STEADY FLOW OF A POLARIZABLE VISCOUS COMPRESSIBLE FLUID PAST A CHARGED ROTATING SPHERE

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الخلاصـــة :

تم دراسة الانسياب المنتظم لمائعٍ لزجٍ قابلٍ للانضغاط وقابل للاستقطاب عبر كرة تدور بسرعه منتظمة ومشحونة كهربياً بانتظام. حولت معادلات الحركة إلى معادلات خطية في بارامتر β . تبين أنه إذا كان محور الدوران للكرة في نفس اتجاه انسياب المائع فإن الدوران لن يؤثر على قوة الإعاقة بينما يؤثر الدوران على عزم الازدواج المؤثر على الكرة ويحدث زيادة من رتبة β (β تعتمد على الشحنة الكهربية) .

ABSTRACT

The flow of a polarizable viscous compressible fluid past a uniformly charged sphere rotating with constant angular velocity is studied. The equations of motion have been linearized by using a small parameter β. It was found that if the axis of rotation has the same direction as the flow, the rotation will not affect the drag force, while the couple on the sphere will receive a correction of order β.

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1. INTRODUCTION

It is well known that if a sphere moves through a viscous compressible fluid, there will be a drag force affecting the motion of the sphere; this has been extensively treated in the literature (see [1] for a review). Recently Smith [2] extended the study to the case of spheroids and spherical caps. If the fluid was polarizable and flowing past a uniformly charged sphere, Schappert [3] showed that there would be a correction term to the well-known Stokes law for the drag force due to the polarizability of the fluid.

Here we study the case of a charged sphere rotating with constant angular velocity Ω. It is found that the drag force does not change because of the rotation of the sphere when the axis of rotation has the same direction as the flow. On the other hand, the couple affecting the rotating sphere will receive a correction to become

$$8\pi\Omega R^3\eta\left[1+\frac{4}{3}\frac{\partial\rho}{\partial p}|_T\,\alpha_0\varepsilon_0\left(\frac{e}{4\pi\varepsilon R^2}\right)^2\right],$$

where p, ρ, η, R, and α_0 are the pressure, density, viscosity, radius of the sphere, and polarizability per atom respectively. These mentioned corrections arise because of the forces of electrostriction derived from the stress tensor [4]:

$$T_{ij}=-p_0(\rho,T)\delta_{iK}-\frac{E^2}{8\pi}\left[\varepsilon-\rho\frac{\partial\varepsilon}{\partial\rho}|_T\right]\delta_{iK}+\frac{\varepsilon E_i E_K}{4\pi},$$

The subscript T means that the derivatives have been evaluated at constant temperature.

We shall make use of the properties of the creeping flow; u_0 the velocity of the steady flow as well as Ω will be considered small and of the same order ($\Omega/u_0=O(1)$). Consequently, the inertia term in the Navier Stokes equation will be neglected. We shall also neglect the gravity effect and other external forces. In addition, we shall assume that no temperature gradient exists.

2. OUTLINE OF THE PROCEDURE

We consider the Z-axis as the axis of rotation, which is, at the same time, the direction of the steady flow. The center of the sphere will be the origin (using spherical polar coordinates (r, θ, ϕ)).

The equation of continuity is

$$\text{div }(\rho\mathbf{u})=0, \tag{1}$$

where $\mathbf{u}$ is the velocity of the fluid element.

The equation of motion for the present case is the same as given by Schappert [3]:

$$p\beta\nabla(R/r)^4-\nabla p+\frac{1}{3}\eta\nabla\text{div}\mathbf{u}+\eta\nabla^2\mathbf{u}=0, \tag{2}$$

where β is the dimensionless constant defined by

$$\beta=\frac{1}{2}\chi(T)\alpha_0\varepsilon_0\left(\frac{e}{4\pi\varepsilon R^2}\right)^2, \tag{3}$$

and χ is defined by the equation of state

$$\rho=\chi(T)p. \tag{4}$$

The solution for Equations (1) and (2) should satisfy the boundary condition $\mathbf{u}(R)=\Omega R\sin\phi\mathbf{1}_\phi$, where $\mathbf{1}_\phi$ is a unit vector in the direction of increasing ϕ.

The zero-order solution in u for p may be of the form

$$p^0(r)=p_0e^{\beta(R/r)^4},\ r\gg R, \tag{5}$$

where p_0 is the pressure at the boundary.

For the first-order solution in u, ρ enters the equation of continuity in its zeroth order, i.e.

$$\text{div }(\rho^0\mathbf{u})=0, \tag{6}$$

which is

$$\text{div } e^{\beta(R/r)^4}\mathbf{u}=0. \tag{7}$$

Introducing the transformation

$$\mathbf{u}=u_0e^{-\beta(R/r)^4}v(r,\theta), \tag{8}$$

we have from Equation (7),

$$\text{div } v=0. \tag{9}$$

v is a solenoidal vector and therefore it can be derived as a curl of some vector $\mathbf{A}$. Since the curl of a polar vector is an axial vector and vice versa, then $\mathbf{A}$ is a polar vector. Accordingly we may follow Landau and Lifshitz's [5] suggestion for the form of $\mathbf{A}$. For the present case we are led to the following expression for $\mathbf{v}$:

$$\mathbf{v}(r,\theta)=\mathbf{1}_r v(r)\cos\theta-\mathbf{1}_\theta\frac{1}{2r}\frac{d}{dr}\left[r^2v(r)\sin\theta\right]$$
$$+\mathbf{1}_\phi e^{\beta\Omega/u^0}W(r)r\sin\theta, \tag{10}$$

and $\mathbf{1}_r$, $\mathbf{1}_\theta$, $\mathbf{1}_\phi$ are unit vectors in the directions of axes of coordinates.

$v(r)$ and $W(r)$ satisfy the following conditions:

$$v(R)=\frac{\partial v(r)}{\partial r}\Big|_{r=R}=0, \qquad W(r)=1 \tag{11}$$

$$v(\infty)=1, \frac{\partial v}{\partial r}\Big|_{\infty}=0, \qquad W(\infty)=0, \frac{\partial W}{\partial r}\Big|_{\infty}=0, \tag{12}$$

It is clear that the solution for $\mathbf{v}$ given by Equation (10) satisfies Equation (9) and all the boundary conditions for Equations (1) and (2).

It is apparent that the solution for p is independent of ϕ, hence we can write

$$P(r,\theta)=P^0(r)+P^1(r,\theta). \tag{13}$$

Substituting from Equations (10) and (13) into (2), we obtain for the r-component:

$$P^1\beta\Big(-\frac{4R^4}{r^5}\Big)-\frac{\partial P^1}{\partial r} +\frac{4}{3}u_0\eta\cos\theta\frac{\mathrm{d}}{\mathrm{d}r}\left[\mathrm{e}^{-\beta(R/r)^4}\left(\frac{4\beta R^4}{r^5}\right)\right] -u_0\eta\mathrm{e}^{-\beta(R/r)^4}\left[\frac{\cos}{r}\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r^2v) +2v-\frac{4\beta R^4}{r^5}\frac{\mathrm{d}}{\mathrm{d}r}(r^2v)\right\}\right]=0, \tag{14}$$

for the θ-component:

$$\frac{1}{r}\frac{\partial P^1}{\partial\theta}-\frac{4}{3}\eta u_0 e^{-\beta(R/r)^4}\left(\frac{4\beta R^4}{r^6}\right)\sin\theta +\frac{\sin\theta}{2r}\left\{\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r^2v) -\frac{4\beta R^4}{r^5}\frac{\mathrm{d}}{\mathrm{d}r}(r^2v)\right]+\frac{4\beta R^4}{r^5}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r^2v) +2v-\frac{4\beta R^4}{r^5}\frac{\mathrm{d}}{\mathrm{d}r}(r^2v)\right]\right\}\eta u_0=0, \tag{15}$$

and for the ϕ-component:

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r^2W)-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4\beta R^4}{r^3}W\right) +2W-\frac{4\beta R^4}{r^5}\left\{\frac{\mathrm{d}}{\mathrm{d}r}(r^2W) +\frac{4\beta R^4}{r^5}W\right\}=0, \tag{16}$$

It is clear that the dependence of P^1 on θ is $\cos\theta$. If we write

$$P^1(r,\theta)=\frac{u_0}{R}\hat{P}(r)\cos\theta \tag{17}$$

and

$$X=\frac{r}{R}, \tag{18}$$

Equations (14) and (15) reduce to Schappert's equations, namely

$$-\beta X^{-5}\hat{P}(X)-\frac{\mathrm{d}}{\mathrm{d}X}\left[\hat{P}(X)-\frac{4}{3}\beta X^{-5}\hat{u}_r(X)\right] +\left[\frac{1}{X^2}\frac{\mathrm{d}}{\mathrm{d}X}\left(X^2\frac{\mathrm{d}\hat{u}_r}{\mathrm{d}X}\right)-\frac{4}{X^3}(\hat{u}_r+\hat{u}_\theta)\right]=0, \tag{19}$$

and

$$\hat{P}(X)-\frac{4}{3}\beta X^{-5}\hat{u}_r(X) +X\left[\frac{1}{X^2}\frac{\partial}{\partial X}\left(X^2\frac{\partial\hat{u}_\theta}{\partial X}\right)-\frac{2}{X^2}(\hat{u}_r+\hat{u}_\theta)\right]=0, \tag{20}$$

where

$$\hat{u}_r=\mathrm{e}^{-\beta/X^4}v(X), \tag{21}$$

$$\hat{u}_\theta=\mathrm{e}^{-\beta/X^4}\frac{1}{2X}\frac{\mathrm{d}v(X)}{\mathrm{d}X}, \tag{22}$$

and

$$v(X)=\left(1-\frac{3}{2X}+\frac{1}{2X^3}\right) +\beta\left(-\frac{2}{X^4}+\frac{66/70}{X^5}+\frac{1/14}{X^7}+\frac{3/35}{X}+\frac{9/5}{X^3}\right). \tag{23}$$

Equation (16) reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}X^2}(X^2W)-2W+4\beta\left\{\frac{\mathrm{d}}{\mathrm{d}X}(X^{-3}W) +X^{-5}\frac{\mathrm{d}}{\mathrm{d}X}(X^2W)\right\}=0. \tag{24}$$

To find the solution for W we notice that the zero-order solution in β for Equation (24) gives $W=1/X^3$, so it can be written in the form

$$W(X)=\frac{1}{X^3}+\beta\left(\frac{A}{X^3}+\frac{B}{X^7}\right). \tag{25}$$

B can be calculated by substituting into Equation (24) and linearizing in β. A can be obtained from the boundary condition (11), hence we obtain

$$W(X)=\frac{1}{X^3}+\beta\left(\frac{1}{X^7}-\frac{1}{X^3}\right). \quad (26)$$

It should be noted that the boundary condition (12) is automatically satisfied.

3. DRAG AND TORQUE

Now we proceed to calculate the couple Q on the sphere. We note that the velocity components $\hat{u}_r$, $\hat{u}_\theta$ and their derivatives vanish on the sphere; it follows that the only component of the stress tensor which contributes to Q is $P_{r\phi}$,

$$P_{r\phi}=\eta\left(-\frac{\hat{u}_\phi}{r}+\frac{\partial\hat{u}_\phi}{\partial r}\right), \quad (27)$$

$$P_{r\phi}|_{X=1}=\eta\Omega\sin\theta(-3-8\beta), \quad (28)$$

$$Q=\int_0^\pi R\sin\theta P_{r\phi}(2\pi R^2\sin\theta)\mathrm{d}\theta, \quad (29)$$

$$Q=8\pi\Omega R^3\left(1+\frac{8}{3}\beta\right), \quad (30)$$

hence

$$Q=8\pi\eta\Omega R^3\left\{1+\frac{4}{3}\frac{\partial\rho}{\partial p}|_T\alpha_0\varepsilon_0\left(\frac{e}{4\pi\varepsilon R^2}\right)^2\right\}. \quad (31)$$

It is clear that the moment has increased because of polarizability by

$$\frac{32}{3}\pi\Omega\eta R^3\frac{\partial\rho}{\partial p}|_T\ \alpha_0\varepsilon_0\left(\frac{e}{4\pi\varepsilon R^2}\right)^2,$$

since the hydrodynamic moment is $8\pi\Omega\eta R^2$. It should be noted that the above approximation is valid only when β is small. Schappert calculated β for the helium atom ($\beta\simeq 2.3$), by the formula

$$\beta\simeq\frac{1}{2}\frac{\rho\alpha_0\varepsilon_0}{MC^2}\left(\frac{e}{4\pi\varepsilon_0 R^2}\right)^2,$$

where C the velocity of sound. This value of β will introduce a remarkable change to the value of Q.

From Equations (19) and (20), the drag force is still that calculated by Schappert,

$$F=6\pi Ru_0\eta\left\{1-\frac{1}{35}\alpha_0\varepsilon_0\ \frac{\partial\rho}{\partial P}|_T\left(\frac{e}{4\pi\varepsilon R^2}\right)^2\right\}, \quad (32)$$

in which F has decreased by the polarizability effect. The drag force decreases owing to the electrostriction, though the stress components P_{rr} and $P_{r\theta}$ increase. Schappert explains the decrease in the drag force that the body force overcompensates for the increase of the stress components. The only stress component that contributes to the torque Q is $P_{r\phi}$ which is decreased by electrostriction. A similar mechanism is therefore responsible for the increase of the torque due to the electrostriction.

If the axis of rotation of the sphere is not in the same direction of the flow, the suggested form of the velocity $\mathbf{u}$:

$$\mathbf{u}=u_0 e^{-\beta(R/r)^4}\left\{\mathbf{1}_r v'(r)\cos\theta-\mathbf{1}_\theta\frac{1}{2r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2v'(r)\sin\theta)\right.$$
$$\left.+\mathbf{1}_\phi e^{\beta\Omega/u_0}W'(r)\Omega\times\mathbf{r}\right\}, \quad (33)$$

the term $\Omega\times\mathbf{r}$ will contribute to the θ-component of the velocity; consequently Equations (19) and (20) will be modified and the drag force will be in general different from that given by Equations (32).

If $u_0=0$, the solution of $\mathbf{u}$ given by Equation (10) satisfies the equation

$$\mathrm{div}\ \mathbf{u}=0, \quad (34)$$

which means that the motion has been reduced to that of an incompressible fluid and the correction given by (13) will be meaningless.

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