The Arabian Journal for Science and Engineering
November 1975, Volume 1, Number 1.

ALGEBRAIC LIE ALGEBRAS

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خــلاصة

لقد عرف س . شيفالي احدى جبر لي الجبري باستعماله مبدأ المجموعات الجبرية ، بعد ذلك وصف هذه الجبريات.
أمــا جاكسبون فقد عرف مبدأ « تكاد أن تكون جبر لي الجبري للتحويلات الخطية » .
وفي هذا البحث تستعمل بعض نتائج الجبر الخطي لنحصل على النظرية التالية :
نظرية : اذا كانت L احدى جبر لي الجبري معرفه على مجال له مميز قيمته تساوي صفر فان L تكاد أن تكون جبرية .

ABSTRACT.

C. Chevalley gives a definition of an algebraic Lie algebra using the concept of algebraic group and characterizes these algebras. Jacobson has defined the concept of an almost algebraic Lie algebra of linear transformations. In this paper we give a purely algebraic definition of an algebraic Lie algebra. Using results from linear algebra we obtain:

Theorem. If L is an algebraic Lie algebra over a field of characteristic zero, then L is almost algebraic.

INTRODUCTION

In this paper F denotes a field of characteristic zero. All vector spaces over F will be finite dimensional. Let L be a vector space over a field F with a binary composition defined from LxL to L. L is called an algebra provided:

(i) (bilinear condition or distributive property) $(x+y)z=xz+yz$, $x(y+z)=xy+xz$ for x, y, z in L

and (ii) $c(xy)=x(cy)$ for $c \in F$ and $x,y \in L$.

L is called a lie algebra if L is an algebra and

(iii) $xx=0$ for all x in L

and (iv) (Jacobi identity) $(xy)z+(yz)x+(zx)y=0$ for $x, y, z \in L$.

The basic structure theory for Lie algebras can be found in [4] and [5].

If V is a vector over F, denote by End (V) the set of all linear transformations of V into V. End (V) forms a ring. In particular End (V) is an associative algebra over F. If we define a new binary operation $[x,y]=xy-yx$ called the bracket of x and y or the Lie product or commutator, we get a Lie algebra structure on End (V). In order to distinguish the Lie algebra structure from the associative algebra structure we denote this algebra by gl(V).

Jacobson [3] has defined almost algebraic. In this paper we give a purely algebraic definition of an algebraic Lie algebra, and show that algebraic implies almost algebraic. The usual definition that an algebraic Lie algebra is the Lie algebra of some algebraic group is seen to be equivalent to our definition; see [1, proposition 2, p. 181].

RESULTS FROM LINEAR ALGEBRA

For completeness we collect some results from linear algebra.

Theorem 1.

Let A be an endomorphism (linear transformation) of a vector space V over F. The following are equivalent:

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(i) A is *semisimple.*

(ii) The minimum polynomial P for A is of the form $P=P_1 P_2 \dots P_k$ where P_i $(i=1,\dots,k)$ are distinct irreducible polynomials of F[x].

(iii) A is diagonalizable over a splitting field K of F.

(iv) Every A-invariant subspace W of V has a complementary A-invariant subspace.

(v) Every subspace W of V is a T-admissible subspace of V. Recall that W is T-admissible means that W is invariant under T and if f(T)u is in W for $f \in F[x]$ and any u in V there exists a vector w in W such that $f(T)u = f(T)w$.

Theorem 2.

Let A be an endormorphism of a vector space V over F. The following are equivalent.

(i) A is *nilpotent.*

(ii) $A^n = 0$ for some positive integer n.

(iii) All characteristic values of A are zero.

The basic ideas of the proofs of the above two Theorems which characterize semisimple and nilpotent endomorphisms may be found in Hoffman and Kunze [3].

Definition 3.

A Lie Algebra L of linear transformations of a vector space over F is called *almost algebraic* if it containts the nilpotent and semisimple components for each $A \in L$.

Recal that a field F of characteristic zero has an algebraic closure.

Theorem 4.

Let F be a field of characteristic zero and V be a vector space over F. If A is an endomorphism on V, then

(i) there exists a semisimple endomorphism S and a nilpotent endomorphism N so that $A=S+N$.

(ii) S and N commute, i.e., $SN=NS$.

(iii) S and N are unique.

(iv) S and N are polynomials in A.

Proof.

See [3, Theorem 13, p. 267].

The above decomposition of $A = S + N$ into semisimple and nilpotents is called the *additive Jordan decomposition* of A.

Definition 5.

A map $\rho: L \to gl(V)$ is a representation of the Lie algebra L if ρ is a F-linear map and

$$\rho(x,y) = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x)$$

for all x,y in L.

Theorem 6.

Let $V^* = \mathrm{Hom}(V,F)$ be the dual space of the vector space V. Define

$$V_{pg} = \underbrace{V \otimes \dots \otimes V}_{p\text{-copies}} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{q\text{-copies}}$$

where p and q are non-negative integers.

The map $\rho: gl(V) \to gl(V_{pq})$ defined by $\rho(A) = A_{pq}$ is a representation of $gl(V)$ in $gl(V_{pq})$.
A_{pq} is defined by

$$\begin{aligned} A_{pq}(v_1 \otimes \dots \otimes vp \otimes v_1^* \otimes \dots \otimes v_q^*) = {} & Av_1 \otimes \dots \otimes v_p \otimes v_1^* \otimes \dots \otimes v_q^* \\ & + v_1 \otimes Av_2 \otimes \dots \otimes v_p \otimes v_1^* \otimes \dots \otimes v_q^* + \dots \\ & + v_1 \otimes \dots \otimes v_p \otimes A^* v_1^* \otimes \dots \otimes v_q^* + \dots \\ & + v_1 \otimes \dots \otimes v_p \otimes v_1^* \otimes \dots \otimes A^* v_q^* \end{aligned} \tag{I}$$

where $v_i \in V$ for $i=1,\dots,p$; $v_j^* \in V^*$ for $j=1,\dots,q$ and A^* is the transpose of A. A very readable treatment of tensor product may be found in [6].

Example 7.

Let $p=q=1$. Thus $\rho: g_l(V) \to g_l(V_{11})$ where $V_{11} = V \otimes V^*$.

Clearly ρ is F-linear. To show
$\rho([A,B]) = [A,B]_{11} = [A_{11}, B_{11}] = [\rho(A), \rho(B)]$.

Now for $v \in V$ and $\lambda \in V^*$,

$$\begin{aligned} [A,B]_{11}(v \otimes \lambda) &= [A,B]v \otimes \lambda + v \otimes [A,B]^* \lambda \\ &= [A,B]v \otimes \lambda + v \otimes [A^*, B^*] \\ &= (ABv - BAv) \otimes \lambda + v\,(A^*B^*\lambda - B^*A^*\lambda) \\ &= ABv \otimes \lambda - BAv \otimes \lambda + v \otimes A^*B^*\lambda - v \otimes B^*A^*\lambda. \end{aligned}$$

Also,

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$$(A_{11}B_{11}-B_{11}A_{11})(v\otimes\lambda)=A_{11}B_{11}(v\otimes\lambda)-B_{11}A_{11}(v\otimes\lambda)$$
$$= A_{11}(Bv\otimes\lambda + v\otimes B^*\lambda) - B_{11}(Av\otimes\lambda) - B_{11}(v\otimes A^*\lambda)$$
$$= ABv\otimes\lambda + Bv\otimes A^*\lambda + Av\otimes B^*\lambda + v\otimes A^*B^*\lambda - BAv$$
$$-BAv\otimes\lambda - Av\otimes B^*\lambda - Bv\ A^*\lambda - v\otimes B^*A^*\lambda$$
$$= ABv\otimes\lambda + v\otimes A^*B_*\lambda - BAv\otimes\lambda - v\otimes B^*A^*\lambda.$$

Comparing the above two equations, we see that ρ is a representation.

ALGEBRAIC LIE ALGEBRAS.

We now give a purely algebraic definition of an algebraic Lie algebra which is independent of the concept of an algebraic group. Then the main result of this paper is established in Theorem 12.

Definition 8.

Let A be an endomorphism of a vector space V. Then an endormorphism B of V is called a *replica* of A if and only if B kills the tensors killed by A, i.e., for each positive integers p,q then $A_{pq}(v)=0$ if and only if $B_{pq}(v)=0$.

Definition 9.

A Lie algebra contained in gl(V) is called an *algebraic Lie algebra* if and only if L contains the replicas for all its elements.

Lemma 10.

Let $A\in End\ (V)$ where $A=S+N$ is the additive Jordan decomposition of A into semisimple and nilpotent components S and N respectively. Then the representation $\rho : A \rightarrow A_{pq}$ preserves Jordan decomposition, i.e., $A_{pq}=S_{pq}+N_{pq}$.

Proof.

Since ρ is a representation $[S_{pq},N_{pq}] = [S,N_{pq}] =0$ Therefore $S_{pq}N_{pq}=N_{pq}S_{pq}$. Let the dimension of V be n. By Theorem 1(i) if and only if (iii), choose a basis $\{v_1,\ldots,v_n\}$ of V consisting of characteristic vectors for S. If F is not a splitting field for S we may extend F to a field K which is a splitting field. Also choose a dual basis $\{v_1^*,\ldots,v_n^*\}$ of V* consisting of characteristic vectors for S*. Thus $\{ v_{i1}\ v_{j1}\ \ldots\ v_{jp}^*\}$ is a basis for V_{pq} and S_{pq} has these basis elements as characteristic vectors. So S_{pq} is semisimple. N_{pq} is a sum of commuting nilpotents of the form

$$1_v\otimes\cdots\otimes N\otimes\ \cdots\otimes 1_v^*\otimes\cdots\otimes 1_v^*$$
$$1_v\otimes\cdots\otimes 1_v\otimes 1_v^*\otimes\cdots\otimes N^*\otimes\cdots\otimes 1_v^*.$$

Therefore, N_{pq} is nilpotent.

Theorem 11.

Let $A_{pq}\in\rho(gl(V))\subset gl(V_{pq})$. Then there exists semisimple S_{pq} in $gl(V_{pq})$ and nilpotent N_{pq} in $gl(V_{pq})$ such that

(i) $A_{pq} = S_{pq}+N_{pq}$

(ii) $S_{pq}N_{pq} = N_{pq}S_{pq}$

(iii) S_{pq} and N_{pq} satisfying (i) and (ii) are unique and each is a polynomial in A_{pq}.

Proof.

Statements (i) and (ii) follow from Lemma 10. The uniqueness part of (iii) is the same as the uniqueness proof in the additive Jordan decomposition (see [3, Theorem 13, p.222]). Let $A\in gl(V)$ and, $A=S+N$ be the additive Jordan decomposition. By [3, Theorem 13, p.267], s is a polynomial in A. Hence there is f in F[x] so that $f(A)=S$. By using equation I we see that $f(A_{pq})=S_{pq}$. Since $N_{pq}=A-S_{pq}$, N_{pq} is also a polynomial in A_{pq} .

Theorem 12.

Let L be an algebraic Lie algebra over a field of characteristic zero. Then L is almost algebraic.

Proof.

Let $A\in L$, then $A\in gl(V)$. Thus A has an additive Jordan decomposition, i.e., $A = S + N$. Suppose $A_{pq}(V)=0$. Since S_{pq} is a polynomial in A_{pq}, we have an f in F[x] so that $S_{pq}=f(A_{pq})$. Consequently, $f(A_{pq})(v)=0$ for each $v\in V_{pq}$ for which $A_{pq}(V)=0$. Otherwise, if $A_{pq}(v)=0$ and $f(A_{pq})(v)\neq 0$ then $f(A_{pq})(v)=cv$ for some $c\in F$. This means v is a characteristic vector for S_{pq} . Hence v is a characteristic vector for A_{pq}. But this would contradict the fact that $A_{pq}(v)=0$. This shows that S_{pq} is a replica of A_{pq}. L algebraic implies S is in L. So $N=A-S$ is in L. Therefore L is almost algebraic.

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Reference Code for AJSE Information Retrieval QA1157DA1. Paper received March 8, 1975.