AN UPPER BOUND ON THE FIBONACCI NUMBER OF A MAXIMAL OUTERPLANAR GRAPH

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الخلاصــة :

إن عدد فيبوناتشي ف(خ) لمخطط بسيط خ هو عدد المخططات الجزئية الكاملة لمكملة المخطط خ . نجد في هذا البحث حداً أعلى للعدد ف لمجموعة أكبر مخططات سطحية ذات رووس خارجية ونبرهن بأن هذا العدد هو أفضل حد ممكن ، وبالتحديد نستنتج أن ف(خ) لا يزيد عن فن + ١ ، حيث أن خ هو أكبر مخطط سطحي ذو رؤوس خارجية لا يقل عددها عن ثلاثة ، وفن هو العدد النوني في متتالية فيبوناتشي .

ABSTRACT

The Fibonacci number $f(G)$ of a simple graph G is the number of complete subgraphs of the complement graph of G. An upper bound on f is established for all maximal outerplanar graphs and this bound is proved to be the best possible. In particular, it is shown that $f(G) \leq F_n + 1$, where G is a maximal outerplanar graph of order $n \geq 3$ and F_n is the nth number in the Fibonacci sequence.

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1. INTRODUCTION

The Fibonacci numbers F_n are defined by

$$F_0 = F_1 = 1,\ F_n = F_{n-1} + F_{n-2}\ (n \geq 2),$$

and the Lucas numbers by

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2} (n \geq 2).$$

In this paper we consider only finite undirected graphs without loops or multiple edges. Such graphs are said to be *simple*. The Fibonacci number $f(X)$ of a simple graph X with vertex set V and edge set E can be regarded [4] as the total number of subsets S of V such that no two vertices of S are adjacent. This is the same as the number of complete (induced) subgraphs of the complement graph of X. This includes the empty graph also. The following Lemmas are helpful [2–4].

Lemma 1
The Fibonacci number $f(P_n)$ of a path P_n is F_{n+1}.

Lemma 2
The Fibonacci number $f(C_n)$ of a cycle C_n is L_n.

Lemma 3
Let $X_1 = (V, E_1)$ and $X_2 = (V, E_2)$ be two graphs with $E_1 \subseteq E_2$, then $f(X_2) \leq f(X_1)$.

Lemma 4
Let $X = (V, E)$ be a graph and $y_1, y_2, \ldots, y_s$ vertices not contained in V. Then $Y = (V_1, E_1)$ denotes the graph with

$$V_1 = V \cup \{y_1, \ldots, y_s\} \text{ and}$$
$$E_1 = E \cup \{\{y_i, v_j\}\ 1 \leq i \leq s, v_j \in V\}.$$

Prodinger and Tichy [4] have shown by a simple induction argument that $f(Y) = f(X) + 2^s - 1$.

Lemma 5
A fan on k vertices, denoted by N_k, is the graph obtained from path $P_{k-1} = 2, 3, \ldots, k$ by making 1 adjacent to every vertex of P_{k-1}. By Lemmas 1 and 4 above we see that $f(N_k) = F_k + 1$.

Lemma 6
If X is a tree on n vertices, then $f(X) \leq 2^{n-1} + 1$.

Lemma 7
If X and Y are disjoint graphs, then $f(X \cup Y) = f(X) \cdot f(Y)$.

We propose in this paper to find an upper bound for $f(g)$ where G is a maximal outerplanar graph. Because maximal outerplanar graphs have a 'tree-like' structure of adjacencies, it is natural that their Fibonacci numbers be investigated as a follow-up to Lemma 6 above.

2. MAXIMAL OUTERPLANAR GRAPHS

A *planar* graph G is one that can be drawn in the plane so that no two edges intersect. We will refer to the regions defined by a planar graphs as its *faces*, the unbounded region being called the *exterior face*. G is *outerplanar* if it can be drawn in the plane so that no two edges intersect, and all of its vertices lie on the same face, which will be assumed to be the exterior face. *A maximal outerplanar graph* G is an outerplanar graph for which $G + (u, v)$ is not outerplanar for any pair u, v of vertices of G such that edge (u, v) is not already in G.

A maximal outerplanar graph is a graph that is isomorphic to a triangulation of a polygon and every such graph is Hamiltonian. All maximal outerplanar graphs can be constructed according to the following recursive rule [5]: (i) the triangle, K_3, is maximal outerplanar, and (ii) a maximal outerplanar graph with $n+1$ vertices can be obtained from some maximal outerplanar graph M with n vertices ($n \geq 3$), by adding a new vertex adjacent to two consecutive vertices on the Hamiltonian cycle of M. We also observe that any two nonadjacent vertices in a maximal outerplanar graph are separated by a pair of adjacent vertices. Moreover, these graphs satisfy the following lemma [3, p. 107].

Lemma 8
A maximal outerplanar graph has at least two vertices of degree 2.

All maximal outerplanar graphs of order 7 are shown in Figure 1.

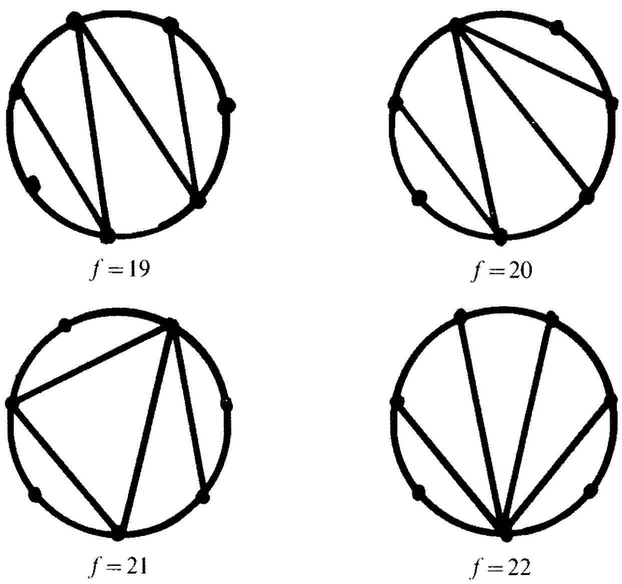

Figure 1. Fibonacci Numbers of Maximal Outerplanar Graphs of Order 7

Now, we state and prove the main result.

Theorem

The Fibonacci number $f(G)$ of a maximal outerplanar graph G of order $n \geq 3$ is bounded above by F_n+1. Moreover, this upper bound is the best possible.

Proof

By Lemma 5, the Fibonacci number of the fan N_k together with the fact that N_k is maximal outerplanar settles the second part of the theorem, that is, the upper bound is attained. Now let G be a maximal outerplanar graph of order n. We complete the proof by induction on n. By Lemma 8, there exists a vertex v of degree 2 in G. We consider two families of subsets of $V(G)$. Each subset in the first family contains v, whereas v is not in any subset in the second family. Let u and w be the neighbors of v in G. Deleting u and w we obtain the outerplanar graph $G_{u,w}$ of order $n-2$ with the isolated vertex v. Since G is a triangulation of a polygon, $G_{u,w}$ contains a path P_{n-3} of length n–3. By Lemma 1, $f(P_{n-3})=F_{n-2}$ and since v is a member of every subset of $V(G)$, $f(P_{n-3} \cup \{v\})=f(P_{n-3})$. Now by Lemma 3 above it follows that $f(G_{u,w}) \leq F_{n-2}$.

Next, we consider those admissible subsets of $V(G)$ not containing v. Let G_v be the remaining graph of order $n-1$ after deleting v. G_v is maximal outerplanar and the hypothesis of the theorem is satisfied. By induction $f(G_v) \leq F_{n-1}+1$. Combining the above results, we have

$$f(G)=f(G_{u,w})+f(G_v) \leq F_{n-2}+F_{n-1}+1=F_n+1.$$

We raise a few related but unsolved problems.

Question 1. Find a lower bound on f.

Question 2. Give a structural characterization of these graphs for which the lower bound holds.

Question 3. Discover a parallel theorem if G is maximal planar.

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