-\frac{1}{2\lambda a}\left\{(1-cx'_s)\ln\left|\frac{D_+}{D_-}\right|+c\lambda a\ln|D_+D_-|\right\},$$

where $D_{\pm}=1-\lambda\tau\phi'(\beta^{\pm})$. When $c=0$, this reproduces Lighthill's expression [1] for the displacement of the shock wave due to diffusion. Also in this particular case, the conditions (39) give the following result for C:

$$C=\frac{bx_s''}{\lambda}-\frac{1}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{(1-cx_s')\ln|D_+D_-|\right.$$

$$\left.+c\lambda a\ln\left|\frac{D_+}{D_-}\right|+cx_s'4\ln 2\right\}.$$

In the particular case $p(u)=\mu u^2+\nu u^3$, $c=0$, the result (40) is in agreement with Equation (37) of [14] when the notation changes $f_3-f_2\rightarrow u_+-u_-$ and $2b\rightarrow b$ are made. We see that the effects of including the cu_{xt} term in the model Equation (2) of [14] are to multiply b by a factor $(1-cx_s')$ and to transform $p'(u)$ into $p'(u)(1-cx_s')/[1+cp(u)]$.

Finally, we should compare the present results with those obtained by Murray [12]. Murray did not determine the higher order inner solution as we have done in Section 4. He obtained an expression for $b(\tau)$ $(=A(t)$ in his notation) by making use of a conserved integral, the result being given in his Equation (84). Because of the error in his outer solution, the middle term in this result should be absent. Also, the first term has the wrong sign, originating from the equation preceding his Equation (82). (The conserved integral approach is described in [13].) With these corrections, Equation (84) of [12] agrees with our result (40).

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