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EXISTENCE THEOREM FOR ABELIAN GROUPS

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خلاصة

هذا البحث يتعلق بعمل إمتداد لبعض التطابقات الذاتية الجزئية لمجموعات إبيليه . ويسمى تطابقاً جزئياً لمجموعة G أي تناظر تطابقي μ يرسم تحت مجموعة $A \subseteq G$ على تحت مجموعة أخرى $B \subseteq G$

ومن المعلوم أن أي تطابق ذاتي جزئي μ لمجموعة G يمكن إمتداده إلى تطابق كلي μ^* لمجموعة G^* تحوى المجموعة G ، والإمتداد هنا يعني أن μ^* معرفه لجميع عناصر تحت المجموعة A وأن $a\mu = a\mu^*$ لكل عنصر $a \in A$.
وقد أمكن بعد ذلك إيجاد شروط كافية لإمتداد تطابقين جزئيين μ, ν لمجموعة إبيليه G إلى تطابقين كليين ν^* ، μ^* لمجموعة ابيلية G^* تحوى المجموعة الابيلية G^* بحيث أن $\nu^*\mu^* = \mu^*\nu^*$.
وفي هذا البحث تمكن من إيجاد شروط كافية لامتداد تطابقات ذاتية جزئية μ, ν, τ لمجموعة ابيلية G إلى تطابقات كلية ω ، $\varnothing$ ، θ لمجموعة ابيلية K تحوى المجموعة الأبيلية G بحيث أن $\theta\omega = \omega\theta$ ، $\varnothing\omega = \omega\varnothing$ ، $\theta\varnothing = \varnothing\theta$.

ABSTRACT

The main purpose of the following paper is to derive conditions which are sufficient for extending three partial automorphisms of an abelian group G to three automorphisms of an abelian supergroup $K \supseteq G$ such that these automorphisms commute among themselves.

INTRODUCTION

By a partial automorphism of a group G we mean an isomorphic mapping μ of a subgroup $A \subseteq G$ onto another subgroup $B \subseteq G$, where B need not be different from A. If μ is defined on the whole of G then it is usually called automorphism on G. It is known [4] that any partial automorphism of a group can always be extended to an automorphism of a supergroup. It is also known [1] that under certain sufficient conditions, two partial automorphisms of an abelian group can be extended to two commutative automorphisms of an abelian supergroup. In this paper we consider a given abelian group G and three partial automorphisms μ, ν and τ of G and derive conditions which are sufficient for μ, ν and τ to be all extendable to automorphisms θ, $\varnothing$, and ω, respectively of one and the same abelian supergroup $K \supseteq G$ such that θ, $\varnothing$ and ω commute among themselves. The principal tool throughout is the direct product of two groups with one amalgamated subgroup [2].

THE CONSTRUCTION

Let G be an abelian group which contains subgroups A, B, C, D, F, and H and three partial automorphisms μ, ν and τ that maps A isomorphically onto B, C isomorphically onto D, and F isomorphically onto H respectively. Assume that:

1) $(A \cap C)\mu = B \cap C$ 2) $(A \cap D)\mu = B \cap D$
3) $(A \cap F)\mu = B \cap F$ 4) $(A \cap H)\mu = B \cap H$
5) $(C \cap A)\nu = D \cap A$ 6) $(C \cap B)\nu = D \cap B$
7) $(C \cap F)\nu = D \cap F$ 8) $(C \cap H)\nu = D \cap H$
9) $(F \cap A)\tau = H \cap A$ 10) $(F \cap B)\tau = H \cap B$
11) $(F \cap C)\tau = H \cap C$ 12) $(F \cap D)\tau = H \cap D$
13) $g\mu\nu = g\nu\mu$, $g\mu\tau = g\mu\tau$ and $g\nu\tau = g\tau\nu$

whenever $(g\mu)$, $(g\tau)$, $(g\nu)$, $(g\mu)\nu$, $(g\mu)\tau$, $(g\nu)\mu$, $(g\nu)\tau$ $(g\tau)\mu$, $(g\tau)\nu$ are defined.

Define for each i in I, the set of all integers, a group G_i isomorphic to G under a fixed isomorphism γ_i :

$$G\gamma_i = G_i$$

Thus each G_i contains subgroups A_i, B_i, C_i, D_i, F_i, and H_i which are isomorphic images of A, B, C, D, F and H under γ_i, and there exist isomorphisms $\mu_i = \gamma_i^{-1}\mu\,\gamma_i$, $\nu_i = \gamma_i^{-1}\nu\gamma_i$ and $\tau_i = \gamma_i^{-1}\tau\gamma_i$ mapping A_i onto B_i, C_i onto D_i and F_i onto H_i respectively

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A_i, B_i, C_i, D_i, F_i, and H_i satisfy the conditions that correspond to 1 — 13.

Now we define a sequence of groups $P_{i,j}$ for all $i, j \in I$ and $i < j$ as follows: we first form the direct product of G_i and G_{i+1} amalgamating $B_i \subseteq G_i$ with $A_{i+1} \subseteq G_{i+1}$ according to the isomorphism $\gamma_i^{-1}\mu^{-1}\gamma_{i+1}$. Call this direct product $P_{i,i+1}$:

$$P_{i,i+1} = \{ G_i \times G_{i+1} ; B_i = A_{i+1} \}$$

Then define $P_{i,j}$ inductively to be the direct product

$$P_{i,j} = \{P_{i,j-1} \times G_j; B_{j-1} = A_j\}$$

amalgamating $B_{j-1} \subseteq P_{i,j-1}$ with $A_j \subseteq G_j$ according to the isomorphism $\gamma_{j-1}^{-1} \mu^{-1} \gamma_j$

If we form $K_1 = \bigcup_{n=1}^{\infty} P_{-n,n}$

then K_1 is evidently abelian.

Using Lemmas 1 and 2 in Reference[1] and through steps similar to those in Lemmas 5-8, Reference [3], we can prove that K_1 possesses an automorphism $\theta_1$that extends each μ_i and partial automorphisms $\varnothing_1$ that extends each ν_i, and ω_i that extends each τ_i such that: $\varnothing_1$ maps the subgroup

$$V = \{..., C_{-1}, C_o, C_1, ...\} \subseteq K_1$$

onto the subgroup

$$W = \{..., D_{-1}, D_o, D_1, ...\} \subseteq K_1$$

and ω_1 maps the subgroup

$$Y = \{..., F_{-1}, F_o, F_1, ... \} \subseteq K_1$$

onto the subgroup

$$Z = \{..., H_{-1}, H_o, H_1, ...\} \subseteq K_1$$

and θ_1 maps V, W, Y, and Z each onto itself.

Lemma.

In K_1 the following holds

(i) $(V \cap Y)\varnothing_1 = W \cap Y$ (ii) $(V \cap Z) \varnothing_1 = W \cap Z$

(iii) $(Y \cap V) \omega_1 = Z \cap V$ (iv) $(Y \cap W)\omega_1 = Z \cap W$

(v) $k_1 \varnothing_1 \omega_1 = k_1 \omega_1 \varnothing_1$ whenever $k_1 \varnothing_1$, $k_1 \omega_1$ $(k_1 \varnothing_1)\omega_1$, $(k_1 \omega_1)\varnothing_1$ are defined.

Proof.

By the use of lemmas 5 and 6 in Reference [3] together with the uniqueness of the normal form of the elements of the direct product of groups with one amalgamated subgroup [2], any element $x \in V \cap Y$ can be written in the form

$$x = x_{\alpha(1)} x_{\alpha(2)} \cdots x_{\alpha(n)},$$

Where each $x_{\alpha(i)} \in C_{\alpha(i)} \cap F_{\alpha(i)}$ and $\alpha(1) < \alpha(2) <, \ldots < \alpha(n)$. Thus

$$x\varnothing_1 = (x_{\alpha(1)} \nu_{\alpha(1)}) (x_{\alpha(2)} \nu_{\alpha(2)}) \ldots (x_{\alpha(n)} \nu_{\alpha(n)}),$$

where each $x_{\alpha(i)} \nu_{\alpha(i)} \in D_{\alpha(i)} \cap F_{\alpha(i)}$ by condition 8. Thus $x\varnothing_1 \in W \cap Y$, and$(V \cap Y) \varnothing_1 \subseteq W \cap Y$. On the other hand, any element $y \in W \cap Y$ can be written in the form

$$y = y_{\beta(1)} y_{\beta(2)} \cdots y_{\beta(m)}$$

where each $y_{\beta(j)} \in D_{\beta(j)} \cap F_{\beta(j)} =$

$$(C_{\beta(j)} \cap F_{\beta(j)}) \nu_{\beta(j)} = (C_{\beta(j)} \cap F_{\beta(j)})\varnothing_1,$$

Thus $y \in (V \cap Y) \varnothing_1$, and hence $(W \cap Y) \subseteq (V \cap Y)\varnothing_1$, which completes the proof of (i). Similarly we prove (ii), (iii), and (iv). The proof of (v) is by direct calculation, putting under consideration condition 13 and that $\varnothing_1$ and ω_1 extend each ν_i and τ_i respectively. This completes the proof of the lemma.

Using the above lemma and lemma (3) in Reference [1] we can prove that: If we replace G; A, B, C, D, F, H; μ, ν, τ by K_1; V, W, Y, Z, K_1, K_1; $\varnothing_1$ ω_1, θ_1 respectively then the conditions that correspond to 1) — 13) will be satisfied.

Thus we can repeat the above procedure, this time embedding K_1 in an abelian group K_2 which posesses an automorphism $\varnothing_2$ that estends $\varnothing_1$ and two partial automorphisms ω_2 and θ_2 extending ω_1 and θ_1 respectively such that $\varnothing_2$, ω_2, θ_2, commute among themselves.

We carry on indefinitely, thus when K_{n-1} is formed we embed it in the abelian group K_n that posesses three mappings θ_n, $\varnothing_n$ and ω_n one of which is an automorphism and the others are parial automorphisms such that θ_n, $\varnothing_n$, and ω_n commute among themselves.

Finally we form the group

$$K = \bigcup_{n=1}^{\infty} K_n$$

which is abelian and define the mappings θ, $\varnothing$ and ω. as follows: For any $k \in K$, $k \in K_i$ for some i and we put

$$k\,\theta = k\,\theta_i \; ; \quad , \quad k\,\varnothing = k\,\varnothing_i \, , \quad k\omega = k\omega_i$$

Thus θ, $\varnothing$ and ω are automorphisms of K which extend each θ_i, $\varnothing_i$ and ω respectively and hence extend μ, ν and τ.

Using Lemma 4 in Reference [1] we can prove that θ, $\varnothing$ and ω commute among themselves.

This completes the proof of the following theorem

Theorem.

Conditions 1 - 13 are sufficient for extending three partial automorphisms μ, ν and τ of an abelian group G to automorphisms θ, $\varnothing$, and ω of an abelian group $K \supseteq G$ such that θ, $\varnothing$, ω commute among themselves.

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