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SOME COMMUTATION PROPERTIES IN THE FREE ALGEBRA

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خلاصة

افرض ان عبارة F عبارة عن نصف مجموعة حرة معرفة على فئة محدودة من المولدات او فئة لا نهائية من المولدات قابلة للعد ، وافرض أن A هي جبر كل الدوال ذات القيم المركبة المعرفة على F بواسطة الجمع المتعلق بالنقط أو الضرب الالتفافي .

في هذا البحث نبرهن على انه اذا كانت $f, g \in A$ حيث f كثيرة حدود متجانسة و $fg = gf$ فانه توجد كثيرة حدود متجانسة $h \in A$ وعدد صحيح موجب n وأرقام مركبة $a, b_o, b_1, \ldots$ بحيث أن $f = ah^n$, $g = \sum_{i=o}^{\infty} b_i h^i$

ومن هذه النتيجة نبرهن أيضاً انه اذا كانت $f, g \in A$ و $f^n g = g f^n$ فان $fg = gf$ أيضاً اذا كان $f^n = g^n$ فان $f = ag$ حيث a عدد مركب معين يحقق العلاقة $a^n = 1$

ABSTRACT

Let F be the free semigroup on a finite or countably infinite set of generators, and let A be the algebra of all complex valued functions on F with pointwise addition and convolution multiplication.

It is shown that if $f, g \in A$, f is a homogeneous polynomial and $fg = gf$; then there is a homogeneous polynomial $h \in A$, a positive integer n and complex numbers $a, b_0, b_1, \ldots$ such that $f = ah^n$ and $g = \sum_{i=o}^{\infty} b_i h^i$. From this it is shown that if $f, g \in A$ and $f^n g = g f^n$ then $fg = gf$, also if $f^n = g^n$ then $f = ag$ for some complex number a with $a^n = 1$.

INTRODUCTION

Let F be the free semigroup on some finite or countably infinite set of generators. Let A be the algebra of complex valued functions on F with pointwise addition and convolution multiplication, i.e., if $s \in F$ and $f, g \in A$, then fg is defined by

$$fg(s) = \sum_{rt=s} f(r)\; g(t)$$

If we let $I(e) = 1$ and $I(s) = 0$ for $s \neq e$ (Where e is the empty word in F), then it is easy to check that I is an identity for A .

Let B be the subalgebra of A consisting of all $f \in A$ with $\sum_{s \in F} | f(s | < \infty$. Then B is a Banach algebra with dentity where $\| f \| = \sum_{s \in F} | f(s) |$

The algebras A and B have been the subject of a lot of research lately, since they provide examples of extreme cases of non-commutativity. This paper is concerned with some commutation relations in A. It is a by product of the author's work with the algebra B [1] . For the interested reader we also refer him to Reference [2] for some impressive results concerning the algebra B.

TERMINOLOGY AND NOTATION.

Given $s \in F$ and a generator c of F we let $d_c(s)$ denote the number of occurences of c in the expression for s. The degree of s, d(s), is defined to be $\sum \{ d_c(s)\text{: c is a generator for F} \}$.
In other words d(s) is the length of s. It is easy to see that $d(st) = d(s) + d(t)$.

Given $f \in A$ we let S_f denote the support of f, i.e., $S_f = \{s \in F: f(s) \neq 0\}$. If S_f is finite we say f is a polynomial and if $d(s) = d(t)$ for all s, $t \in S_f$ we say f is a homogeneous polynomial. If f is a homogeneous polynomial we let $d(f) = d(s)$ for $s \in S_f$. It is clear that in this case $d(fg) = d(f) + d(g)$

Given $f \in A$, let $m_1(f)$ denote the set of words of minimum degree in $S_f \setminus \{e\}$. After defining $m_1(f), \ldots, m_k(f)$, define $m_{k+1}(f)$ to be the set of words of minimum degree in S_f subject to the condition $d(s) < d(r)$ for $s \in m_k(f)$ and $r \in m_{k+1}(f)$; i.e., $m_{k+1}(f)$ contains the words in S_f. which are next in length to those in $m_k(f)$. If no such words exist, let $m_{k+1}(f) = \varnothing$. Note that $m_n(f)=0$ for some n if and only if f is a polynomial.

For each $n > 0$, let $f_n = f \backslash m_n(f)$, i.e., f_n is the restriction of f to $m_n(f)$. Thus f_n is a homogeneous polynomial.

The letter C will denote the field of complex numbers.

Commutation Properties in A.

By a homogeneous polynomial we will always mean a non-trivial one, i.e., we exclude the case $f=1$. The following lemma is easy to prove (by induction on k) and we leave it to the reader.

1. *Lemma.*

Let $f_1, \ldots, f_k$ be homogeneous polynomials and let $s_1, \ldots, s_k$ be elements of F such that $d(f_i) = d(s_i)$; $i = 1, \ldots, k$. Then $f_1 \ldots f_k (s_1 \ldots s_k) = f_1(s_1) \ldots f_k(s_k)$.

2. *Lemma.*

Let f and g be homogeneous polynomials and suppose that $d(f) = d(g)$. If $fg = gf$ then there exists a number α such that $f = \alpha g$.

Proof.

Let $s \in S_f$ and $t \in S_g$. Then, by lemma 1.
$0 \neq f(s)\, g(t) = fg(st) = gf(st) = g(s)\, f(t)$. Hence $g(s) \neq 0 \neq f(t)$.

This says that $S_f = S_g$.
Now let $s,t \in S_f = S_g$.

Let $\alpha = \dfrac{f(s)}{g(s)}$ and $\beta = \dfrac{f(t)}{g(t)}$.

Then $\beta\, g(s)\, g(t) = g(s)\, f(t) = gf(st) = fg(st) = f(s)\, g(t) = \alpha g(s)\, g(t)$.

Hence $\beta = \alpha$. Thus $\dfrac{f(s)}{g(s)} = \alpha$ for all $s \in S_f$ and thus $f = \alpha g$.

3. *Lemma.*

Let f and g be homogeneous polynomials with $fg=gf$. Then there exists a homogeneous polynomial $h \in A$ such that $f = \alpha\, h^n$ and $g = \beta\, h^m$ for some $\alpha, \beta \in C$ and some positive integers n and m.

Proof.

Without loss of generality we may assume that $d(f) \geqslant d(g)$.

Claim 1.

There exists a non negative integer k such that for each $s \in S_f$ there exists $r_1, \ldots, r_k \in S_g$ and $t \in F$ with $d(t) < d(g)$ and $s = r_1 \ldots r_k t$.

Proof.

Let $s \in S_f$ and let $r \in S_g$. We have $0 \neq f(s)\, g(r) = fg(sr) = gf(sr)$. Hence there exists $r \in S_g$ and $t_1 \in F$ such that $d(t_1 r) = d(s) = d(f)$ and $r_1 t_1 = s$.

Now $0 \neq f(s)\, g(r) = fg(sr) = gf(sr) = gf(r_1 t_1 r) = g(r_1)\, f(t_1 r)$. Hence $f(t_1 r) \neq 0$.

Suppose $d(t_1) \leqslant d(g)$, then by repeating the same argument with s replaced by $t_1 r$, we get r_2 and t_2 such that $t_1 = r_2 t_2$ and $f(t_2 r_1 r) \neq 0$, so $s = r_1 r_2 t_2$.
Repeating the same procedure until $d(t_k) < d(g)$ we get $s = r_1 \ldots r_k t_k$.

Claim 2.

Let s and t be as in the conclusion of claim 1. Then given any $r_1, \ldots, r_k \in S_g$; we have $r_1 \ldots r_j\, t\, r_{j+1} \ldots r_k \in S_f$ for any $j = 0, \ldots, k$. Moreover, $f(r_j \ldots r_k t r_1 \ldots r_{j-1}) = f(r_1 \ldots r_k t)$ for all j.

Proof.

By claim 1, there exists $r_1, \ldots, r_k \in S_g$ such that $s = r_1 \ldots r_k t$ for some t with $d(t) < d(g)$.

We have $g(r_1)\, f(r_2 \ldots r_k t r_1) = gf(r_1 \ldots r_k t r_1)$
$= fg(r_1 \ldots r_k t r_1) = f(r_1 \ldots r_k t)\, g(r_1) = f(s)\, g(r_1) \neq 0$.

Repeating this argument establishes the second statement of the claim. We now show that.

(1) $r\, r_{j+1} \ldots r_k\, t\, r_1 \ldots r_{j-1} \in S_f$ for all j and all $r \in S_g$ and
(2) If $r_j' \ldots r_k'\, t\, r_1' \ldots r_{j-1}' \in S_f$ then $r_{j-1}' \ldots r_k' t r_1' \ldots r_{j-2}' \in S_f$ for all $r_1', \ldots, r_k' \in S_g$.

This will establish the claim.

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Let $r \in S_g$. Then

$0 \neq g(r)\, f(r_{j+1} \dots r_k t r_1 \dots r_j) = gf(r\, r_{j+1} \dots r_k\, t\, r_1 \dots r_j) =$
$= fg(r\, r_{j+1} \dots r_k\, t\, r_1 \dots r_j) = f(r\, r_{j+1} \dots r_k\, t\, r_1 \dots r_{j-1})\; g(r_j)$
which proves (1).

Let $f(r_j' \dots r_k' t r_1' \dots r_{j-1}') \neq 0$.

Then $0 \neq g(r_{j-1}')\, f(r_j' \dots r_k'\, t\, r_1' \dots r_{j-1}') =$
$= gf(r_{j-1}' r_j' \dots r_k'\, t\, r_1 \dots r_{j-1}') =$
$= fg(r_{j-1}'\, r_j' \dots r_k'\, t\, r_1' \dots r_{j-1}') =$
$= f(r_{j-1}' \dots f_k'\, t\, r_1' \dots r_{j-2}')\, g(r_{j-1}')$.
This finishes the proof of the claim

Claim 3.

Let $T_1 = \{ t:\ d(t) < d(g)$ and there exists $r_1, \dots, r_k \in S_g$ such that $r_1 \dots r_k t \in S_f \}$.
Then given $t \in T_1$ and $r_1, \dots, r_k;\ r_1', \dots, r_k' \in S_g$

we have $\dfrac{f(r_1 \dots r_k\, t)}{g(r_1) \dots g(r_k)} = \dfrac{f(r_1' \dots r_k'\, t)}{g(r_1') \dots g(r_k')}$

Hence the map $t \rightarrow \dfrac{f(r_1 \dots r_k\, t)}{g(r_1) \dots g(r_k)}$ is a well defined function of T_1 into C.

Proof.

$g(r_1') \dots g(r_k')\, f(r_1 \dots r_k\, t) =$
$= g^k(r_1' \dots r_k')\, f(t\, r_1 \dots r_k') = g^k f\,(r_1' \dots r_1'\, t\, r_1 \dots r_k) =$
$= fg^k\,(r_1' \dots r_k'\, t\, r_1 \dots r_k) = f(r_1' \dots f_k' t)\, g(r_1) \dots g(r_k)$.

Claim 4.

Given $t \in T_1$ and $r \in S_g$ there exists $t' \in T_1$ and $r' \in S_g$ such that $tr = r't'$.

Proof.

Choose any $r_1, \dots, r_k \in S_g$. Then $r_2, \dots, r_k r t \in S_f$ and hence $0 \neq g(r_1)\, f(r_2 \dots r_k r t) = gf\,(r_1 \dots r_k r t) = f\, g\,(r_1 \dots r_k r t)$.
Hence there exists $r' \in S_g$ and $t' \in F$ such that $d(t') = d(t)$ and $rt = t'r'$.
But $r_1 \dots r_k t' \in S_f$, thus $t' \in T_1$.

Claim 5.

Choose $r_1 \dots r_k \in S_g$ and let $h_1(t) = \dfrac{f(r_1 \dots r_k t)}{g(r_1) \dots g(r_k)}$

for $t \in T_1$. Let $h_1(s) = 0$ for $s \in T_1$. Then $f = g^k h_1 = h_1 g^k$, $gh_1 = h_1 g$ and $fh_1 = h_1 f$. (Note that h_1 is independent of the choice of $r_1 \dots r_k$ by claim 3.)

Proof.

It is obvious that $f = g^k\, h_1 = h_1\, g^k$.
Now, by claim 4, $S_{gh_1} = S_{h_1 g}$.

Let $tr \in S_{h_1 g}$ with $t \in S_{h_1}$ and $r \in S_g$. Then, by claim4, there exists $r' \in S_g$ and $t' \in S_{h_1}$ such that $tr = r't'$.

We have $h_1\, g(tr) = h_1(t)\, g(r) = \dfrac{f(r_1 \dots r_{k-1}\, rt)\ g(r)}{g(r_1) \dots g\,(r_{k-1})\, g\,(r)}$
$= \dfrac{f(r_1 \dots r_{k-1}\, r\, t)}{g(r_1) \dots g(r_{k-1})} = \dfrac{f(r_1 \dots r_{k-1}\, t'\, r')}{g(r_1) \dots g\,(r_{k-1})}$
$= \dfrac{f\,(r_1 \dots r_{k-1}\, t'\, r')\ g(r')}{g\,(r_1) \dots g(r_{k-1})\, g(r')}$
$= h_1(t')\, g(r') = g(r')\, h_1(t') = gh_1(r't') = gh_1\,(tr)$.
Hence $h_1\, g = g\, h_1$

We also have $h_1 f(t' r_1 \dots r_k t) = h_1(t')\, f(r_1 \dots r_k t) =$

$= \dfrac{f(r_1\ \dots\ r_k t')}{g(r_1) \dots\ g(r)}\, f(r_1 \dots r_k t) = \dfrac{f(r_1\ \dots\ r_k t)}{g(r_1) \dots g(r_k)}\, f(r_1 \dots r_k t')$

$= h_1(t)\, f(r_1 \dots r_k t') = f(t' r_1 \dots r_k) h_1(t) = f h_1(t' r_1 \dots r_k t)$
Hence $h_1\, f = f\, h_1$. This establishes the claim.

Claim 6.

There exists a homogeneous polynomial $h \in A$ such that $g = \alpha h^n$ and $f = \beta h^m$ for some $\alpha, \beta \in C$ and some positive integers n and m.

Proof.

If $T_1 = \{1\}$ then $s \in S_f$ if and only if $s = r_1 \dots r_k$ for some $r_1 \dots r_k \in S_g$. Hence $S_f = S_g^k$, and since $f\, g^k = g^k\, f$ it follows by lemma 2 that $f = \alpha g^k$ for some $\alpha \in C$
If $T_1 \neq \{1\}$, then replacing (f,g) by (g, h_1) in the previous claims we obtain a set T_2 and $h_2 \in A$ such that $S_{h_2} = T_2$, $d(h_2) > d(h_1)$, $g = h_1^{k_1} h_2$ for some positive integer k_1 and h_2 commutes with h_1 and g.

If $T_2 \neq \{1\}$ repeat the same argument to obtain T_3 and h^3
This process will stop at the n-th stage when $T_{n+1} = \{1\}$.
Let $h = h_n$. Then it is easy to verify that h satisfies the conclusion of the claim. This finishes the proof.

Lemma 4.

Let f and $g \in A$ be homogeneous polynomials. Suppose that $f^n = g^n$ for some positive integers n and m. Then $f\,g = g\,f$.

Proof.

We may assume that $d(f) \geqslant d(g)$. Given $s \in S_f$ we have $s^n \in Sg^m = (S_g)^m$. Hence there exists $r, r' \in S_g$ and $u, u' \in F$ such that $s = ru = u'r$.

Claim 1.

If $ur \in S_f$ then $r'u \in S_f$ for all $r' \in S_g$, where $r' \in S_g$.

Proof.

We have $0 \neq g(r')\, f(ur)^n = gf^{\,n}(r'(u\, r)^n) = f^{\,n}\, g(r'(ur)^n)$

$=f^n(r' u(r u)^{n-1}) g(r) = f(r' u) f(u r)^{n-1} g(r)$.
Hence $r' u \in S_f$.

Claim 2.

If $u r \in S_f$ with $r \in S_g$ then $f(u r) = f(r u)$.

Proof.

Fix $\bar{u} \bar{r} \in S_f$ where $\bar{r} \in S_g$.
Let $s \in S_f$ be arbitrary. Let $s = u r$ where $r \in S_g$. Then
$f(r \bar{u}) f(\bar{r} \bar{u})^{n-1} = f^n(r \bar{u} (\bar{r} \bar{u})^{n-1}) = g^m (r \bar{u}(\bar{r} \bar{u})^{n-1}) =$
$= g(r) g^{m-1} ((\bar{u} \bar{r})^{n-1} \bar{u}) = g^{m-1} ((\bar{u} \bar{r})^{n-1} \bar{u}) g(r) =$
$= g^m ((\bar{u} \bar{r})^{n-1} \bar{u} r) = f^n ((\bar{u} \bar{r})^{n-1} \bar{u} r) = f(\bar{u} \bar{r})^{n-1} f(\bar{u} r)$.

Hence

$$\frac{f(\bar{u} r)}{f(r \bar{u})} = \left(\frac{f(\bar{r} \bar{u})}{f(\bar{u} \bar{r})}\right)^{n-1}$$

Now $f(ru) f(r u)^{n-1} = f^n (ru (r u)^{n-1}) = f^n(u(ru)^{n-1} r)$
(by going through g^m)
$= f^n(u r (u r)^{n-1}) = f(u r) f(u r)^{n-1}$

Hence $$\frac{f(r u)}{f(u r)} = \left(\frac{f(\bar{u} r)}{f(r \bar{u})}\right)^{n-1} = \left(\frac{f(\bar{r} \bar{u})}{f(\bar{u} \bar{r})}\right)^{(n-1)^2} = \alpha$$

Let $U = \{ u \in F : r u \in S_f$ for some $r \in S_g \}$.
Then $S_f = \{ r u : r \in S_g , u \in U\} = \{u r : r \in S_g , u \in U\}$

Hence $$\sum_{\substack{u \in U \\ r \in S_g}} f(u,r) = \sum_{g \in F} f(s) = \sum_{\substack{u \in U \\ r \in S_g}} f(r,u).$$

Thus $\alpha = 1$, establishing the claim.

Claim 3.

$f g = g f$.

Proof.

Let $s \in S_f$ and $r \in S_g$. Then there exists $s_1 \in S_f$ and $r_1 \in S_g$ such that $s r = r_1 s_1$, and hence there exists $u \in F$ such that $s = r_1 u$, so $ur = s_1$ since $r_1 ur = r_1 s_1$. Thus we have $f(u r_1)^n g(r) = f(r_1 u)^n g(r) = g^{m+1} ((r_1 u)^n r) =$
$= g(r_1) g^m (u(r_1 u)^{n-1} r) = g(r_1) f^n((u r_1)^{n-1} u r) =$
$= g(r_1) f(u r_1)^{n-1} f(u r)$.
Hence $f(u r_1) g(r) = g(r_1) f(u r)$. Thus $fg(sr) = fg(r_1 ur) =$
$= f(r_1 u) g(r) = f(u r_1) g(r) = g(r_1) f(u r) = g(r_1) f(s_1) =$
$= g f(r_1 s_1) = g f(s r)$.
Hence $f g = g f$. This finishes the proof.

5. *Lemma.*

Let f and g be homogeneous polynomials in A. If $f g^k = g^k f$ for some positive integer k, then there exists $\alpha \in C$ and positive integers m and n such that $f^m = \alpha g^n$

Proof.

Let $h_1 = f^{k\, d(g)}$ and let $h_2 = g^{k\, d(f)}$.
Then $h_1 h_2 = h_2 h_1$ and $d(h_1) = k\, d(g)\, d(f) = d(h_2)$.
Hence by Lemma2 there exists $\alpha \in C$ such that $h_1 = \alpha h_2$.
Hence $f^{k\, d(g)} = g^{k\, d(f)}$.

The following theorem summarizes the results in the previous lemmas.

6. *Theorem.*

Let f and g be homogeneous polynomials in A. Then the following statements are equivalent.

(a) $f g = g f$.

(b) $f g^k = g^k f$ for some positive integer k.

(c) $f^m = \alpha g^n$ for some $\alpha \in C$ and some positive integers m and n.

(d) There exists $h \in A$ such that $f = \alpha h^m$ and $g = \beta h^n$ for some $\alpha, \beta \in C$ and some positive integers m and n.

Proof.

(a) $\Rightarrow$ (b) : trivial
(b) $\Rightarrow$ (c) : by Lemma 5
(c) $\Rightarrow$ (a) : by Lemma 4
(a) $\Rightarrow$ (d) : by Lemma 3
(d) $\Rightarrow$ (a) : trivial.

7. *Corollary.*

If f, g, and h are homogeneous polynomials such that $f h = h f$ and $g h = hg$, then $f g = g f$.

Proof.

By theorem 6, $f^n = ah^m$ for some $n, m > 0$ and $a \in C$. Hence $f^n g = g f^n$ and thus $f g = g f$ by (b) of theorem 6

8. *Corollary.*

If $f, g \in A$ with f a homogeneous polynomial and $f g = g f$, then there is a homogeneous polynomial h such that $f = a h^n$ and $g = \sum_{i=0}^{\infty} b_i h^i$ for some $n > 0$ and complex numbers $a, b_0, b_1, \ldots$

Proof.

Note that for every positive integer m, we have

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$f g_m = (f g)_m = (g f)_m = g_m f$. Hence by corollary 7, the elements f, g_1, g_2, ... are pairwise commutative.

Let B be a maximal commutative subalgebra containing f (such subalgebra exists by Zorn's lemma). Then $g_m \in B$ for all $m > 0$.

Choose $h \in B$ to be a homogeneous polynomial of minimal degree. Then there exists $k \in A$ such that $f = ak^n$ and $h = k^m$. By corollary 7, $g k_m = g_m k$ for all $m>0$. Hence, $k \in B$ by maximality of B and thus $k = h$ by minimality of d(h).

Repeat the above argument with g_m in place of f, to get $g_m = c_m h^{k_m}$ for some $k_m \geqslant 0$ and $c_m \in C$. Therefore $f = a\ h^n$ and $g = \sum_{i=0}^{\infty} g_i = \sum_{i=0}^{\infty} c_i\, h^{k_i}$.

9. *Lemma.*

A has no zero divisors.

Proof.

Suppose $f \neq 0 \neq g$. Let $s \in m_1(f)$ and $t \in m_1(g)$. Then $f g(s\, t) = f(s)\, g(t) \neq 0$.

10. *Theorem.*

Let $f, g \in A$ with $f^n g = g f^n$ for some positive integer n. Then $f g = g f$.

Proof

We have $f^n(fg - gf) = (fg - gf)f^n$.
Therefore, $f_1^n(fg - f)_1 = (f^n(fg - gf))_1 = ((fg - gf)f^n)_1 = (fg - gf)_1 f_1^n$ (1)

From the fact that $f^n g = g f^n$, we get,

$$\sum_{k=0}^{n-1} f^{n-k-1}(fg - gf)f^k = 0.$$

Hence $0 = \left[\sum_{k=0}^{n-1} f^{n-k-1}(fg - gf)_f f^k\right]_1 = \sum_{k=0}^{n-1} f_1^{n-k-1}(fg - gf)_1 f_1^k = n f_1^{n-1}(fg - gf)_1$; by (1).

Hence, by lemma 9, $f_1 = 0$, or $(fg - gf)_1 = 0$ and therefore $f = 0$ or $f g = g f$.

11. *Theorem.*

If $f, g \in A$ with $f^n = g^n$ for some positive integer n. Then $f = a g$ for some $a \in C$ such that $a^n = 1$. (This result shows uniqueness, in some sense, of n-th roots, when they exist).

Proof.

If $f^n = g^n$, then $f g = gf$ by theorem 10, and the result follows by factoring out $f^n - g^n$ and applying lemma 9.

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