BASIC METHODS FOR STABILITY ANALYSIS OF NONLINEAR OSCILLATIONS AND WAVES

F. M. Hamzeh*

Department of Physics,
University of Petroleum and Minerals,
Dhahran, Saudi Arabia

الخلاصة :

إن التحليل الخطي للاستقرارات ، وتحليل ليابوتوف ، ومخطط نوكوست طرق ذات جدوى لدراسة استقرار البلازما . فتطبيقات هذه الطرق عامة وليست مقتصرة على فيزياء البلازما . تعطي هذه المقالة دراسة مكثفة لطبيقات هذه الطرق في تحليل استقرار الأمواج والإهتززات اللاخطية .

ABSTRACT

Linear stability analysis, Lyapunov analysis, and the Nyquist diagram are powerful tools with which to study plasma stability. Their applications, however, are not limited to the field of plasma physics. This paper presents a comprehensive (nonspecialized) discussion of these methods for stability analysis of nonlinear oscillations and waves.

* Present address: *Department of Systems Science, University of California, Los Angeles, CA 90024, U.S.A.*

0377–9211/83/020087–11$01.10

BASIC METHODS FOR STABILITY ANALYSIS OF NONLINEAR OSCILLATIONS AND WAVES

1. EQUILIBRIUM AND STABILITY

An equilibrium is a state in which all forces are balanced. The equilibrium is stable or unstable according to whether small perturbations are damped or amplified. To illustrate, consider a simple one-dimensional system in a potential field, such as a marble in a vertical gravitational field, e.g. a marble on a surface represented by Figure 1. The positions A, B, C, E, F, and G are positions of equilibrium. In these positions, the force, which is proportional to the slope, is zero. D is not an equilibrium position; the marble if placed there is acted on by a force accelerating it toward C.

The equilibria B and C are, however, very different. The slightest perturbation of the marble from position B results in a force which accelerates it away from the equilibrium. At position C if the marble is perturbed it is acted on by a force which returns it toward C; the marble here executes oscillations around the equilibrium position. Since a small perturbation of the marble can never be completely avoided, it is clear that for practical purposes the unstable position B is no better than the nonequilibrium position D.

In position E the marble is stable; however, if it is moved beyond a threshold, it is in an unstable state. This is called an 'explosive instability'. A similar situation occurs in position G. At E we have linear stability and nonlinear instability, while at G we have a state of metastable equilibrium. In position F the marble is unstable, but it cannot go too far; it has limited excursions. At F we have a linear instability and a nonlinear stability. In position A the marble is said to be neutrally stable.

In general, if the solution of an ordinary or partial differential equation describing a given process becomes infinite for an increasing independent variable (mainly the time), then we call the solution unstable and we speak of an instability. Sometimes an instability occurs during a finite time t_0,

$$\lim_{t \to t_0} \Psi(t) = \infty, \text{ e.g. for } \Psi(t) \simeq \frac{1}{t - t_0},$$

where Ψ represents, for example, the amplitude of the oscillation or wave, then we speak of an explosive instability.

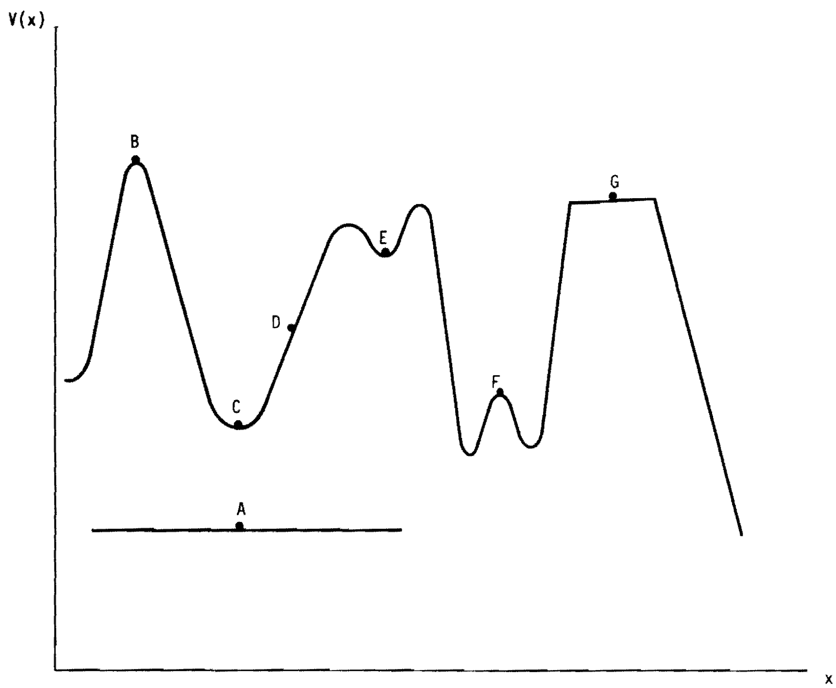

Figure 1.

2. OSCILLATIONS AND WAVES

Oscillations act on a single particle; the space coordinates $x(t)$, $y(t)$, $z(t)$ of an oscillating particle obey ordinary differential equations. Because of the interaction between particles the oscillatory motion of a single particle is spread over space and waves are generated. Waves amplitudes $\Psi(x,y,z,t)$ depend on space and time and are therefore described by partial differential equations. Both oscillations and waves are characterized by three basic quantities: the amplitude, the frequency, and the phase.

If the ordinary (or partial) differential equation describing a given process is linear, then the process and the instabilities are also called linear. Terms containing powers of the dependent variable (amplitude) or of its derivatives make the differential equation nonlinear. The process and the instabilities are then called nonlinear. (Nonlinear dispersion relations belong to nonlinear wave equations and contain the wave amplitude.)

3. OSCILLATIONS: A SIMPLE ONE-DIMENSIONAL PROBLEM

Consider the equation of motion in the immediate neighborhood of equilibrium. If the coordinate of equilibrium is x_0 and the force $f(x)$, one finds

$$m\ddot{x}=f(x)=f(x_0)+f'(x_0)(x-x_0)+\ldots, \tag{1}$$

where primes and dots refer to derivatives with respect to the x coordinate and the time respectively. At the equilibrium position, $f(x_0)=0$. If we introduce the displacement $\xi=(x-x_0)$, for small ξ we can neglect higher-order terms in (1), the resulting equation of motion,

$$m\ddot{\xi}=f'(x_0)\xi \tag{2}$$

yields the solution

$$\xi=\xi_0 e^{-i\omega t}, \tag{3}$$

where

$$\omega^2=-\frac{f'(x_0)}{m}. \tag{4}$$

The frequency* ω is determined by the slope $f'(x_0)$ of the restoring force. For $f'(x_0)<0$, ω is real and ξ oscillates; the equilibrium is stable. For $f'(x_0)>0$, ω is imaginary and instability may occur.

These results are easily understood from the energy principle. For a conservative system the sum of the kinetic and potential energies is constant. At equilibrium, since $f(x)=-V'(x)$, the potential gradient $V'(x_0)=0$ and if $\omega^2=V''(x_0)/m$ (from Equation (4)) is positive it follows that a displacement in either direction increases the potential energy $(\xi^2V''(x)_0)>0)$, and therefore decreases the kinetic energy. This is the case of the marble in a potential well, the marble in static equilibrium at the bottom of the well has a zero kinetic energy and therefore cannot climb out of the well without help. On top of a potential hill, however, where $\omega^2=V''(x_0)/m$ is negative, the farther the marble rolls the more kinetic energy it acquires and the faster it runs away.

In the presence of friction the motion is damped, the process is described by Equation (3) with complex ω and Im $\omega<0$. One can also contruct a system with positive feedback. For example, an external energy pump which feeds energy into an oscillating system. If the process is linear it is described by Equation (3) with complex ω, but here Im $\omega>0$. Thus, to introduce linear damping (positive or negative) in Equation (1), it is convenient to consider solutions of the form (3) with complex ω:

$$\omega\equiv\omega_r+i\omega_i \tag{5}$$

where obviously $\omega_r\equiv\mathrm{Re}\,\omega$ and $\omega_i\equiv\mathrm{Im}\,\omega$. The various possible cases are given in Table 1. $\gamma\equiv|\omega_i|$ is the

*Customary name for ω by plasma physicists; it is the angular frequency measured in radians per second.

Table 1.

Type	ω_r	ω_i	Equation		The solution is
1	$\gtrless 0$	<0	$e^{-\gamma t}e^{-i\omega_r t}$	Stable	Oscillatory, damped
2	$\gtrless 0$	>0	$e^{\gamma t}e^{-i\omega_r t}$	Unstable	Oscillatory, increasing
3	0	<0	$e^{-\gamma t}$	Stable	Aperiodic, damped
4	0	>0	$e^{\gamma t}$	Unstable	Aperiodic, increasing
5	$\lessgtr 0$	0	$e^{-i\omega_r t}$	Stable	Periodic
6	0	0	Constant	Stable	Marginal, no motion

growth rate. Needless to say that unstable solutions are determined by $\omega_i > 0$.

4. OSCILLATIONS: THE DIFFERENTIAL EQUATION

An extremely high number of oscillatory processes in science and engineering are described by equations of the type

$$\ddot{x} + q(t)g(\dot{x})h(x) + p(t)f(x) = F(t). \tag{6}$$

The first l.h.s. term is the acceleration. The second l.h.s. term describes damping or dissipation. The third l.h.s. term is the restoring force. (The restoring force describes the tendency of the system to return to or deviate from equilibrium.) The r.h.s. term describes the external force. $q(t)$ and $p(t)$ contain parametric effects. The same characteristic types of solutions will be shown to appear for the nonlinear Equation (6) as for the linear equation discussed above.

5. LINEAR STABILITY ANALYSIS

The linearized analysis of equilibria consists in the transformation of Equation (6) into three first-order equations. If the time does not appear explicitly in the equation, i.e. if $p =$ constant, $q =$ constant, and $F(t) = 0$, then Equation (6) is called autonomous and two equations are sufficient.

$$\dot{x} \equiv Q(x, \dot{x}) = \dot{x}, \tag{7}$$

$$\ddot{x} \equiv P(x, \dot{x}) = -qg(\dot{x})h(x) - pf(x). \tag{8}$$

In order to simplify our calculations we consider explicit forms for the functions $g(\dot{x})$, $h(x)$, and $f(x)$. For physical reasons, we choose symmetric attenuation in the form

$$g(\dot{x}) = x, \tag{9}$$

$$qh(x) = -a - bx^2, \tag{10}$$

and antisymmetric restoring force

$$pf(x) = -cx - dx^3. \tag{11}$$

This choice comprises not only the Van der Pol equation but also Duffing's equation and Lashingsky's equation. In this case, the equation for the acceleration becomes

$$P(x, \dot{x}) = a\dot{x} + bx^2\dot{x} + cx + dx^3, \tag{12}$$

which with Equation (7) forms the desired set of two first-order equations. This set of equations has the following equilibria:

$$\text{E1: } x = 0,\ \dot{x} = 0 \tag{13}$$

and

$$\text{E2: } x = \pm(-c/d)^{1/2},\ \dot{x} = 0. \tag{14}$$

We look for a solution of the form

$$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \tag{15}$$

where $\lambda_{1,2}$ are the roots of the characteristic equation

$$\lambda^2 - (P_{\dot{x}} + Q_x)\lambda + P_{\dot{x}}Q_x - P_x Q_{\dot{x}} = 0. \tag{16}$$

In Equation (16), the various quantities are defined as follows:

$$P_x \equiv \frac{\partial P}{\partial x} = 2bx\dot{x} + c + 3dx^2, \tag{17}$$

$$P_{\dot{x}} \equiv \frac{\partial P}{\partial \dot{x}} = a + bx^2, \tag{18}$$

$$Q_x \equiv \frac{\partial Q}{\partial x} = 0, \tag{19}$$

$$Q_{\dot{x}} \equiv \frac{\partial Q}{\partial \dot{x}} = 1. \tag{20}$$

The roots $\lambda_{1,2}$ of Equation (16) characterize the stability of the respective equilibrium solutions. If $\lambda_{1,2}$ are real then the solution is nonoscillatory and for both roots positive we have unstable solution type 4, for both roots negative the solution is stable type 3, and for roots of different signs we have unstable solution type 4 (see Table 1). If $\lambda_{1,2}$ are complex conjugate then the solution is oscillatory and for the positive real part we have unstable solution type 2, for the negative real part the solution is stable type 1, and for purely imaginary roots we have stable periodic solution type 5.

5.1. Stability at Equilibria Type E1

For E1 we have $\dot{x} = 0$ and $x = 0$. If we substitute in Equations (17)–(20) we obtain the coefficients

$$P_x = c;\ P_{\dot{x}} = a;\ Q_x = 0;\ Q_{\dot{x}} = 1, \tag{21}$$

and hence, the characteristic equation

$$\lambda^2 - a\lambda - c = 0. \tag{22}$$

Equation (22) leads to solutions of the types 1 to 6 found for the simple one-dimensional problem. $\lambda_{1,2} = (a/2) \pm ((a^2/4) + c)^{1/2}$. The various possibilities are shown in Table 2.

Table 2

c	a	Stability	Type
	<0	Unstable	4
>0	0	Unstable	4
	>0	Unstable	4
	<0	Stable	1,3
<0	0	Stable	5
	>0	Unstable	2,4
	<0	Stable	3
0	0	Marginal	6
	>0	Unstable	4

The Stability at E1 is established when the two conditions

$$a \leq 0 \tag{23a}$$

$$c \leq 0 \tag{23b}$$

are both satisfied.

5.2 Stability at Equilibria Type E2

For E2 we have $\dot{x}=0$ and $x=\pm(-c/d)^{1/2}$; by substituting in Equations (17)–(20) we obtain

$$P_x = -2c;\ P_{\dot{x}} = A;\ Q_x = 0;\ Q_{\dot{x}} = 1, \tag{24}$$

where

$$A = a - \frac{bc}{d}. \tag{25}$$

The resulting characteristic equation for E2

$$\lambda^2 - A\lambda + 2c = 0 \tag{26}$$

yields the roots

$$\lambda_{1,2} = \frac{A}{2} \pm \left(\frac{A^2}{4} - 2c\right)^{1/2}. \tag{27}$$

We therefore have stability (negative real part) if*

$$A \leq 0 \tag{28a}$$

$$c \geq 0. \tag{28b}$$

The solution is oscillatory if

$$A^2 < 8c. \tag{29}$$

*Equation (26) is of the same form as Equation (22). The conditions (28a, b) can therefore be deduced from the conditions (23a, b) by letting $a \to A$ and $c \to -c$.

5.3. Stabilization by Saturation due to Nonlinear Terms

The calculations above demonstrate possible stabilization by saturation due to the nonlinear terms. For better clarification of this point compare the two sets of conditions (23a, b), where the equation is linear, and (28a, b), where the nonlinear terms are introduced into the equation. Consider for example the linear equation with $a \leq 0$, $c \geq 0$, and all the nonlinear terms equal to zero, then from (23a, b) the system has an unstable equilibrium at the origin. If the nonlinear terms are introduced and if the coefficients b and d are chosen to give $(-c/d) \geq 0$ and $A \leq 0$, the nonlinear system has two new* equilibria at $x = \pm\sqrt{(-c/d)}$ which are stable. To illustrate, consider the values $a=-2$, $c=+1$, and all the nonlinear terms equal to zero, then the differential equation is

$$\ddot{x} + 2\dot{x} - x = 0, \tag{30}$$

which has an equilibrium at the origin that is unstable nonoscillatory of the type 4 (see it from (23a, b) or from $\lambda_{1,2} = -1 \pm \sqrt{2}$). Let us now add the nonlinear terms, and consider precisely $b=+1$ and $d=-1$, the differential equation becomes

$$\ddot{x} + 2\dot{x} - x - x^2\dot{x} + x^3 = 0. \tag{31}$$

In this case we have $A=-1<0$, and since $c=+1>0$, then, from the conditions (28a, b) and (29) we have a stable oscillatory solution of the type 1 ($\lambda_{1,2} = -1/2 \pm i\sqrt{7}/4$) at $x = \pm 1$.

6. THE LYAPUNOV ANALYSIS†

We consider here the more general Levinson–Smith equation:

$$\ddot{x} + g(x,\dot{x})\dot{x} + f(x) = 0. \tag{32}$$

For the case treated above, we have

$$g(x,\dot{x}) = -a - bx^2, \tag{33}$$

$$f(x) = -cx - dx^3. \tag{34}$$

For equilibria of the type E1 we construct the

*For the system described by Equations (7) and (12) the nonlinear effect is negligible at the origin.

†The equilibrium of an autonomous system is stable in the Lyapunov sense, if in a certain region of phase space, which includes the equilibrium and its neighborhood, there exists a function $V(x,\dot{x})$ of definite (positive or negative) sign, which admits an infinitely small upper bound, and is such that the total derivative dV/dt is semidefinite of opposite sign to V (or identically zero).

Lyapunov function:

$$V(x,\dot{x})=\frac{1}{2}\dot{x}^2+\int_0^x f(x)\mathrm{d}x$$
$$=\frac{1}{2}\dot{x}^2-c\frac{x^2}{2}-d\frac{x^4}{4}, \qquad (35)$$

where x here is a small displacement from the origin.

A necessary condition for stability is $xf(x)\geq 0$. This condition is implied by the fact that a perturbed stable system has a tendency to return to equilibrium. To understand this condition physically, consider the case of a simple spring for which $f(x)=kx$ and $xf(x)=kx^2$ (all simple spring oscillations are stable). This necessary condition for stability yields $c\leq 0$, which is the same as condition (23b).

Consider now

$$\dot{V}=\dot{x}\frac{\partial V}{\partial x}+\ddot{x}\frac{\partial V}{\partial \dot{x}}=\dot{x}f(x)+\ddot{x}\dot{x}=-\dot{x}^2 g(x,\dot{x}), \qquad (36)$$

where we have used the Equations (32) and (35). From Equation (33) we have

$$\dot{V}=a\dot{x}^2+bx^2\dot{x}^2. \qquad (37)$$

From Equation (35) and $xf(x)\geq 0$ one can easily show that $V=(\dot{x}^2+xf(x))/2+O(x^4)$ is positive definite. Lyapunov's theorem for stability thus implies that $\dot{V}\leq 0$ (or, more precisely, that $\dot{V}$ be negative semi-definite or identically zero). This condition yields $a\leq 0$, which is the same as condition (23a). Thus, the same conditions for stability at E1 are obtained from the Lyapunov and linear stability analyses.

6.1. How to Apply the Lyapunov Analysis to Equilibria Type E2

The Lyapunov function defined in Equation (35) is valid for the origin*, i.e. at $x=0$ (E1 is an equilibrium at the origin). For E2, we need therefore to shift the singularity to the origin by the following transformation:

$$v=x-w;\ w=\pm(-c/d)^{1/2}. \qquad (38)$$

We thus obtain the following differential equation:

$$\ddot{v}+g(v,\dot{v})\dot{v}+f(v)=0. \qquad (39)$$

This equation is of the Levinson–Smith type where the functions

$$g(v,\dot{v})=-A-bv^2-2bvw, \qquad (40)$$

*V should admit an infinitely small upper bound. One may satisfy this condition at E2 by taking $V(v,\dot{v})$, where $v=x\pm(-c/d)^{\frac{1}{2}}$ and V as defined in Equation (35).

$$f(v)=2cv-3dv^2w-dv^3, \qquad (41)$$

and A is defined in Equation (25).

The transformation (38) transforms E2 to the origin of the v-axis. A necessary condition for stability at E2 is $vf(v)\geq 0$. This condition yields $c\geq 0$ which is the same as the condition (28 b) obtained from the linear stability analysis.

As for E1, we construct for E2 the Lyapunov function

$$V(v,\dot{v})=\tfrac{1}{2}\dot{v}^2+\int_0^v f(v)\,\mathrm{d}v. \qquad (42)$$

The derivative of V with respect to time is $\dot{V}=-\dot{v}^2 g(v,\dot{v})$. By substituting from (40) we obtain

$$\dot{V}=\dot{v}^2(A+bv^2+2bvw). \qquad (43)$$

From Lyapunov's theorem ($\dot{V}\leq 0$ as $V=(\dot{v}^2+vf(v))/2+O(v^3)$ is positive definite) we have stability at E2 if $A\leq 0$. This condition is the same as the condition (28a) obtained from the linear stability analysis. Thus the same conditions for stability at E2 are obtained from the Lyapunov and linear stability analyses.

7. WAVES: THE DIFFERENTIAL EQUATION

We consider the linear partial differential equation:

$$\nabla^2\Psi=\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}+g\frac{\partial\Psi}{\partial t}+b\Psi, \qquad (44)$$

where $\Psi(x,y,z,t)$ is the wave amplitude. Here b, g, and v are constants characterizing the wave. v is the speed of whatever the wave is — in the case of sound, it is the sound speed; in the case of light, it is the speed of light; etc.

8. WAVES: THE ONE-DIMENSIONAL PROBLEM

We look for solutions of the form

$$\Psi=\Psi_0 e^{\mathrm{i}(kx-\omega t)}. \qquad (45)$$

If we substitute (45) in (44) we obtain the dispersion relation:

$$H(\omega,k)=\omega^2-k^2v^2-bv^2+\mathrm{i}\omega g v^2=0. \qquad (46)$$

Since (46) belongs to a linear wave equation it is sometimes called a linear dispersion relation, although ω^2 and k^2 appear. Nonlinear dispersion relations belong to nonlinear wave equations and contain the wave amplitude. If $g=0$ (no dissipation) and $b=0$, [46] becomes identical to

$$v_\phi^2 \equiv \frac{\omega^2}{k^2} = v^2, \tag{47}$$

where v_ϕ is the phase velocity. If v_ϕ is constant the wave is called nondispersive, while if v_ϕ depends on ω(and hence k) the wave is called dispersive.

For problems with dissipation ($g \neq 0$) ω and H become complex. We consider $H = H_r + iH_i$ where H_r and H_i are respectively the real and imaginary parts of $H(\omega, k)$, and ω is defined in (5). In this case the phase velocity equals ω_r/k. From Equation (46) one finds

$$H_i = (2\omega_i + gv^2)\omega_r = 0 \tag{48}$$

and, for the real part,

$$H_r = \omega_r^2 - \omega_i^2 - k^2v^2 - bv^2 - \omega_i g v^2 = 0. \tag{49}$$

From Equation (45) an instability is obtained if $\omega_i > 0$. Equation (48) yields $\omega_i = -gv^2/2$, so that the instability occurs when $g < 0$ i.e. for negative damping (of course). To obtain the phase and group velocities of the wave, one eliminates ω_i in (49) by replacing from (48), one finds

$$H_r = \omega_r^2 - k^2v^2 - bv^2 + \tfrac{1}{4}g^2v^4 = 0, \tag{50}$$

which yields the phase speed

$$v_\phi \equiv \left(\frac{\omega_r}{k}\right) = v\left(1 + \frac{b}{k^2} - \frac{g^2v^2}{4k^2}\right)^{1/2}. \tag{51}$$

The group velocity, $v_g(\equiv d\omega_r/dk)$, is obtained from (50)

$$v_g = \frac{v^2}{(\omega_r/k)} = \frac{v^2}{v_\phi}. \tag{52}$$

An alternative form for Equation (52) is the well-known relation $v_g v_\phi = v^2$. Precisely, $v_\phi \geq v$ and v_g (the signal speed) $\leq v$.

9. THE NYQUIST METHOD

The Nyquist diagram is a convenient method for discovering instabilities. An instability is a root ω_0 of $H(\omega, k)$ with Im $\omega_0 > 0$. The number of unstable modes (unstable waves) therefore equals the number of zeros of $H(\omega, k)$ in the upper half of the complex ω-plane. For plasma waves the form of $H(\omega, k)$ and location of its roots are determined by the equilibrium fields and plasma distribution.

Assume $H(\omega, k)$ to be an analytic function of ω in the upper half of the complex ω-plane. The function $(dH(\omega, k)/d\omega)/H(\omega, k)$ clearly has poles at those values of ω for which $H(\omega, k)$ has zeros. The number N_0 of points where $H(\omega, k) = 0$ in the upper half of the complex ω-plane is given by the residue theorem

$$N_0 = \frac{1}{2\pi i} \int_C \left(\frac{1}{H}\frac{dH}{d\omega}\right) d\omega. \tag{53}$$

The integration contour C is shown in Figure 2(a). Equation (53) can be generalized to functions $H(\omega, k)$ analytic in the upper half of the complex ω-plane except for a finite number of poles.

The counter-clockwise contour C starts at Re $\omega = -\infty$, follows the real axis through $\omega_r = 0$, and goes to $\omega_r = +\infty$, then it closes back to $\omega_r = -\infty$ on the half circle ($\omega = Re^{i\theta}$ with $R \to \infty$) over $\omega_i = +\infty$. All the poles with $\omega_i > 0$ are enclosed.

In many cases $H(\infty) =$ constant, so that

$$\int_0^\pi \left(\frac{i\omega}{H}\frac{dH}{d\omega}\right) d\theta \to 0, \qquad R \to \infty. \tag{54}$$

For such cases the contribution from the great half circle vanishes and the integral in (53) is evaluated merely by finding the change in the phase of H as ω changes from $-\infty$ to $+\infty$ along the real axis

$$\int_{-\infty}^{+\infty} \left(\frac{1}{H}\frac{dH}{d\omega}\right) d\omega = \ln\frac{H(+\infty)}{H(-\infty)}. \tag{55}$$

The number of instabilities is therefore obtained from (53), (54), and (55)

$$N_0 = \frac{1}{2\pi i} \ln\frac{H(+\infty)}{H(-\infty)}. \tag{56}$$

Equation (56) holds if $H(|\omega| \to \infty) \to$ constant, thus $|H(\infty)| = |H(-\infty)|$. Since the problem is unchanged by multiplying $H(\omega, k)$ by a constant, we arbitrarily set $|H(\infty)| = |H(-\infty)| = 1$. The phase of H can be chosen arbitrarily at $\omega = -\infty$; take $H(-\infty) = 1$ (i.e. the phase at $-\infty$ to be zero). This choice, however, determines the phase at $\omega = +\infty$.

$$H(+\infty) = e^{2\pi i n},\ H(-\infty) = 1. \tag{57}$$

The change of the phase is obtained by plotting H_i versus H_r for all values of ω from $-\infty$ to $+\infty$ on the real axis; n is the number of times the curve H_i versus H_r encircles the origin. This curve in the complex H-plane is the conformal mapping of the real ω-axis. The great half circle maps into $H = 1$. From Equations (56)

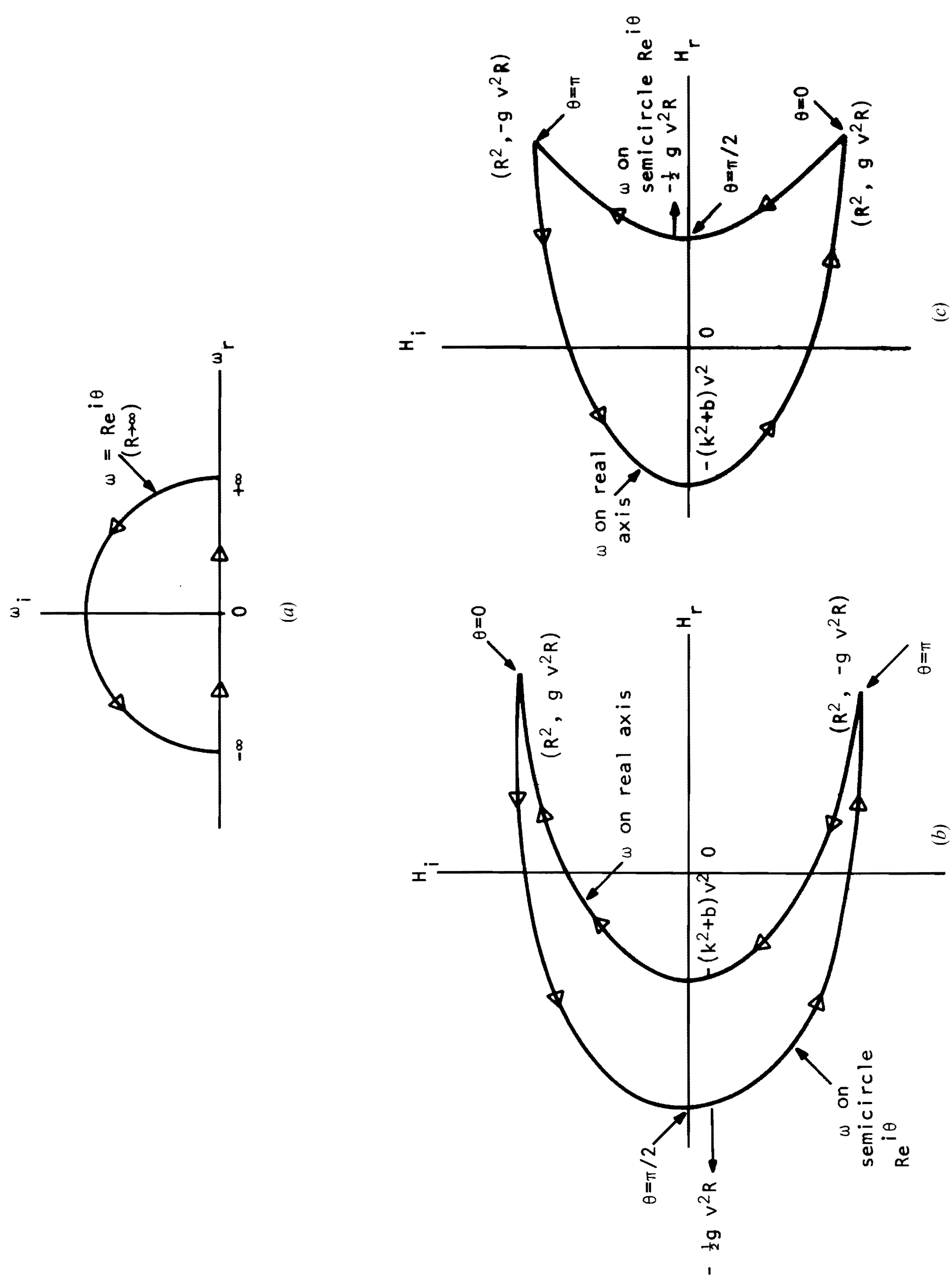

Figure 2. (*a*)*The Integration Contour C.* (*b*) *Nyquist Plot for* $g>0$. (*c*) *Nyquist Plot for* $g<0$.

and (57) the number of instabilities is easily shown to equal n;

$$N_0 = n. \tag{58}$$

The Nyquist technique is very useful since the dispersion curve $H(\omega, k)$ is simpler for real ω than for complex ω. Furthermore, to keep track of the encircling of the origin only H_r at ω_0 such that $H_i(\omega_0)=0$ is required. Note that the behavior of $H(\omega)$ away from the points $H_i(\omega)=0$ is irrelevant to the stability analysis.

The Nyquist method is a powerful tool with which to study stability because it makes it possible to predict stability by calculating the sign of H_r for a few particular values of ω_r instead of having to solve an equation $H(\omega,k)=0$. A simple example is presented below.

10. A SIMPLE EXAMPLE

We consider here the one-dimensional problem treated in Section 8. The dispersion relation is given by Equation (46). In this case $(1/H\ \mathrm{d}H/\mathrm{d}\omega)\mathrm{d}\omega$ does not vanish for $|\omega|\to\infty$. Thus the sum of $(1/H\ \mathrm{d}H/\mathrm{d}\omega)\mathrm{d}\omega$ over the contour C cannot be replaced by $\int_{-\infty}^{+\infty} (1/H\ \mathrm{d}H/\mathrm{d}\omega)\mathrm{d}\omega$. This poses no real difficulty; writing $\omega = R\mathrm{e}^{i\theta}$, we sketch $H(\omega)$ as ω follows the contour C; this gives the number of instabilities directly.

The imaginary part of H is given by Equation (48); the real part of H by Equation (49). ω_i and g are related at the pole by the Equation (48). In order to simplify this example we substitute from Equation (48) once for ω_i as function of gv^2 in the second term on the r.h.s. of Equation (49) only. This is equivalent to adding to Equation (49) a quantity that equals zero at the singularity. This is perfectly legitimate since it leaves the dispersion equation unchanged. We thus write the real part of the dispersion equation:

$$H_r = \omega_r^2 - k^2v^2 - bv^2 - \tfrac{1}{2}g\omega_i v^2 = 0. \tag{59}$$

On the real ω axis ($\omega_i=0$) and for $\omega_r = (-R, 0, +R)$ we have $H_i = (-gv^2R,\ \mathrm{O},\ gv^2R)$ and $H_r = (R^2, -(k^2+b)v^2, R^2)$* respectively. On the semicircle $\omega = R\mathrm{e}^{i\theta}$ and for $\theta = (\mathrm{O}, \pi/2, \pi)$ we have respectively $H_i = (gv^2R, \mathrm{O}, -gv^2R)$ and $H_r = (R^2, -\tfrac{1}{2}gv^2R, R^2)$*.

For $g>0$ the wave is damped and therefore stable. Accordingly the corresponding Nyquist plot in Figure 2(b) does not encircle the origin *to predict stability*. For $g<0$ the wave (negatively damped) is increasing and unstable. Accordingly the corresponding Nyquist plot in Figure 2(c) encircles the origin once to predict the existence of one instability.

11. THE MULTIDIMENSIONAL CONSERVATIVE SYSTEM

For a system to be conservative, it is required that the kinetic part of the energy integral be a function of the time derivative of coordinates only and the potential part of the coordinates only. For such a system the total energy, kinetic plus potential, is conserved. An unexpected example is a magnetohydrodynamic system.

Consider the one-dimensional conservative system described by the potential $V(x_1)$. An example of such system was illustrated in Section 1 and studied at the beginning of Section 3: $(f(x_1) = -\mathrm{d}V/\mathrm{d}x_1)$. In the vicinity of an equilibrium x_1^0we found one equation of motion for $(x_1 - x_1^0)$ (Equation (2)) and one frequency of oscillation, ω, where $\omega^2 = m^{-1}(\mathrm{d}^2v/\mathrm{d}x_1^2|x_1^0)$ (Equation (4)). Among the equilibria of this system are: potential hilltops, where $(\mathrm{d}^2V/\mathrm{d}x_1^2)<0$, and potential wells, where $(\mathrm{d}^2V/\mathrm{d}x_1^2)>0$. Clearly the stable equilibria are the potential wells where ω is real.

The two-dimensional conservative system is described by the potential $V(x_1,x_2)$. Among the equilibria are: potential hilltops $((\partial^2V/\partial x_1^2)<0, (\partial^2V/\partial x_2^2)<0)$; potential wells $((\partial^2V/\partial x_1^2)>0, (\partial^2V/\partial x_2^2)>0)$; and two types of saddle points $((\partial^2V/\partial x_1^2)>0, (\partial^2V/\partial x_2^2)<0$ and $(\partial^2V/\partial x_1^2)<0, (\partial^2V/\partial x_2^2)>0)$. In the vicinity of an equilibrium (x_1^0, x_2^0) we have two equations of motion one for $(x_1 - x_1^0)$ and one for $(x_2 - x_2^0)$. With the proper choice of coordinates (normal coordinates) the two equations are uncoupled, and two frequencies of oscillations are sufficient to describe the motion about each equilibrium. If both frequencies are real the system is stable, otherwise it is unstable. These frequencies are related to the second derivatives of V by equations similar to (4). Clearly then the stable equilibria are potential wells.

Further generalization to more then two degrees of freedom follows the same lines: the condition for stability is that all values of ω are real. Since a real ω is associated with a positive second derivative of v, then a necessary and sufficient condition for stability of a conservative multidimensional system is that all the second derivatives of V with respect to all the independent variables, $\partial^2V/\partial x_i^2$ where $i=1,2,3\ldots$, be positive.

*Stated values sometimes take into account the fact that $R\to\infty$.

If we assume all equilibria to be equally probable: in the one-dimensional case (we have one constraint for stability $\partial^2 V/\partial x^2 > 0$) the chances are that roughly half the equilibria are stable. In the two-dimensional case, we have two constraints and roughly one fourth of the equilibria are stable. By extrapolating the above reasoning to the multidimensional conservative system, one finds that an arbitrarily chosen equilibrium configuration in that system has almost nil chance to be stable. It is therefore desirable to look directly for the stable equilibrium configuration in these systems through general stability criteria. Examples of these criteria are the energy principle illustrated below and Gardner's theorem illustrated in Section 12.

Consider two incompressible fluids in hydrostatic equilibrium in a gravitational field. Take the two-fluid interface to be in a horizontal plane and assume the specific weights of the two fluids to be different. The slight perturbation of the interface (for example a ripple) causes the potential energy of the system to change*. Now if the lower fluid is heavier than the upper fluid then the change in the potential energy is a net increase causing the kinetic energy (if any) to decrease and therefore resulting in stable oscillations of the interface. (The system composed of the two-fluids is assumed to be a conservative system. The conditions under which this assumption hold are irrelevant here.) If the upper fluid is the heavier one, the potential energy decreases and the kinetic energy increases; the perturbation grows leading finally to the two fluids exchanging places. The instability of a heavy fluid supported under gravity by a lighter fluid is known as the Rayleigh–Taylor instability. An example from the field of controlled fusion is a plasma (the heavy fluid) in a gravitational field, supported by magnetic field lines (the lighter fluid!).

The above example illustrates how the energy principle permits a quick answer to the question of stability. In the following section, where the stability of a monotone-decreasing distribution is demonstrated, this stability criterion is generalized to include, on top of the energy principle, the principle of volume conservation in phase space, also known as Liouville's theorem.

12. GARDNER'S THEOREM

This theorem states the stability of a field-free plasma with an equilibrium distribution that decreases monotonically with speed. The stability of a monotone-decreasing distribution (Figure 3(a)) is a consequence of two constraints: conservation of energy and conservation of volume in phase space (Liouville's theorem). Consider a one-dimensional model for a general initial distribution $F(x,u^2)$. In Figure 3(b) we plot regions of approximately constant F in (x,u^2) phase space, indicating the size F in each region by the ordering $F(a) > F(b) > F(c) > F(d)$.... Stability is determined for a given initial state if no other state with lower internal (kinetic) energy can be reached from that state. What then is the minimum value, consistent with Liouville's theorem, of the energy integral $\int 1/2mu^2 F(x,u^2)\,dx du$?

* $\Delta V \sim (\rho_L - \rho_U)\xi$, where L, U, ρ, and ξ are for lower, upper, fluid density, and interface excursion respectively.

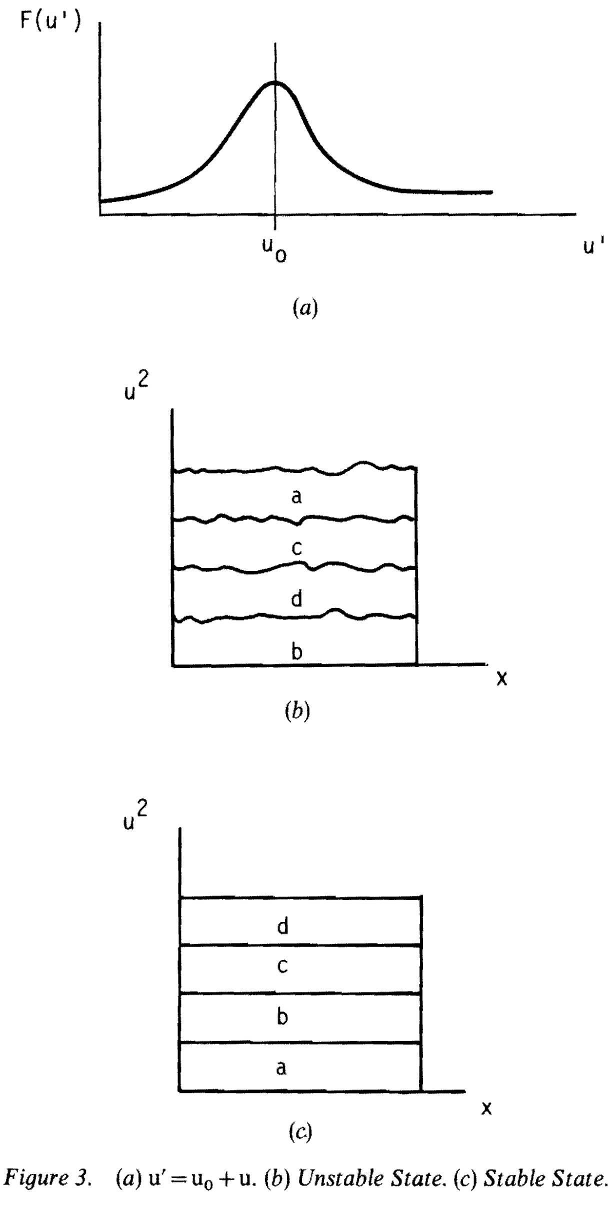

Figure 3. (*a*) $u' = u_0 + u$. (*b*) *Unstable State.* (*c*) *Stable State.*

The constraint from Liouville's theorem allows the constant F zones to be rearranged and reshaped but not changed in area. Clearly, then, because of the weighting factor u^2, the lowest value of the energy integral is achieved when the zones of greatest F lie nearest $u^2=0$. Like a mixture of fluids of different specific gravity, the heaviest layer sinks to the bottom. The result is Figure 3(c), a function distribution independent of x and monotone decreasing in u^2.

If the initial state is already a monotone-decreasing state, the proof above shows that small disturbances do not grow. In particular, the maximum change in kinetic energy transfer to fields cannot be greater than any initial kinetic energy perturbation. By energy conservation, changes in the field energy are similarly bounded. The extension to three dimensions is straightforward. This proof is due to Gardner [7].

REFERENCES

[1] N. Minorsky, *Nonlinear Oscillations*, Princeton: Van Nostrand, 1962.

[2] H. A. Antosiewics. in *Contributions to the Theory of Nonlinear Oscillations*, edited by S. Lefschetz. Princeton University Press, 1958.

[3] B. Noble, *Numerical Methods: Differences, Integration and Differential Equations*, New York: Interscience, 1964.

[4] E. A. Barbashin, *Introduction to the Theory of Stability*, Groningen: Noordhoff, 1970.

[5] N. N. Krasovskii, *Stability of Motion*, Stanford University Press, 1963.

[6] W. Eckhaus, *Studies in Nonlinear Stability Theory*, Springer–Verlag, 1965.

[7] C. S. Gardner, *Physics of Fluids*, **6** (1963), p. 839.

[8] N. A. Krall and A. W. Trievelpiece, *Principles of Plasma Physics*, McGraw–Hill, 1973.

Paper Received 13 March 1982; Revised 15 December 1982.