PARAMETRIC EXCITATION OF ELECTROHYDRODYNAMIC SURFACE WAVES

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الخلاصة :

أمكن تكبير الموجات الكهروهيدروديناميكية التي تنتشر خلال سطح أفقي يفصل مائعين عازلين بوضع مجال كهربي أفقي دوري ولقد وجد أن النظام يصل إلى حالة رنين عند قيم معينة لتردد المجال الكهربي . وقد أمكن الوصول إلى علاقة رياضية توضح إعتماد هذه التحت توافقيات على مختلف بارامترات النظام . وقد عرضنا أيضا دراسات مفصلة لبعض الحالات الخاصة عندما يكون التدعيم صغيراً أو كبيرا .

ABSTRACT

The electrohydrodynamic surface wave propagating through a horizontal interface between two- dielectrics can be amplified by introducing a horizontal periodic electric field. For certain values of the frequency of the field the system is set into a state of resonance. An expression for the dependence of such subharmonics on the various parameters of the system is obtained. Detailed studies for limiting cases of small and large modulation are introduced.

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I. INTRODUCTION

The theory of parametric excitation is associated with systems in which the parameters which define the natural frequencies when constant vary with time in a periodic manner. The behavior of such systems is described by an equation of the Hill or Mathieu type. It is well known [1–4] that the stability of such solutions may be described by means of the characteristic curves of the Mathieu functions which admit regions of resonance instability. It turns out that the resonance (instability) situation occurs if the natural frequency of a normally stable system is approximately an integral multiple of the frequency at which one of the parameters of the system varies periodically with time.

Such phenomena of excitation were first observed in fluid mechanics by Faraday [5] in the case of waves at an interface near a vibrating elastic surface. Recently Benjamin and Ursell [6] applied the theory of Mathieu equations to the problem of excitation of surface waves in a container which is partially filled with a fluid and which oscillates vertically. Kelley's investigation [7] concerned the stability of an interface between two fluids of different densities which flow parallel to each other in a periodic manner. He deduced that when the differences in the mean speed are below the steady, critical speed for instability but are large compared with the amplitude of the fluctuations, parametric amplification of waves at the interface occurs and the interface exhibits a resonance of a subharmonic nature.

Montgomery and Harding [8] and Vahala and Montgomery [9] have shown that Alfven waves in a plasma can be parametrically excited if a d.c. magnetic field impressed on a plasma is given a small low-frequency modulation. Crowley [10], Reynolds [11], Yih [12], and Mohamed and Nayyar [3] have studied experimentally and theoretically the excitation of surface waves of a capillary liquid jets and conducting fluids stressed by a time-dependent electric field. In electrohydrodynamics it is known [13, 14] that if the electric field parallel to the interface between two dielectric media satisfies certain conditions, there exists a possibility of having electrohydrodynamic surface waves at the interface of the two dielectric media. In this paper we shall study the excitation of such electrohydrodynamic surface waves propagating along the interface between two semi-infinite dielectric liquids by superimposing a tangential time-dependent electric field on the already existing tangential electrostatic field. Thus one can generate the necessary subharmonics required for the parametric excitations of the electrohydrodynamic surface waves. For certain values of the frequency of the superimposed time-dependent electric field the system is set in a state of resonance and an expression giving the dependence of such subharmonics in terms of the various parameters of the system is obtained. The limiting cases when the strength of the superimposed electric field is small or large relative to the electrostatic field are examined in detail. On the other hand, the above model is considered as the electrohydrodynamic analog to the hydrodynamic periodic flow introduced by Kelley to amplify surface waves.

2. FORMULATION OF THE PROBLEM

The system discussed here consists of two semi-infinite homogeneous dielectric fluids of densities ρ_u and ρ_l and dielectric constants ε_u and ε_l (Figure 1). The two fluids are separated by a horizontal interface $z=0$ and the subscripts or superscripts u and l refer to quantities in the upper fluid and lower fluid respectively. It is known that if a tangential constant electric field $\bar{E}$ acting in the x direction is applied to the system, the field has a stabilizing effect on the system when subjected to a small disturbance [11]. For an electric field intensity exceeding a critical value, electrohydrodynamic surface waves propagate through the interface. The critical value of the electric field will be evaluated in the coming analysis (Section 3). No volume changes are present in the bulk of the fluids. Also, because of the continuity of the tangential

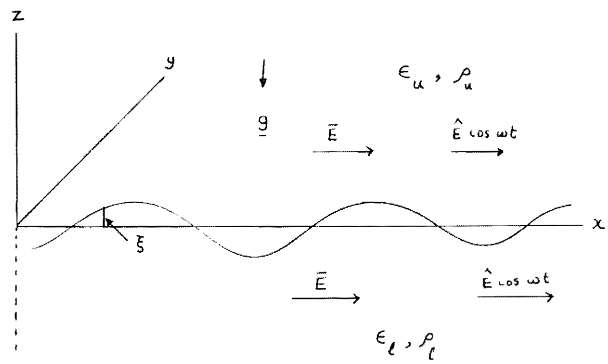

Figure 1. Sketch of the System under Consideration

electric field, no surface changes are present at the interface in the equilibrium state and will therefore vanish during the perturbations [15]. In order to produce parametric excitations in the electrohydrodynamic surface waves, we superimpose a modulated electric field $\hat{E}\cos\omega t$ in the x direction on the already existing constant field $\bar{E}$.

In our analysis, we assume that the quasi-static approximation is valid and there exists an electrostatic potential ϕ such that

$$\mathbf{E}=-\nabla\phi$$

The equations of motion for an inviscid incompressible fluid are given by

$$\rho\frac{d\mathbf{v}}{dt}=-\nabla p+\mathbf{g}, \tag{1}$$

with

$$\nabla.\mathbf{v}=0, \tag{2}$$

$\mathbf{v}$, $\mathbf{g}$ and p are the fluid velocity, the acceleration due to gravity in the negative z direction, and the pressure respectively.

The potential ϕ satisfies the Laplace equation

$$\nabla^2\phi=0. \tag{3}$$

It is readily seen that the equilibrium state solution is

$$p_0^{u,l}=-p^{u,l}g_z+c^{u,l}. \tag{4}$$

The subscript 0 refers to the equilibrium state. If we imagine a small departure from the equilibrium state, the linearized equations of motion become

$$\rho\frac{\partial\mathbf{v}_1}{\partial t}=-\nabla p_1, \tag{5}$$

$$\nabla.\mathbf{v}_1=0, \tag{6}$$

$$\nabla^2\phi_1=0. \tag{7}$$

The subscript 1 refers to the perturbed quantities. From Equations (5) and (6) we obtain

$$\nabla^2 p_1=0. \tag{8}$$

We assume that the various perturbed quantities have the following space and time dependence:

$$F(x,y,z,t)=f(z,t)\ e^{i(k_x x+k_y y)}. \tag{9}$$

As a result of the perturbation, the equilibrium plane interface between the two fluids becomes deformed and we assume that the equation of the deformed surface for the present case is given by

$$z=\xi, \tag{10}$$

where

$$\xi=\delta\gamma(t)\ e^{i(k_x x+k_y y)}, \tag{11}$$

and $\gamma(t)$ is a function of time to be solved, and δ is small.

Making use of the dependence given by Equation (9) in Equations (7) and (8) we obtain the following differential equations:

$$(D^2-k^2)\phi_1=0, \tag{12}$$

$$(D^2-k^2)p_1=0, \tag{13}$$

where

$$D=\frac{d}{dz}\ \text{and}\ k^2=k_x^2+k_y^2.$$

These equations admit the following type of solutions for ϕ_1 and p_1:

$$\phi_1^u=A_u(t)\ e^{kz+i(k_x x+k_y y)}, \tag{14}$$

$$p_1^u=B_u(t)\ e^{kz+i(k_x x+k_y y)}, \tag{15}$$

and for the lower fluid

$$\phi_1^l=A_l(t)\ e^{-kz+i(k_x x+k_y y)}, \tag{16}$$

$$p_1^l=B_l(t)\ e^{-kz+i(k_x x+k_y y)}, \tag{17}$$

where A_u, A_l, B_u, and B_l are time-dependent constants of integration which are to be evaluated by making use of the boundary conditions.

The unit normal vector $\mathbf{N}$ to the deformed interface between the two fluids is given by

$$\mathbf{N}=\frac{\nabla F}{|\nabla F|}, \tag{18}$$

where

$$F=z-\xi.$$

Thus, from Equations (11) and (18), we obtain

$$\mathbf{N}=\mathbf{e}_z-i\delta\gamma(t)\{k_x\mathbf{e}_x+k_y\mathbf{e}_y\}e^{i(k_x x+k_y y)}, \tag{19}$$

where $(\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)$ are the unit vectors along the x, y, and z axes respectively.

Equation (19) along with the continuity of the electric potential and the normal component of the electric displacement at the interface lead to the following relations:

$$A_0=A_l=\frac{ik_x\gamma(t)}{k}\frac{(\varepsilon_u-\varepsilon_l)}{\varepsilon_u+\varepsilon_l}\{\bar{E}+\hat{E}\cos\omega t\}. \tag{20}$$

Again the continuity of normal component of velocity at the interface along with its compatibility with the assumed surface deformation given by Equation (11) lead to

$$B_u = \frac{\rho_u}{k}\frac{d^2\gamma}{dt^2}, \tag{21}$$

$$B_l = -\frac{\rho_l}{k}\frac{d^2\gamma}{dt^2}. \tag{22}$$

Finally the continuity of the normal component of the stress tensor at the interface $z=\xi$ implies that

$$p^u(\xi) - p^l(\xi) = \tfrac{1}{2}\varepsilon_u E^{u^2} - \tfrac{1}{2}\varepsilon_l E^{l^2} + T\nabla^2\xi, \tag{23}$$

where T is the surface tension.

Note that

$$\begin{aligned} p^{u,l}(\xi) &= p_0^{u,l}(\xi) + p_1^{u,l}(\xi) \\ &= p_0^{u,l}(0) + \xi\left.\frac{\partial p_0^{u,l}}{\partial z}\right|_{z=0} + p_1^{u,l}(\xi) \\ &= c^{u,l} - \rho^{u,l} g\xi + p_1^{u,l}(\xi). \end{aligned} \tag{24}$$

Substituting from Equations (14)–(17) and (24) into Equation (23) we obtain for the first-order problem the following differential equation for $\gamma(t)$:

$$\begin{aligned} &\frac{d^2\gamma}{dt^2} + \frac{k}{\rho_u+\rho_l}\left[k^2 T - g(\rho_u - \rho_l)\right] \\ &+ \frac{k_x^2(\varepsilon_u-\varepsilon_l)^2}{k(\varepsilon_u+\varepsilon_l)}\{\bar{E}^2 + 2\bar{E}\hat{E}\cos\omega t \\ &+ \hat{E}^2\cos^2\omega t\}\gamma = 0. \end{aligned} \tag{25}$$

Equation (25) is the well-known Hill's differential equation. The nature of the solution of this differential equation will govern the fluctuations of the amplitude of the disturbed interface, and it will therefore determine the parametric excitation of the electrohydrodynamic surface waves.

3. ELECTROHYDRODYNAMIC SURFACE WAVES

If we exclude the modulated field $\hat{E}\cos\omega t$ for a while and consider only the original field $\bar{E}$ we obtain from Equation (24), as $\hat{E}\to 0$, the following differential equation:

$$\frac{d^2\gamma}{dt^2} + \sigma^2\gamma = 0 \tag{25a}$$

where

$$\begin{aligned} \sigma^2 = &\frac{Tk^3}{\rho_u+\rho_l} - gk\frac{\rho_u-\rho_l}{\rho_u+\rho_l} \\ &+ \frac{(\varepsilon_u-\varepsilon_l)^2}{\varepsilon_u+\varepsilon_l}\frac{k_x^2\bar{E}^2}{\rho_u+\rho_l}. \end{aligned} \tag{26}$$

Equation (26) is the dispersion relation for the present case. The solution of Equation (25a) is $\gamma(t)=\text{const. } e^{\pm i\sigma t}$ and therefore the system is stable if σ is real. In other words, the stability implies that

$$\sigma^2 \geq 0. \tag{27}$$

It is seen that the electric field is stabilizing. Moreover, for values of $\bar{E} \geq \bar{E}^*$, where

$$\bar{E}^{*2} = \frac{\varepsilon_u+\varepsilon_l}{k_x^2(\varepsilon_u-\varepsilon_l)^2}\left[gk(\rho_u-\rho_l) - Tk^3\right], \tag{28}$$

the system is stable and electrohydrodynamic surface waves propagate through the interface. We note that when $k_x=0$ (i.e. the component of the wave vector parallel to the electric field is zero), Equation (26) shows that the tangential electric field has no effect on the Rayleigh–Taylor instability.

This is the electrohydrodynamic analogy to the Kruskal–Schwarzschild Rayleigh–Taylor instability for a plasma supported against gravity by a uniform magnetic field [16].

In what follows we shall examine the possibility of amplifying the electrohydrodynamic surface waves by superimposing the modulated field $\hat{E}\cos\omega t$.

4. WAVE EXCITATIONS

The Hill's differential equation (25) can be written in the following standard form:

$$\frac{d^2\gamma}{dt^2} + \{\theta_0 + 2\theta_2\cos 2\tau + 2\theta_4\cos 4\tau\}\gamma = 0, \tag{29}$$

where

$$\tau = \tfrac{1}{2}\omega t, \tag{30}$$

$$\theta_0 = \frac{4\sigma^2}{\omega^2} + \frac{\hat{E}^2 k_x^2(\varepsilon_u-\varepsilon_l)^2}{\omega^2(\varepsilon_u+\varepsilon_l)(\rho_u+\rho_l)}, \tag{31}$$

$$\theta_2 = \frac{4k_x^2(\varepsilon_u-\varepsilon_l)^2\hat{E}\bar{E}}{\omega^2(\varepsilon_u+\varepsilon_l)(\rho_u+\rho_l)}, \tag{32}$$

$$\theta_4 = \frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\hat{E}^2}{\omega^2(\varepsilon_u+\varepsilon_l)(\rho_u+\rho_l)}. \tag{33}$$

It can be shown [2] that the solutions of Equation (29) can be represented as

$$\gamma(\tau)=e^{\mu\tau}\sum_{r=-\infty}^{\infty}C_{2r}e^{2i\tau}, \tag{34}$$

where C_{2r} are constant coefficients and μ satisfies the relation

$$\sin^2\tfrac{1}{2}\mu\pi=\Delta(0)\sin^2\tfrac{1}{2}\pi\sqrt{\theta_0} \tag{35}$$

$\Delta(0)$ is an infinite determinant depending on θ_0, θ_2, and θ_4; for small values of θ_2 and θ_4 it takes the form

$$\Delta(0)\approx 1+\frac{\pi\cot\frac{1}{2}\pi\sqrt{\theta_0}}{4\sqrt{\theta_0}}\left[\frac{\theta_2^2}{1-\theta_0}+\frac{\theta_4^2}{2^2-\theta_0}\right] \tag{36}$$

It is apparent from Equation (34) that the time dependence of the perturbation will depend on the nature of the parameter μ(whether real or complex). Because the Hill's determinant is infinite, the analysis for finding out the nature of μ is quite complicated.

The Hill's differential equation can also be written in the following equivalent form:

$$\frac{d^2\gamma}{dt^2}+[a-2q\psi(\tau)]\gamma=0, \tag{37}$$

where q is defined such that $|\psi(\tau)|^{\max.}_{\min.}=1$ and a is the same as θ_0 defined by Equation (31). It can be shown [2] that the (a,q) plane may be divided into stable and unstable regions for positive values of a in a manner similar to the characteristic curves of the Mathieu functions in the (a,q) plane. Figure 2 shows the regions in the (a,q) plane in which the values of a and q yield imaginary values for μ. These regions are the stable regions. On the other hand, if μ is real the solution for γ will tend to ∞ as $t\to\infty$. The unstable regions correspond to real values of μ. The boundary curves are the characteristic curves of the Mathieu functions. Thus, in general, instability occurs even though $\sigma^2>0$. Moreover, for negative values of a the solution of Equation (37) is unstable. Further, it may be remarked that a is negative if and only if σ^2 is negative and it follows therefore that a system which is unstable in the presence of a steady electrostatic field cannot be stabilized by superimposing on it an alternating electric field.

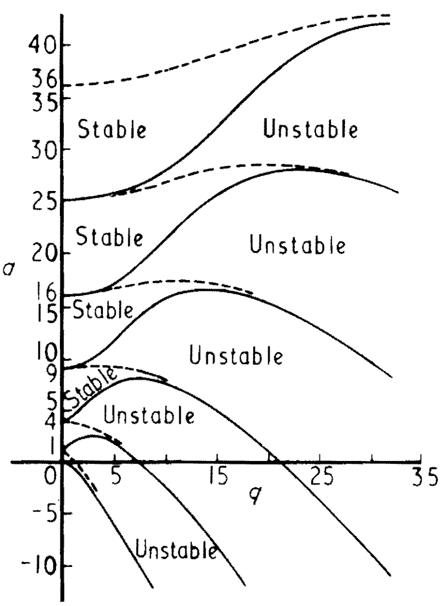

Figure 2. The Different Regions of Stability and Instability of the Characteristic Curves of the Mathieu Functions $y''+(a-2q\cos 2\omega t)y=0$

It can be shown [1] that for a small value of q, the solution of a differential equation of the type represented by Equation (37) corresponds to a state of chief resonance if $a=1$. The overtone resonances occur when $a=n^2(n=2,3,4,\ldots)$. Thus parametric excitations of the electrohydrodynamic waves take place when the frequency of the modulated field equals a fractional part of the natural frequency of the system. It is clear that the different values of n correspond to the different regions of instabilities in the (a,q) plane. The condition for parametric resonance can now be expressed as

$$4\sigma^2+\frac{\hat{E}^2k_x^2(\varepsilon_u-\varepsilon_l)^2}{(\varepsilon_u+\varepsilon_l)(\rho_u+\rho_l)}\approx n^2\omega^2,\ n=1,2,3,\ldots \tag{38}$$

Equation (38) gives the dependence of ω on $\hat{E},\bar{E}$ and the other parameters of the system for parametric excitation of electrohydrodynamic surface waves.

In the following sections we shall study in detail some limiting cases.

5. CASES OF SMALL MODULATION

We consider here the case when the strength of the modulated electric field E is small compared with the electrostatic field, i.e. $\hat{E}\ll\bar{E}$. The terms of order $\hat{E}^2$ can be neglected from Equation (25) and the resulting equation becomes

$$\frac{d^2\gamma}{dt^2}+\frac{k}{\rho_u+\rho_l}\left[k^2T-g(\rho_u-\rho_l)\right.$$

$$\left.+\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2}{\varepsilon_u+\varepsilon_l}\{\bar{E}^2+2\bar{E}\,\hat{E}\cos\omega t\}\right]\gamma=0. \tag{39}$$

If we denote

$$a'=\frac{4k}{\omega^2(\rho_u+\rho_l)}\left[k^2T-g(\rho_u-\rho_l)+\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\bar{E}^2}{k(\varepsilon_u+\varepsilon_l)}\right], \tag{40}$$

$$q'=-\frac{4k_x^2(\varepsilon_u-\varepsilon_l)^2\bar{E}\hat{E}}{\omega^2(\rho_u+\rho_l)(\varepsilon_u+\varepsilon_l)}, \tag{41}$$

Equation (35) can be written as

$$\frac{d^2\gamma}{dt^2}+(a'-2q'\cos 2\tau)\gamma=0, \tag{42}$$

where τ is defined by Equation (30).

The parametric excitation of electrohydrodynamic surface waves occurs near the cusp points in the unstable regions of the characteristic curves of the Mathieu functions in the (a',q') plane. The boundary curves of the first region of instability in the (a',q') plane are given by [2],

$$a_1'(q')=1-q'-\frac{1}{8}q'^2+\frac{1}{64}q'^4+\ldots, \tag{43}$$

and

$$a_2'(q')=1+q'-\frac{1}{8}q'^2+\frac{1}{64}q'^4+\ldots, \tag{44}$$

where $a_1'(q')$ and $a_2'(q')$ are the equations defining the two boundary curves of the first region of instability in the (a',q') plane. For small values of q Equations (43) and (44) can be written as

$$a'=1\pm q', \tag{45}$$

which is equivalent to

$$\frac{\omega^2}{4}=\frac{k}{\rho_u+\rho_l}\left[k^2T-g(\rho_u-\rho_l)+\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\bar{E}^2}{k(\varepsilon_u+\varepsilon_l)}\pm\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\bar{E}\hat{E}}{k(\varepsilon_u+\varepsilon_l)}\right]. \tag{46}$$

Equation (46) gives the range required for ω in terms of $\bar{E}$ and $\hat{E}$ to maintain the system at a state of chief resonance.

For values of $\hat{E}$ such that these boundaries are surpassed one solution of Equation (42) will grow exponentially with time. Clearly for very small values of $\bar{E}$, the instability condition (46) becomes

$$\frac{\omega^2}{4}=\frac{k}{\rho_u+\rho_l}\left[\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\bar{E}^2}{k(\varepsilon_u+\varepsilon_l)}+k^2T+g(\rho_u-\rho_l)\right]. \tag{47}$$

Hence, given ω and $\bar{E}$, one can find a corresponding k to give the subharmonic response. Therefore the interface is always unstable under the present case. Letting $k_x=k$ in Equation (47) we find that ω^2 is cubic in k. Thus, for a given ω, Equation (47) will have either one real root for k or three real roots. The differentiation of Equation (47) with respect to k when replacing k_x by k yields.

$$\frac{1}{4}\frac{d\omega^2}{dk}=\frac{1}{\rho_u+\rho_l}\left[3k^2T+g(\rho_u-\rho_l)+\frac{2k(\varepsilon_u-\varepsilon_l)\bar{E}^2}{\varepsilon_u+\varepsilon_l}\right]\ldots \tag{48}$$

It is seen from Equation (44) that ω^2 attains its maximum or minimum value if k satisfies the equation

$$3k^3T-g(\rho_u-\rho_l)+\frac{2k(\varepsilon_u-\varepsilon_l)\bar{E}^2}{\varepsilon_u+\varepsilon_l}=0. \tag{49}$$

It follows therefore that, for values of $\bar{E}$ satisfying the inequality

$$\bar{E}^2>\bar{E}^{**2}, \tag{50}$$

where $\bar{E}^{**}$ is given by

$$\bar{E}^{**2}+3Tg(\rho_u-\rho_l)\left[\frac{\varepsilon_u+\varepsilon_l}{(\varepsilon_u-\varepsilon_l)^2}\right]^2=0, \tag{51}$$

there will be three real values for k satisfying equation (47) for a given ω. The growth rate would then determine which wavenumber will predominate. It is clear that the inequality (50) is trivially satisfied for $\rho_u>\rho_l$. On the other hand, if $\bar{E}^2<\bar{E}^{**2}$ one can easily show that ω^2 increases monotonically with k and therefore Equation (47), for a given value of ω, admits one positive root for k. Also from Equation (47) we see that the resonance frequency ω for a given wavenumber k will decrease with the decrease of $\bar{E}$.

It is known [2] that the solution of the Mathieu differential equation (42) can also be represented by the series given by Equation (34). The parameter μ appears in the series is now defined by the following Equation [2]:

$$\Delta(0)\sin^2\frac{\pi}{2}a^{\frac{1}{2}}\leq 1.$$

For small values of q', μ approaches the value $\sqrt{a'}$ and the series given by Equation (34) can be approximated as

$$\gamma(t)=A_0 e^{\sqrt{a'}\tau} \tag{52}$$

Near $a=1$, the solution of the Mathieu equation can therefore be represented by

$$\gamma(t)=\gamma_0 e^{\pm\frac{1}{4}q'\tau}$$

or

$$\gamma(t)=\gamma_0 e^{\pm\frac{1}{4}q'\omega t}. \tag{53}$$

If $q^*=\frac{1}{4}q'\omega$ denote the growth rate, then from Equation (44)

$$q^*=\frac{k^2(\varepsilon_u-\varepsilon_l)^2\bar{E}\hat{E}}{\tilde{\omega}(\rho_u+\rho_l)(\varepsilon_u+\varepsilon_l)}, \tag{54}$$

where ω is given by Equation (47). From Equation (43) we see that the growth rate is greater for disturbances with higher wavenumbers. Therefore, for the case $\bar{E}>\bar{E}^{**}$ when for a given ω, three values of the wavenumber k are possible, the disturbance corresponding to the smallest wavelength will predominate.

It is interesting to note that the growth rate per cycle of oscillation (q^*/ω), namely

$$\frac{q^*}{\omega}=\frac{k(\varepsilon_u-\varepsilon_l)^2\bar{E}\hat{E}}{\varepsilon_u+\varepsilon_l} \times\left[k^2T-g(\rho_u-\rho_l)+\frac{k(\varepsilon_u-\varepsilon_l)^2\bar{E}^2}{\varepsilon_u+\varepsilon_l}\right], \tag{55}$$

has a maximum for all values of E, when $k^2=g(\rho_u-\rho_l)/T$, i.e. for the disturbances with the minimum phase velocity in the absence of any electric field.

The preceding discussions in this section were confined only to the first region of instability (the natural resonance) and similar arguments can be applied to the other regions of instability.

The boundary curves for the second region of instability of the characteristic curves of the Mathieu functions are given by [2]

$$a'_1=4-\frac{1}{12}q'^2+\frac{5}{13824}q'^4+\ldots$$

and

$$a'_2=4+\frac{5}{12}q'^2+\frac{763}{13824}q'^4+\ldots$$

and for the nth region $(n\geq 7)$

$$a'_1,\ a'_2=n^2+\frac{1}{2(n^2-1)}q^2+\frac{5n^2+7}{32(n^2-1)(n^2-4)}q^4+\ldots$$

Thus for small values of q' (i.e. neglecting terms of order q'^2 and higher-order terms) the overtone resonance occurs if

$$\frac{4k}{\rho_u+\rho_l}\left[k^2T-g(\rho_u-\rho_l)+\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\bar{E}^2}{k(\varepsilon_u+\varepsilon_l)}\right]\approx n^2\omega^2, \tag{56}$$

where n is an integer ($n=2,3,4,\ldots$). Equation (56) is a limiting case of Equation (38).

6. THE CASE OF LARGE MODULATION

If we assume that the electrostatic field is small compared to the modulated field, i.e. $\bar{E}\ll\hat{E}$, we can neglect terms involving $\bar{E}_0$ in Equation (25) and we finally obtain the following differential equation:

$$\frac{d^2\gamma}{dt^2}+\frac{k}{\rho_u+\rho_l}\left[K^2T-g(\rho_u-\rho_l)+\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\hat{E}^2\cos^2\omega t}{k(\varepsilon_u+\varepsilon_l)}\right]\gamma=0, \tag{57}$$

which can be written in the following canonical form:

$$\frac{d^2\gamma}{d\eta^2}+(a''-2q''\cos 2\eta)\gamma=0\ldots, \tag{58}$$

where

$$\eta=\omega t,$$

$$a''=\frac{k}{\omega^2(\rho_u+\rho_l)}\left[k^2T-g(\rho_u-\rho_l)+\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\hat{E}^2}{k(\varepsilon_u+\varepsilon_l)}\right], \tag{59}$$

$$q''=-\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\hat{E}^2}{4\omega^2(\varepsilon_u+\varepsilon_l)(\rho_u+\rho_l)}\ldots \tag{60}$$

The Mathieu differential equation (58) is similar to that discussed extensively in an earlier paper [3] in the sense that the term explicitly dependent upon time has a frequency twice that of the superimposed electric field. A general remark which follows immediately is that the subharmonic response of Equation (58) will actually be isochronous with respect to the modulation.

In what follows we shall be interested in studying the excitation mechanism corresponding to the first region of instability of the characteristic curves of the Mathieu functions in the (a'',q'') plane. The first region of instability is bounded by the two lines:

$$a''=1-q'' \text{ and } a''=1+q'',$$

which are equivalent to

$$\omega^2=\frac{k}{\rho_u+\rho_l}\left[k^2T-g(\rho_u-\rho_l)+\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\hat{E}^2}{4k(\varepsilon_u+\varepsilon_l)}\right] \tag{61}$$

and

$$\omega^2=\frac{k}{\rho_u+\rho_l}\left[k^2T-g(\rho_u-\rho_l)-\frac{3k_x^2(\varepsilon_u-\varepsilon_l)^2\hat{E}^2}{4k(\varepsilon_u+\varepsilon_l)}\right]. \tag{62}$$

From these two equations, it is clear that instability (or excitation) will occur only if

$$\omega^2>\frac{k}{\rho_u+\rho_l}[k^2T-g(\rho_u-\rho_l)]$$

or

$$\omega^2>\sigma^2, \tag{63}$$

i.e. ω^2 should exceed the frequency of surface waves propagating along the interface in the absence of any electric field.

For the other regions of instability the overtone resonances are given by

$$\frac{k}{\rho_u+\rho_l}\left[k^2T-g(\rho_u-\rho_l)+\frac{k_x^2(\varepsilon_u-\varepsilon_l)^2\hat{E}^2}{4k(\varepsilon_u-\varepsilon_l)}\right]\approx n^2\omega^2, \tag{64}$$

$$n=1,2,3,4,\ldots$$

7. CONCLUSION

The previous discussion emphasizes the role of electrohydrodynamics as a substitution of pure hydrodynamic motion in some physical situation. Thus, the introduction of an electric field $\bar{E}+\hat{E}\cos\omega t$ to surface waves propagating through the interface between two dielectrics is equivalent to imposing a parallel oscillatory flow on the fluids as introduced by Kelley [7]. The dual role of electrohydrodynamics and magnetohydrodynamics for some physical phenomena has been already emphasized [13].

The main feature of periodicity is a loss mechanism. In the limit as $\omega\to 0$, $\hat{E}$ will be added to $\bar{E}$ in Equation (26) and it therefore plays a stabilizing role. The presence of ω contributes a loss to this role and the result is instability. Therefore, as proved in Section 4 the periodicity of the modulation cannot stabilize an otherwise unstable system. The loss mechanism is related to the average of $\hat{E}\cos\omega t$ over a cycle which is zero while for $\hat{E}$ only the average is $\hat{E}$. The above explanation of the periodicity mechanism is supported by the results of Wesson [17] and Zrnik and Hendricks [18]. Wesson considered the Rayleigh–Taylor instability where the gravitational force is destabilizing. He superimposed a dynamical periodic force in the same direction of gravity ($F\cos\omega t$ say). In the limit $\omega\to 0$ the force acts as gravity and hence destabilizing. The presence of ω causes a loss to this mechanism and hence stability is achieved. He could also, instead of the periodic force, apply a magnetic modulation since the constant magnetic field is destabilizing. The latter suggestion, with a slight modification was used by Zrnik and Hendricks who applied a magnetic feedback proportional to the surface displacement which in turn is periodic. Although their analysis is different from ours, their results achieved stabilization with magnetic feedback.

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