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Unreliable Server M/G/1 Queueing System with Bernoulli Feedback, Repeated Attempts, Modified Vacation, Phase Repair and Discouragement

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Abstract. This paper deals with unreliable M/G/1 queueing system with modified vacation, repeated attempts and K-phase repair. The jobs arrive in Poisson fashion. On finding the server busy, under setup, under repair or on vacation, the jobs either join to the orbit or balk from the system. The jobs in the orbit repeat their request for service after some random time. The inter retrial time of each job is general distributed. The jobs are served according to FCFS discipline. After receiving the unsuccessful service, the job may immediately join the tail of the original queue with some probability p or may depart from the system with probability q(= 1-p). Accidental (operational) breakdown of the server is also considered. There is a provision of k-phase repairs to restore the server to the state as before failure. First phase repair is essential whereas other (K-1) phases are optional. The repairman, who restores the server, requires some time to start the first phase of repair; this time is called as setup time. The service time, setup time and repair time of each phase are independent and general distributed. On finding the orbit empty, the server goes on at most J vacations repeatedly until at least one job is recorded in the orbit. The probability generating function of steady state queue size at random epoch is obtained using supplementary variable technique. Various models studied earlier are discussed as special cases of our model, by appropriate choice of parameter values. Some queueing as well as reliability indices to predict the behaviour of the system are also derived. The effects of various system parameters on the system performance indices are also examined numerically.

Keywords: M/G/1 retrial queue, General retrial policy, Discouragement, Bernoulli feedback, Modified vacations, Phase optional repair.

Introduction

In data transmission, a packet transmitted from the source may not successfully reach to the destination and returned back; it may retry for transmission until the packet is finally transmitted. These repeated attempts of jobs have been analyzed via retrial queueing model and Bernoulli feedback model. Such queueing situations may arise in many real time congestion systems such as telecommunication, data/voice transmission, computer system, manufacturing system, *etc*.

Segmented message transmission is a practical application of retrial queue in real life. We consider an M/G/1 retrial queue where blocked jobs on finding the server busy or broken down leave the service area and enter the retrial group in accordance with FCFS discipline. We assume that only the job at the head of the queue is allowed to occupy the server if it is free. After some random time the blocked jobs return to repeat their request. Several studies on retrial queues have been made by many researchers working in the area of applied probability theory, from time to time. In almost all models of retrial queues, the time between retrials for any job is assumed to be exponentially distributed. The general retrial time policy arises naturally in many congestion problems related to many service systems where, after each service completion, the server spends a random amount of time in order to find the next job to be processed. In recent years there has been an increasing interest in the study of retrial queue with general retrial time[1,2]. Atencia and Moreno[3] analyzed a discrete time $Geo^{[X]}/G_H/1$ retrial queue where each call after service either immediately returns to the orbit to complete unsuccessful service with probability θ or leaves the system forever with probability $1-\theta$. A single server retrial queue with general retrial times and Bernoulli schedule was discussed by Atencia and Moreno[4].

In Bernoulli feedback queueing model, if the service of the job is unsuccessful, it may try again and again until a successful service is completed. Takacs[5] was the first to study feedback queueing model. Studies on queue length, the total sojourn time and the waiting time for an M/G/1 queue with Bernoulli feedback were provided by Vanden Berg and Boxma[6]. Choudhury and Paul[7] derived the queue size distribution at random epoch and at a service completion epoch for M/G/1 queue with two phases of heterogeneous services and Bernoulli feedback system.

Krishna Kumar *et al.*[8] considered a generalized M/G/1 feedback queue in which customers are either "positive" or "negative".

Queueing systems with vacations have also found wide applicability as realized in retrial and feedback models for the analysis of computer and communication network and several other engineering systems. Vacation models are explained by their scheduling disciplines, according to which when a service stops, a vacation starts. Wortman *et al.*[9] discussed feedback as a mechanism for scheduling customer's service; in systems in which customers bring work that is divided into a random number of stages. Li and Yang[10] suggested a single server retrial queueing system with server vacations, no waiting space and finite population of the customers. Choi *et al.*[11] considered M/G/1 queueing system with multiple types of feedback, gated vacations and obtained joint probability generating function of the system size at steady state. Wenhui[12] analyzed ergodic condition and probability generating functions for M/G/1 queue with Bernoulli vacation, general retrial, vacation and setup time. Feedback queueing system with single vacation policy was examined by Madan and Al-Rawwash[13].

Discouragement behaviour of the jobs has been realized in many retrial congestion situations. Non-Markovian multiple vacation queueing models with balking was discussed by Thomo[14]. Artalejo and Herrero[15] determined the limiting distribution of the number of customers in the system by using recursive approach based on regenerative theory for the single server retrial queue with balking. Hyperexponential queueing model in case of impatient customers with retrial attempts was discussed by Jain and Rakhee[16].

In many waiting line systems, the role of server is played by mechanical/electronic device, such as computer, pallets, ATM, traffic light, *etc*., which is subject to accidental random failures; until the failed server is repaired, it may cause a halt in service. Ke[17] studied the control policy of the N policy M/G/1 queue with server vacations, startup and breakdowns, where arrival forms a Poisson process and service times are generally distributed. Gharbi and Ioualalen[18] gave a detailed analysis of finite source retrial systems with multiple servers subject to random breakdowns and repairs using generalized stochastic model and shown how this model is capable to cope up with the complexity of such retrial system involving the unreliability of the servers. The steady state performance and reliability indices were also derived. Sherman and Kharoufeh[19] analyzed unreliable M/M/1 retrial queue with infinite capacity orbit.

The present investigation is concerned with single unreliable server queueing model by incorporating Bernoulli feedback, general retrial time, K-phase optional repair along with modified vacation policy. To refer a practical application on the model under investigation, we cite an illustration of reception counter at any organization wherein the operator (receptionist) uses a telephone/answering machine. If any customer makes a call and finds the operator busy, then the customer either leaves message on the answering machine (retrial queue) or may balk due to impatience. Moreover the service of the incoming calls may be interrupted when the operator is not well or there may occur some problem in the telephone set. The operator/telephone set is immediately recovered in phases. In case when the operator is free, he goes on the vacation, however he may be restricted to take finite number of vacations.

The study of queueing model is organized as follows. The model along with notations is described in Section 2. Steady state behaviour of the system by constructing the Chapman Kolmogorov equations is outlined in Section 3. By using generating function method, the queue size distribution has been obtained in Section 4. Some queueing indices to predict the behaviour of the system are derived in Section 5. In Section 6, the results for some of the well-established models as special cases of our model, are deduced. Reliability indices are obtained in Section 7. To validate the analytical results and to facilitate the sensitivity analysis, we present some numerical results for system performance indices in Section 8. Finally, we have concluded our work in Section 9.

2. Model Description

We consider a single unreliable server retrial queueing system at which jobs arrive according to Poisson process with mean arrival rate λ. On finding the server busy, under setup, under repair or on vacation either job goes to some virtual place referred as an orbit or job may balk from the system with some probability $\bar{h}$ ($\bar{h}$=1-h; h being joining probability). From the orbit, the jobs repeat their request for service after some random time. The inter retrial time of each job is i.i.d. general distributed with distribution function A(u).

On finding server idle only one job at the head of the queue (orbit) is allowed to access the server. The jobs are served according to FCFS discipline. After completion of service if the job is not satisfied with its service for any reason or if it receives incomplete service, in that case the job may immediately join the tail of the queue (orbit) with some probability p ($0 \le p \le 1$) *i.e.* feedback to have another regular service or job may depart from the system forever with probability q(= 1-p). The service times of the jobs follow the i.i.d. general distribution with cumulative distribution function B(u).

When the server is working, he may meet unpredictable breakdowns. We assume that server's life time is exponentially distributed with rate α. When the server fails, it is immediately send for repair at a repair facility. The repair facility requires some time before starting the repair; this time is known as setup time for the repairman. The repair is assumed to be performed in K phases. The repairman provides the first phase (essential) repair (ER) to the server. As soon as the ER of a server is completed, the server may join the system for service or may immediately go into second phase of repair with probability r_1. Similarly after completing second phase of repair, which is optional, the repairman immediately starts subsequent third phase of repair with probability r_2, otherwise the server goes to working state. On a similar pattern, the server may goes into a maximum of K phases of repair (including first ER phase) with different probabilities r_{k-1} (k = 2,3,…, K) for moving from $(k-1)^{th}$ phase to k^{th} phase of repair. Setup time and k^{th} phase (k = 2,3,…, K) repair time are mutually independent and identically general distributed.

As soon as the service of the jobs is completed, the server deactivates and takes at most J vacations repeatedly until at least one job recorded in the orbit upon returning from a vacation. If one or more jobs are present in the orbit in case when the server returns from a vacation, the server reactivates otherwise goes back for subsequent vacation. If no job is present in the orbit at the end of J^{th} vacation, the server remains dormant until at least one job arrives in the orbit. We assume that j^{th} phase (j = 1,2,…, J) vacation time follows general distribution law.

For distribution of retrial time, service time, j^{th} (j = 1,2,…, J) phase vacation time , setup time and k^{th} (k = 1,2,…, K) phase repair time, we use the following notations:

$a(u), b(u), v^j(u)$	Probability density functions for retrial time, service time and j^{th} (j = 1,2,…, J) phase vacation time, respectively.
$A(u), B(u), V^j(u)$	Distribution functions for retrial time, service time and j^{th} (j = 1,2,…, J) phase vacation time, respectively.
$A^*(.), b^*(.), v^{j*}(.)$	Laplace transform of a(.), b(.) and $v^j(.)$, respectively.
$S(v), g^{(k)}(v)$	Probability density functions for setup time and k^{th} (k = 1,2,…, K) phase repair time, respectively.
$S(v), G^{(k)}(v)$	Distribution functions for setup time and k^{th} (k = 1,2,…, K) phase repair time, respectively.
$S^*(.), g^{(k)*}(.)$	Laplace transform of s(.) and $g^{(k)}(.)$, respectively.

3. Steady State Equations

We analyze the system state at time t by means of Markov process $X(t) = \{N(t), \psi(t), \xi_0^j(t), \eta_0^{(k)}(t), t \geq 0\}$, where N(t) denotes the number of jobs in the orbit at time, $\Psi(t)$ stands for the elapsed retrial time, $\xi_0(t)$ and $\xi^j(t)$ stand for the elapsed time of service and elapsed vacation time of j^{th}(j = 1,2,…, J) phase at time t, respectively. $\eta_0(t)$ and $\eta^{(k)}(t)$ stand for the elapsed time of setup and k^{th}(k = 1,2,…, K) phase elapsed repair time, respectively. We introduce the supplementary variables corresponding to retrial time, service time, setup time, phase repair time and vacation time. We define the following limiting probabilities corresponding to different states:

For retrial: $A_n(u)du = \lim_{t\to\infty} P\{N(t) = n, u < \psi(t) \leq u + du\}, \ \ n \geq 1$

For server's busy state: $B_n(u)du = \lim_{t\to\infty} P\{N(t) = n, u < \xi_0(t) \leq u + du\}, \ \ n \geq 0$

For setup state: $S_n(u, v)dv = \lim_{t\to\infty} P\{N(t) = n, \xi_0(t) = u, v < \eta_0(t) \leq v + dv\}, \ \ n \geq 0$

For kth phase repair:

$$R_n^{(k)}(u,v)dv = \lim_{t\to\infty} P\{N(t)=n, \xi_0(t)=u, v<\eta^{(k)}(t)\le v+dv\},\ n\ge 0,\ 1\le k\le K$$

For jth phase vacation:

$$V_n^j(u)du = \lim_{t\to\infty} P\{N(t)=n, u<\xi^j(t)\le u+du\},\ \ n\ge 0,\ \ 1\le j\le J$$

Since A(u), B(u), S(v), $G^{(k)}(v)$ and $V^j(u)$ are distribution functions, A(0) = 0, A(∞) = 1; B(0) = 0, B(∞) = 1; S(0) = 0, S(∞) = 1; $G^{(k)}(0) = 0$, $G^{(k)}(\infty) = 1$ and $V^j(0) = 0$, $V^j(\infty) = 1$. Also

$$\bar{a}(u)=\frac{a(u)}{\bar{A}(u)},\ \bar{b}(u)=\frac{b(u)}{\bar{B}(u)},\ \bar{s}(v)=\frac{s(v)}{\bar{S}(v)},\ \bar{g}^{(k)}(v)=\frac{g^{(k)}(v)}{\bar{G}^{(k)}(v)},\ \bar{\nu}^{j}(u)=\frac{\nu^{j}(u)}{\bar{V}^{j}(u)}$$

are the hazard rate functions of A(u), B(u), S(v), $G^{(k)}(v)$ and $V^j(u)$, respectively.

In all phases of vacation, we consider identical hazard rate, so that

$$\bar{\nu}^{j}(u)=\bar{\nu}(u)=\frac{\nu(u)}{\bar{V}(u)}$$

Now i^{th} moments for retrial time, service time, setup time, repair time and vacation time are defined as:

$$a_i=(-1)^i a^{*(i)}(0), b_i=(-1)^i b^{*(i)}(0),\ s_i=(-1)^i s^{*(i)}(0),\ g_i^{(k)}=(-1)^i g^{*(k)(i)}(0) \text{ and }$$

$$\nu_i=(-1)^i \nu^{*j(i)}(0).$$

Here θ, μ, s, γ, ν denote the retrial rate, service rate, setup rate, repair rate, vacation rate, respectively.

For steady state, we obtain the following differential difference equations governing the model:

$$\lambda A_0=\int_0^\infty V_0^J(u)\bar{\nu}(u)\,du \tag{1}$$

$$\frac{dA_n(u)}{du}=-[\lambda+\bar{a}(u)]A_n(u),\qquad n\ge 1 \tag{2}$$

$$\frac{dB_n(u)}{du}=-[\lambda h+\bar{b}(u)+\alpha]B_n(u)+\lambda h B_{n-1}(u)+\int_0^\infty \bar{g}^{(K)}(v)R_n^{(K)}(u,v)\,dv+\sum_{k=1}^{K-1}\int_0^\infty (1-r_k)\bar{g}^{(k)}(v)R_n^{(k)}(u,v)\,dv,$$

$$n \geq 0, \; 1 \leq k \leq K-1 \tag{3}$$

$$\frac{dS_n(u,v)}{dv} = -[\lambda h + \bar{s}(v)]S_n(u,v) + \lambda h S_{n-1}(u,v), \quad n \geq 0 \tag{4}$$

$$\frac{dR_n^{(k)}(u,v)}{dv} = -[\lambda h + \bar{g}^{(k)}(v)]R_n^{(k)}(u,v) + \lambda h R_{n-1}^{(k)}(u,v), \; n \geq 0, \; 1 \leq k \leq K \tag{5}$$

$$\frac{dV_n^j(u)}{du} = -[\lambda h + \bar{v}(u)]V_n^j(u) + \lambda h V_{n-1}^j(u), \quad n \geq 0, \; 1 \leq j \leq J \tag{6}$$

The boundary conditions are:

$$A_n(0) = \sum_{j=1}^{J} \int_0^\infty V_n^j(u)\bar{v}(u)du + q\int_0^\infty B_n(u)\bar{b}(u)du + p\int_0^\infty B_{n-1}(u)\bar{b}(u)du, \quad n \geq 1 \tag{7}$$

$$B_0(0) = \int_0^\infty A_1(u)\bar{a}(u)du + \lambda A_0 \tag{8}$$

$$B_n(0) = \int_0^\infty A_{n+1}(u)\bar{a}(u)du + \lambda\int_0^\infty A_n(u)du, \quad n \geq 1 \tag{9}$$

$$S_n(u,0) = \alpha B_n(u), \quad n \geq 0 \tag{10}$$

$$R_n^{(1)}(u,0) = \int_0^\infty S_n(u,v)\bar{s}(v)dv, \quad n \geq 0 \tag{11}$$

$$R_n^{(k)}(u,0) = \int_0^\infty r_{k-1} R_n^{(k-1)}(u,v)\bar{g}^{(k-1)}(v)dv, \quad n \geq 0, \; k \geq 2 \tag{12}$$

$$V_n^1(0) = \begin{cases} \int_0^\infty q B_n(u)\bar{b}(u)du, & n = 0 \\ 0, & n \geq 1 \end{cases} \tag{13}$$

$$V_n^j(0)=\begin{cases}\int_0^\infty V_n^{j-1}(u)\bar{v}^{(j-1)}(u)du, & n=0,\quad 2\le j\le J\\ 0, & n\ge 1\end{cases} \tag{14}$$

The normalizing condition is

$$A_0+\sum_{n=1}^{\infty}\int_0^\infty A_n(u)du+\sum_{n=0}^{\infty}\int_0^\infty B_n(u)du+\sum_{n=0}^{\infty}\int_0^\infty\int_0^\infty S_n(u,v)dudv$$
$$+\sum_{k=1}^{K}\sum_{n=0}^{\infty}\int_0^\infty\int_0^\infty R_n^{(k)}(u,v)dudv+\sum_{j=1}^{J}\sum_{n=0}^{\infty}\int_0^\infty V_n^j(u)du=1 \tag{15}$$

4. Queue Size Distribution

We use method of generating function to solve above steady state equations. Define the following probability generating functions:

$$A(z,u)=\sum_{n=1}^{\infty}A_n(u)z^n,\quad B(z,u)=\sum_{n=0}^{\infty}B_n(u)z^n,\quad S(z,u,v)=\sum_{n=0}^{\infty}S_n(u,v)z^n,$$

$$R^{(k)}(z,u,v)=\sum_{n=0}^{\infty}R_n^{(k)}(u,v)z^n,\ V^j(z,u)=\sum_{n=0}^{\infty}V_n^j(u)z^n,\quad |z|\le 1$$

Now from eqs (2) to (12), we obtain

$$\frac{\partial A(z,u)}{\partial u}=-[\lambda+\bar{a}(u)]A(z,u) \tag{16}$$

$$\frac{\partial B(z,u)}{\partial u}=-[\lambda h(1-z)+\bar{b}(u)+\alpha]B(z,u)+\int_0^\infty \bar{g}^{(K)}(v)R^{(K)}(z,u,v)dv+\sum_{k=1}^{K-1}\int_0^\infty(1-r_k)\bar{g}^{(k)}(v)R_n^{(k)}(z,u,v)dv,$$
$$1\le k\le K-1 \tag{17}$$

$$\frac{\partial S(z,u,v)}{\partial v}=-[\lambda h(1-z)+\bar{s}(v)]S(z,u,v) \tag{18}$$

$$\frac{\partial R^{(k)}(z,u,v)}{\partial v}=-[\lambda h(1-z)+\bar{g}^{(k)}(v)]R^{(k)}(z,u,v),\quad 1\le k\le K \tag{19}$$

$$\frac{\partial V^j(z,u)}{\partial u} = -[\lambda h(1-z) + \bar{v}(u)]V^j(z,u), \quad 1 \le j \le J \tag{20}$$

$$A(z,0) = \sum_{j=1}^{J} \int_0^\infty V^j(z,u)\bar{v}(u)du + (q+pz)\int_0^\infty B(z,u)\bar{b}(u)du - \lambda A_0 - \sum_{j=1}^{J} V_0^j(0) \tag{21}$$

$$B(z,0) = \frac{1}{z}\int_0^\infty A(z,u)\bar{a}(u)du + \lambda \int_0^\infty A(z,u)du + \lambda A_0 \tag{22}$$

$$S(z,u,0) = \alpha B(z,u) \tag{23}$$

$$R^{(1)}(z,u,0) = \int_0^\infty S(z,u,v)\bar{s}(v)dv \tag{24}$$

$$R^{(k)}(z,u,0) = \int_0^\infty r_{k-1} R^{(k-1)}(z,u,v)\bar{g}^{(k-1)}(v)dv, \quad k \ge 2 \tag{25}$$

Normalizing condition (15) yields

$$A_0 + A(z) + \int_0^\infty B(z,u)du + \int_0^\infty\int_0^\infty S(z,u,v)dudv + \sum_{k=1}^{K}\int_0^\infty\int_0^\infty R^{(k)}(z,u,v)dudv + \sum_{j=1}^{J}\int_0^\infty V^j(z,u)du = 1 \tag{26}$$

Theorem 1: The partial probability generating functions when the server is in idle, busy, under setup, k^{th} (k = 1,2,..., K) phase repair and on j^{th} (j = 1,2,… J) phase vacation state respectively, are given by

$$A(z,u) = \frac{\lambda A_0 z\{N(z) - 1 + (q+pz)b^*(H(z))\}}{z - (q+pz)\{a^*(\lambda) + z(1-a^*(\lambda))\}b^*(H(z))} \exp\{-\lambda u\}\bar{A}(u) \tag{27}$$

$$B(z,u) = \frac{\lambda A_0[\{N(z)-1\}\{a^*(\lambda) + z(1-a^*(\lambda)\} + z]}{z - (q+pz)\{a^*(\lambda) + z(1-a^*(\lambda))\}b^*(H(z))} \exp\{-H(z)u\}\bar{B}(u) \tag{28}$$

$$S(z,u,v) = \alpha \frac{\lambda A_0[\{N(z)-1\}\{a^*(\lambda) + z(1-a^*(\lambda)\} + z]}{z - (q+pz)\{a^*(\lambda) + z(1-a^*(\lambda))\}b^*(H(z))} \exp\{-H(z)u\}\exp\{-\lambda h(1-z)v\}\bar{B}(u)\bar{S}(v) \tag{29}$$

$$R^{(1)}(z,u,v)=\alpha\frac{\lambda A_0[\{N(z)-1\}\{a^*(\lambda)+z(1-a^*(\lambda)\}+z]}{z-(q+pz)\{a^*(\lambda)+z(1-a^*(\lambda))\}b^*(H(z))}s^*\{\lambda h(1-z)\}$$

$$\times\exp\{-H(z)u\}\exp\{-\lambda h(1-z)v\}\overline{B}(u)\overline{G}^{(1)}(v) \tag{30}$$

$$R^{(k)}(z,u,v)=\alpha\frac{\lambda A_0[\{N(z)-1\}\{a^*(\lambda)+z(1-a^*(\lambda)\}+z]}{z-(q+pz)\{a^*(\lambda)+z(1-a^*(\lambda))\}b^*(H(z))}s^*\{\lambda h(1-z)\}$$

$$\times\exp\{-H(z)u\}\prod_{n=1}^{k-1}[r_n g^{(n)*}\{\lambda h(1-z)\}]\exp\{-\lambda h(1-z)v\}\overline{B}(u)\overline{G}^{(k)}(v)\ ,2\le k\le K \tag{31}$$

$$V^j(z,u)=\frac{\lambda A_0}{[v^*\{\lambda h\}]^{J-j+1}}\exp\{-\lambda h(1-z)u\}\overline{V}(u)\ ,\quad 1\le j\le J \tag{32}$$

where

$$N(z)=\frac{1-[v^*(\lambda h)]^J}{[v^*(\lambda h)]^J[1-v^*(\lambda h)]}[v^*\{\lambda h(1-z)\}-1]$$

$$H(z)=\lambda h(1\text{-}z)+\alpha[1-s^*\{\lambda h(1-z)\}\{\prod_{k=1}^{K-1}[r_n g^{(n)*}\{\lambda h(1-z)\}]g^{(K)*}\{\lambda h(1-z)\}$$

$$+(1-r_1)g^{(1)*}\{\lambda h(1-z)\}+\sum_{k=1}^{K-2}(1-r_{K-k})\prod_{n=1}^{K-(k+1)}[r_n g^{(n)*}\{\lambda h(1-z)\}]g^{(K-k)*}\{\lambda h(1-z)\}\}]$$

Proof: For proof see Appendix A.

Theorem 2: Probability generating functions for the number of jobs in the orbit and in the system (one is in service) are

$$\Theta(z)=A(z)+B(z)+S(z)+\sum_{k=1}^{K}R^{(k)}(z)+\sum_{j=1}^{J}V^j(z) \tag{33}$$

$$\Pi(z)=A(z)+zB(z)+zS(z)+\sum_{k=1}^{K}zR^{(k)}(z)+\sum_{j=1}^{J}V^j(z) \tag{34}$$

where

$$A(z) = \frac{A_0 z\{1 - a^*(\lambda)\}\{N(z) - 1 + (q + pz)b^*(H(z))\}}{z - (q + pz)\{a^*(\lambda) + z(1 - a^*(\lambda))\}b^*(H(z))} \tag{35}$$

$$B(z) = \frac{\lambda A_0[\{N(z) - 1\}\{a^*(\lambda) + z(1 - a^*(\lambda)\} + z]}{z - (q + pz)\{a^*(\lambda) + z(1 - a^*(\lambda))\}b^*(H(z))} \times \frac{[1 - b^*(H(z))]}{H(z)} \tag{36}$$

$$S(z) = \alpha \frac{\lambda A_0[\{N(z) - 1\}\{a^*(\lambda) + z(1 - a^*(\lambda)\} + z]}{z - (q + pz)\{a^*(\lambda) + z(1 - a^*(\lambda))\}b^*(H(z))} \times \frac{[1 - b^*(H(z))]}{H(z)} \times \frac{[1 - s^*\{\lambda h(1 - z)\}]}{\lambda h(1 - z)} \tag{37}$$

$$R^{(1)}(z) = \alpha \frac{\lambda A_0[\{N(z) - 1\}\{a^*(\lambda) + z(1 - a^*(\lambda)\} + z]}{z - (q + pz)\{a^*(\lambda) + z(1 - a^*(\lambda))\}b^*(H(z))} \times \frac{[1 - b^*(H(z))]}{H(z)} s^*\{\lambda h(1 - z)\} \times \frac{[1 - g^{(1)*}\{\lambda h(1 - z)\}]}{\lambda h(1 - z)}$$

$$R^{(k)}(z) = \alpha \frac{\lambda A_0[\{N(z) - 1\}\{a^*(\lambda) + z(1 - a^*(\lambda)\} + z]}{z - (q + pz)\{a^*(\lambda) + z(1 - a^*(\lambda))\}b^*(H(z))} \times \frac{[1 - b^*(H(z))]}{H(z)} s^*\{\lambda h(1 - z)\} \times \prod_{n=1}^{k-1} [r_n g^{(n)*}\{\lambda h(1 - z)\}] \times \frac{[1 - g^{(k)*}\{\lambda h(1 - z)\}]}{\lambda h(1 - z)} \quad ,2 \le k \le K \tag{38}$$

$$V^j(z) = \frac{A_0}{[v^*\{\lambda h\}]^{J-j+1}} \times \frac{[1 - v^*\{\lambda h(1 - z)\}]}{h(1 - z)}, \qquad 1 \le j \le J \tag{39}$$

where

$$A_0 = \frac{h[a^*(\lambda) - p - \rho\zeta]}{ha^*(\lambda)[q - \rho(\zeta - 1) + \alpha\rho\{s_1 + g_1^{(1)} + \prod_{n=1}^{k-1} r_n \sum_{k=2}^{K} g_1^{(k)}\}] + N'(1)[h - p + a^*(\lambda)(1 - h) + \rho(h - \zeta) + \alpha\rho h\{s_1 + g_1^{(1)} + \prod_{n=1}^{k-1} r_n \sum_{k=2}^{K} g_1^{(k)}\}]} \tag{40}$$

$$\rho = \lambda b_1, \quad \zeta = h\left[1 + \alpha\left\{s_1 + (1 - r_1)g_1^{(1)} + \prod_{k=1}^{K-1} r_k \sum_{k=1}^{K} g_1^{(k)} + \sum_{k=1}^{K-2} (1 - r_{K-k}) \prod_{n=1}^{K-(k+1)} r_n \sum_{n=1}^{K-k} g_1^{(n)}\right\}\right]$$

Proof: For proof see Appendix A.

5. Performance Indices

We derive some performance indices to predict the behaviour of the system using the probability generating functions obtained in previous section as follows:

- Expected number of jobs in the orbit: (for three (K = 3) phase of repair) $E(L) = \underset{z\to 1}{Lt}\dfrac{d\Theta(z)}{dz}$

$$= A_0 \frac{\begin{array}{l} N''(1)\{H'(1)[a^*(\lambda)-p+b_1H'(1)][1-a^*(\lambda)+b_1H'(1)/h]+H'(1)[a^*(\lambda)-p+b_1H'(1)]^2/h\} \\ +\{[p-a^*(\lambda)]b_2H'(1)-b_1^2H''(1)\}\{[a^*(\lambda)+N'(1)][-H'^2(1)/h]\}+2H'(1)a^*(\lambda)[1-a^*(\lambda)][p+N'(1)-b_1H'(1)]\{q-b_1H'(1)/h+b_1H'(1)\} \\ -2p[N'(1)+a^*(\lambda)][H'(1)]^2b_1\{1-a^*(\lambda)-b_1H'(1)/h\}+[1-a^*(\lambda)][a^*(\lambda)+N'(1)H''(1)][b_2(H'(1))^2-b_1H''(1)] \\ +\alpha\lambda^2hb_1H'(1)[a^*(\lambda)-p+b_1H'(1)][N'(1)+a^*(\lambda)]\{s_2+g_2^{(1)}+r_1g_2^{(2)}+2r_1r_2[g_1^{(1)}g_1^{(3)}+g_1^{(2)}g_1^{(3)}]+r_1r_2g_2^{(3)}+2r_1g_1^{(1)}g_1^{(2)}+2s_1[g_1^{(1)}+r_1g_1^{(2)}+r_1r_2g_1^{(3)}]\} \end{array}}{2H'(1)[a^*(\lambda)-p+b_1H'(1)]^2} \tag{41}$$

where $H'(1) = -\lambda h[1+\alpha(s_1+g_1^{(1)}+r_1g_1^{(2)}+r_1r_2g_1^{(3)})]$

$$\begin{aligned} H''(1) = -\alpha(\lambda h)^2\{ & s_2+2s_1r_1r_2(g_1^{(1)}+g_1^{(2)}+g_1^{(3)})+2s_1(1-r_1)g_1^{(1)}+2s_1r_1(1-r_2)(g_1^{(1)}+g_1^{(2)})+2r_1r_2(g_1^{(1)}g_1^{(3)}+g_1^{(2)}g_1^{(3)}+g_1^{(1)}g_1^{(2)}) \\ & +r_1r_2(g_2^{(1)}+g_2^{(2)}+g_2^{(3)})+(1-r_1)g_2^{(1)}+(1-r_2)r_1(g_2^{(1)}+g_2^{(2)}+2g_1^{(1)}g_2^{(2)})]\} \end{aligned}$$

- Expected number of jobs in the system:

$$E(M) = \underset{z\to 1}{Lt}\frac{d\Pi(z)}{dz}$$

$$= \frac{h\rho[N'(1)+a^*(\lambda)]}{ha^*(\lambda)[q-\rho(\zeta-1)+\alpha\rho\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]+N'(1)[h-p+a^*(\lambda)(1-h)+\rho(h-\zeta)+\alpha\rho h\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]}+E(L) \tag{42}$$

- Long run fraction of the time when the server is idle state during retrial time: $P(A) = Lt_{z\to 1}A(z)$

$$= \frac{h[1-a^*(\lambda)][N'(1)+p+\rho\zeta]}{ha^*(\lambda)[q-\rho(\zeta-1)+\alpha\rho\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]+N'(1)[h-p+a^*(\lambda)(1-h)+\rho(h-\zeta)+\alpha\rho h\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]} \tag{43}$$

- Long run fraction of the time when the server is in busy state:

$$P(B) = Lt_{z\to 1}B(z)$$

$$= \frac{h\rho[N'(1)+a^*(\lambda)]}{ha^*(\lambda)[q-\rho(\zeta-1)+\alpha\rho\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]+N'(1)[h-p+a^*(\lambda)(1-h)+\rho(h-\zeta)+\alpha\rho h\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]} \quad (44)$$

- Long run fraction of the time when the server is under setup state:

$$P(S) = Lt_{z\to 1}S(z)$$

$$= \frac{\alpha h\rho s_1[N'(1)+a^*(\lambda)]}{ha^*(\lambda)[q-\rho(\zeta-1)+\alpha\rho\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]+N'(1)[h-p+a^*(\lambda)(1-h)+\rho(h-\zeta)+\alpha\rho h\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]} \quad (45)$$

- Long run fraction of the time when the server is under k^{th} phase repair state: $P(R^{(k)}) = Lt_{z\to 1}R^{(k)}(z)$

$$= \frac{\alpha h\rho[N'(1)+a^*(\lambda)][g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}]}{ha^*(\lambda)[q-\rho(\zeta-1)+\alpha\rho\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]+N'(1)[h-p+a^*(\lambda)(1-h)+\rho(h-\zeta)+\alpha\rho h\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]} \quad (46)$$

- Long run fraction of the time when the server is on j^{th} phase vacation state: $P(V^j) = Lt_{z\to 1}V^j(z)$

$$= \frac{N'(1)[a^*(\lambda)-p-\rho\zeta]}{ha^*(\lambda)[q-\rho(\zeta-1)+\alpha\rho\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]+N'(1)[h-p+a^*(\lambda)(1-h)+\rho(h-\zeta)+\alpha\rho h\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]} \quad (47)$$

where $N'(1) = \frac{1-[\nu^*(\lambda h)]^J \lambda h \nu_1}{[\nu^*(\lambda h)]^J[1-\nu^*(\lambda h)]}$, $N''(1) = \frac{1-[\nu^*(\lambda h)]^J (\lambda h)^2 \nu_2}{[\nu^*(\lambda h)]^J[1-\nu^*(\lambda h)]}$

6. Special Cases

In this section, we shall examine whether by setting appropriate parameters, our results are consistent with known results for some specific cases.

9. ***Model with Bernoulli feedback, repeated attempts, modified vacation with discouragement and reliable server*** : Setting $\alpha = 0$, Eq. (41) becomes

$$E(L) = \frac{\begin{array}{l} N''(1)\left\{[a^*(\lambda)-p-\lambda h b_1][1-a^*(\lambda)+\lambda b_1]+[a^*(\lambda)-p-\lambda h b_1]^2/h\right\}+\left\{(a^*(\lambda)-p)\lambda h b_2+2p\lambda h b_1^2\right\}\left\{[a^*(\lambda)+N'(1)]\lambda\right\} \\ +[N'(1)+a^*(\lambda)]\lambda h[1-a^*(\lambda)][\lambda h b_2+2pb_1)]+2a^*(\lambda)[1-a^*(\lambda)][p+N'(1)+\lambda h b_1]\left\{q+\lambda b_1(1-h)\right\} \end{array}}{2[a^*(\lambda)-p-\lambda h b_1]^2}$$
$$\times \frac{h[a^*(\lambda)-p-\lambda h b_1]}{N'(1)[a^*(\lambda)(1-h)+h-p]+ha^*(\lambda)[1-p+(1-h)\lambda b_1]} \tag{48}$$

(II) ***Model with modified J-vacations and reliable server:*** Putting $\alpha=0$, $p=0$,
$h=1$, $a^*(\lambda)\to 1$ in eq. (41), we get

$$E(L) = \frac{N''(1)}{2[N'(1)+1]} + \frac{\lambda^2 b_2}{2[1-\lambda b_1]} \tag{49}$$

This result agrees with Takagi's modified vacation model[20].

(III) ***Model with single vacation and reliable server:*** Substituting $\alpha = 0$, $p = 0$,

$h = 1$, $a^*(\lambda)\to 1$, $J = 1$, eq. (41) reduce to

$$E(L) = \frac{\lambda^2 v_2}{2[v^*(\lambda)+\lambda v_1]} + \frac{\lambda^2 b_2}{2[1-\lambda b_1]} \tag{50}$$

which coincides with the results obtained by Takagi for single vacation system[20].

(IV) Model with multiple vacations and reliable server: Substituting $\alpha = 0$, $p = 0$, $h = 1$, $a^*(\lambda)\to 1$, $J\to\infty$, eq. (41) provides

$$E(L) = \frac{\lambda v_2}{2v_1} + \frac{\lambda^2 b_2}{2[1-\lambda b_1]} \tag{51}$$

In this case, our result tally with the result of Takagi's multiple vacation model[20].

(V) Model with Bernoulli feedback and reliable server: On putting $\alpha=0$, $h=1$, $a^*(\lambda)\to 1$, $J=0$, eq. (41) yields

$$E(L) = \frac{\lambda^2 q b_2 + 2p\lambda^2 b_1^2}{2[1-\lambda b_1]} \tag{52}$$

This result matches with Madan and Al-Rawwash's[13]model with without bulk and vacation.

(VI) Model with general retrial attempts and reliable server: On setting $\alpha = 0$, $p = 0$, $h = 1$, $J = 0$, eq. (41) turns into

$$E(L) = \frac{\lambda^2 b_2 + 2\lambda\, b_1(1-a^*(\lambda))}{2[a^*(\lambda)-\lambda b_1]} \tag{53}$$

which is same as that of Gomez-Corral[1].

(VII) Model with exponential retrial attempts and reliable server: Substituting

$\alpha=0$, $h=1$, $J=0$, $a^*(\lambda) = \dfrac{\theta}{\theta+\lambda}$, eq. (41) converts into $E(L) = \dfrac{\lambda^2 b_2 + 2\lambda^2 b_1}{2[\theta-(\theta+\lambda)\lambda b_1]}$ (54)

which coincide with the results obtained by Choi *et al.*[21].

7. Reliability Indices

Since reliability shows failure free behaviour of the server, in order to obtain reliability indices we consider setup state and repair state as absorbing states. Now we shall construct the transient state equations for idle state, busy state and vacation state as follows:

$$\left[\frac{d}{dt}+\lambda\right]A_0(t)=\int_0^\infty V_0^J(t,u)\bar{v}(u)\,du \tag{55}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial u}\right]A_n(t,u)=-[\lambda+\bar{a}(u)]A_n(t,u), \quad n\geq 1 \tag{56}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial u}\right]B_n(t,u)=-[\lambda h+\bar{b}(u)+\alpha]B_n(t,u)+\lambda h B_{n-1}(t,u), \quad n\geq 0 \tag{57}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial u}\right]V_n^j(t,u)=-[\lambda h+\bar{v}(u)]V_n^j(t,u)+\lambda h V_{n-1}^j(t,u), \quad n\geq 0, \ 1\leq j\leq J \tag{58}$$

The transient state boundary conditions are:

$$A_n(t,0)=\sum_{j=1}^{J}\int_0^\infty V_n^j(t,u)\bar{v}(u)du+q\int_0^\infty B_n(t,u)\bar{b}(u)du+p\int_0^\infty B_{n-1}(t,u)\bar{b}(u)du, \quad n\geq 1 \tag{59}$$

$$B_0(t,0)=\int_0^\infty A_1(t,u)\bar{a}(u)du+\lambda A_0(t) \tag{60}$$

$$B_n(t,0)=\int_0^\infty A_{n+1}(t,u)\bar{a}(u)du+\lambda\int_0^\infty A_n(t,u)du, \quad n\geq 1 \tag{61}$$

$$V_n^1(t,0)=\begin{cases}\int_0^\infty q\,B_n(t,u)\bar{b}(u)du, & n=0\\ 0, & n\geq 1\end{cases} \tag{62}$$

$$V_n^j(t,0)=\begin{cases}\int_0^\infty V_n^{j-1}(t,u)\bar{v}^{(j-1)}(u)du, & n=0,\ 2\leq j\leq J\\ 0, & n\geq 1\end{cases} \tag{63}$$

Initial condition is

$$A_n(0)=\begin{cases}1, & n=0\\ 0, & n>0\end{cases}$$

Taking Laplace transform of equations (55-63), we get

$$(s+\lambda)A_0^*(s)-1=\int_0^\infty V_0^{J*}(s,u)\bar{v}(u)du \tag{64}$$

$$\frac{\partial}{\partial u}A_n^*(s,u)=-[s+\lambda+\bar{a}(u)]A_n^*(s,u) \tag{65}$$

$$\frac{\partial}{\partial u}B_n^*(s,u)=-[s+\lambda h+\alpha+\bar{b}(u)]B_n^*(s,u)+\lambda h B_{n-1}^*(s,u) \tag{66}$$

$$\frac{\partial}{\partial u}V_n^{j*}(s,u)=-[s+\lambda h+\bar{v}(u)]V_n^{j*}(s,u)+\lambda h V_{n-1}^{j*}(s,u),\quad 1\le j\le J \tag{67}$$

$$A_n^*(s,0)=q\int_0^\infty B_n^*(s,u)\bar{b}(u)du+p\int_0^\infty B_{n-1}^*(s,u)\bar{b}(u)du+\sum_{j=1}^{J}V_n^{j*}(s,u)\bar{v}(u)du \tag{68}$$

$$B_0^*(s,0)=\int_0^\infty A_1^*(s,u)\bar{a}(u)du+\lambda A_0^*(s) \tag{69}$$

$$B_n^*(s,0)=\int_0^\infty A_{n+1}^*(s,u)\bar{a}(u)du+\lambda\int_0^\infty A_n^*(s,u)du \tag{70}$$

$$V_n^{1*}(s,0)=\begin{cases}\int_0^\infty q B_n^*(s,u)\bar{b}(u)du, & n=0\\ 0, & n\ge 1\end{cases} \tag{71}$$

$$V_n^{j*}(s,0)=\begin{cases}\int_0^\infty V_n^{(j-1)*}(s,u)\bar{v}^{(j-1)}(u)du, & n=0,\ 2\le j\le J\\ 0, & n\ge 1\end{cases} \tag{72}$$

Define probability generating functions in the form of Laplace transforms as

$$A^*(z,s,u)=\sum_{n=1}^{\infty}A_n^*(s,u)z^n,\ B^*(z,s,u)=\sum_{n=0}^{\infty}B_n^*(s,u)z^n,\ V^{j*}(z,s,u)=\sum_{n=0}^{\infty}V_n^{j*}(s,u)z^n,\quad |z|\le 1$$

Multiplying eqs. (64)-(70) with appropriate power of z^n and summing, we get

$$\frac{\partial}{\partial u}A^*(s,z,u) = -[s+\lambda+\bar{a}(u)]A^*(s,z,u) \tag{73}$$

$$\frac{\partial}{\partial u}B^*(s,z,u) = -[s+\lambda h(1-z)+\alpha+\bar{b}(u)]B^*(s,z,u) \tag{74}$$

$$\frac{\partial}{\partial u}V^{j*}(s,z,u) = -[s+\lambda h(1-z)+\bar{v}(u)]V^{j*}(s,z,u) \tag{75}$$

$$A^*(s,z,0) = \sum_{j=1}^{J}\int_0^{\infty}V^{j*}(s,z,u)\bar{v}(u)du + (q+pz)\int_0^{\infty}B^*(s,z,u)\bar{b}(u)du - (s+\lambda)A_0^*(s) + 1 - \sum_{j=1}^{J}V_0^{j*}(0) \tag{76}$$

$$B^*(s,z,0) = \frac{1}{z}\int_0^{\infty}A^*(s,z,u)\bar{a}(u)du + \lambda\int_0^{\infty}A^*(s,z,u)du + \lambda A_0^*(s) \tag{77}$$

Theorem 3: Reliability of the server in Laplace form is given by

$$R^*(s) = A_0^*(s) + \frac{[1-a^*(s+\lambda)][1-(s+\lambda)A_0^*(s)) + b^*(s+\alpha)\lambda A_0^*(s)]}{(s+\lambda)-\{(s+\lambda)a^*(s+\lambda)+\lambda[1-a^*(s+\lambda)]\}b^*(s+\alpha)}$$

$$+\frac{[\{1-(s+\lambda)A_0^*(s)\}\{sa^*(s+\lambda)+\lambda\}+(s+\lambda)\lambda A_0^*(s)][1-b^*(s+\alpha)]}{[s+\alpha][(s+\lambda)-\{(s+\lambda)a^*(s+\lambda)+\lambda[1-a^*(s+\lambda)]\}b^*(s+\alpha)]}$$

$$+\frac{[(s+\lambda)A_0^*(s)-1][1-V^*(s)]}{s[V^*(\lambda h)]^{J-j+1}} \tag{78}$$

Proof: For proof see Appendix B.

Theorem 4: Availability of the server is obtained as

$$AV=\frac{h[a^*(\lambda)\{1+\rho\}-(p+\rho\zeta)a^*(\lambda)+N'(1)\{1+\rho-a^*(\lambda)\}]+N'(1)[a^*(\lambda)-p-\rho\zeta]}{ha^*(\lambda)[q-\rho(\zeta-1)+\alpha\rho\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]+N'(1)[h-p+a^*(\lambda)(1-h)+\rho(h-\zeta)+\alpha\rho h\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1}r_n\sum_{k=2}^{K}g_1^{(k)}\}]} \tag{79}$$

Proof: $AV = A_0 + \underset{z\to 1}{Lt}\int_0^{\infty} A(z,u)\,du + \underset{z\to 1}{Lt}\int_0^{\infty} B(z,u)\,du + \underset{z\to 1}{Lt}\sum_{j=1}^{J}\int_0^{\infty} V^j(z,u)du$

Theorem 5: Failure frequency of the server is:

$$FR = \frac{h\alpha\rho[N'(1)+a^*(\lambda)]}{ha^*(\lambda)[q-\rho(\zeta-1)+\alpha\rho\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1} r_n \sum_{k=2}^{K} g_1^{(k)}\}]+N'(1)[h-p+a^*(\lambda)(1-h)+\rho(h-\zeta)+\alpha\rho h\{s_1+g_1^{(1)}+\prod_{n=1}^{k-1} r_n \sum_{k=2}^{K} g_1^{(k)}\}]} \tag{80}$$

Proof: $FR = \underset{z\to 1}{Lt}\int_0^{\infty} \alpha\, B(z,u)\,du$

8. Numerical Results

In this section, we present numerical illustrations in order to justify the implementation of analytical results. We have used software MATLAB to develop computational program. Reliability indices such as availability and failure frequency of the server are summarized in Tables 1-3. The effect of different parameters on the expected number of jobs in the orbit E(L) are displayed in Fig. 1-4 by varying arrival rate (λ), failure rate (α), retrial rate (θ) & set up rate (s). Fig. 1-3 are for the model when service time, setup time, repair time and vacation time follow Erlangian (k = 4) distribution, whereas graphs for the exponential distributions (*i.e.* k = 1) are taken in Fig. 2-4. The numerical results corresponding to Erlangian retrial time (k = 4) and exponential retrial time have been shown through discrete and continuous lines, respectively in all figures. For numerical experiments, we set default parameters as $\lambda = .7$, $\alpha = 1$, $\theta = 2$, $s = 1$, $p = .3$, $h = .4$, $r_1 = r_2 = r = .5$, $g = .8$, $\mu = 6$, $\nu = .5$, $J = 10$.

In Tables 1-3, we have examined the sensitivity of the feedback parameter (p) and failure rate (α), arrival rate (λ) & service rate (μ) and joining probability (h), retrial rate (θ) and repair rate (γ), respectively on availability (AV) and failure frequency (FR) of the server. It is noticed that the availability (failure frequency) increases (decreases) with μ, γ and θ whereas on increasing p, h, λ and α, the availability (failure frequency) of the server decreases (increases).

Table 1. Effect of feedback parameter (p) and failure rate (α) of the server on availability and failure frequency of the server.

A		Exponential (k = 1) retrial rate				Erlangian (k = 4) retrial rate			
		θ = 1		θ = 3		θ = 1		θ = 3	
		AV	FR	AV	FR	AV	FR	AV	FR
0	p=0	1	0	1	0	1	0	1	0
1		0.8960	0.0620	0.9116	0.0527	0.8904	0.0653	0.9108	0.0532
2		0.7920	0.1240	0.8232	0.1054	0.7809	0.1306	0.8216	0.1064
3		0.6880	0.1860	0.7348	0.1581	0.6714	0.1959	0.7324	0.1596
4		0.5841	0.2480	0.6465	0.2108	0.5619	0.2612	0.6432	0.2127
0	p=.3	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
1		0.8271	0.1031	0.8664	0.0797	0.8113	0.1125	0.8645	0.0808
2		0.6543	0.2061	0.7328	0.1593	0.6226	0.2250	0.7291	0.1616
3		0.4816	0.3091	0.5994	0.2389	0.4340	0.3375	0.5937	0.2423
4		0.3090	0.4120	0.4660	0.3184	0.2455	0.4499	0.4584	0.3230

Table 2. Effect of arrival rate (λ) and service rate (μ) of the server on availability and failure frequency of the server.

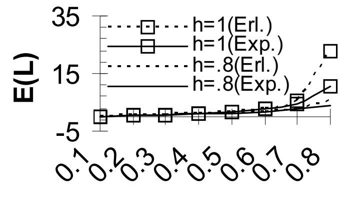
A		Exponential (k = 1) retrial rate				Erlangian (k = 4) retrial rate			
		θ = 1		θ = 3		θ = 1		θ = 3	
		AV	FR	AV	FR	AV	FR	AV	FR
0.2	μ=6	0.9594	0.0242	0.9625	0.0224	0.9590	0.0244	0.9625	0.0224
0.4		0.9145	0.0510	0.9277	0.0431	0.9114	0.0528	0.9274	0.0433
0.6		0.8586	0.0843	0.8879	0.0669	0.8484	0.0904	0.8867	0.0676
0.8		0.7936	0.1231	0.8440	0.0930	0.7703	0.1370	0.8412	0.0947
1.0		0.7205	0.1667	0.7967	0.1212	0.6766	0.1928	0.7912	0.1245
0.2	μ=12	0.9796	0.0122	0.9812	0.0112	0.9794	0.0123	0.9812	0.0112
0.4		0.9572	0.0255	0.9638	0.0216	0.9557	0.0264	0.9637	0.0217
0.6		0.9293	0.0422	0.9439	0.0334	0.9242	0.0452	0.9433	0.0338
0.8		0.8968	0.0616	0.9220	0.0465	0.8851	0.0685	0.9206	0.0474
1.0		0.8602	0.0833	0.8983	0.0606	0.8383	0.0964	0.8956	0.0622

Table 3. Effect of joining probability (h) and repair rate γ of the server on availability and failure frequency of the server.

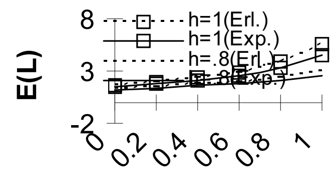
H		Exponential (k = 1) retrial rate				Erlangian (k = 4) retrial rate			
		θ = 2		θ = 4		θ = 2		θ = 4	
		AV	FR	AV	FR	AV	FR	AV	FR
0.2	γ=8	0.9184	0.0487	0.9303	0.0415	0.9150	0.0507	0.9295	0.0420
0.4		0.8561	0.0858	0.8717	0.0765	0.8519	0.0883	0.8706	0.0772
0.6		0.8031	0.1174	0.8167	0.1093	0.7997	0.1195	0.8157	0.1099
0.8		0.7585	0.1440	0.7665	0.1393	0.7566	0.1452	0.7659	0.1396
1		0.7205	0.1667	0.7205	0.1667	0.7205	0.1667	0.7205	0.1667
0.2	γ=16	0.9348	0.0487	0.9444	0.0416	0.9321	0.0507	0.9437	0.0420
0.4		0.8851	0.0858	0.8976	0.0765	0.8818	0.0883	0.8967	0.0772
0.6		0.8429	0.1174	0.8537	0.1093	0.8401	0.1195	0.8529	0.1099
0.8		0.8073	0.1440	0.8136	0.1393	0.8057	0.1452	0.8131	0.1396
1		0.7769	0.1667	0.7769	0.1667	0.7769	0.1667	0.7769	0.1667

From Fig. 1(a) & 1(b), we study the effect of arrival rate (λ) and failure rate (α), respectively on the expected number of jobs in the orbit E(L) for different sets of joining probability (h). We observe that initially E(L) increases gradually but later on increases sharply with the arrival rate (λ) and failure rate (α). The effect of retrial rate (θ) and setup rate (s) have been displayed in Fig. 1(c) and 1(d), respectively. We can easily see that initially E(L) decreases rapidly as θ and s increase but after some time there is a linear decrement. In Fig. 2(a-d), a slight change occurs in comparison to Fig. 1(a-d). A remarkable increment has been seen in E(L) on increasing joining probability (h) in Fig. 1(a-d) and 2(a-d).

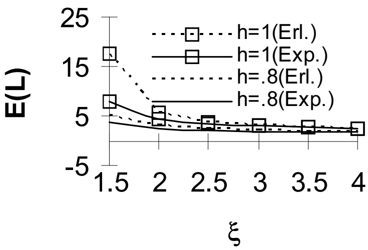
Figures 3(a-d) and 4(a-d) depict the same pattern with arrival rate (λ), failure rate (α), retrial rate (θ) and setup rate (s) as we have obtained in Fig. 1(a-d) and 2(a-d). As we increase the feedback parameter (p), the expected number of jobs in the orbit E(L) also increases. Significant change occurs in case of Erlangian retrial time as comparison to exponential retrial time.



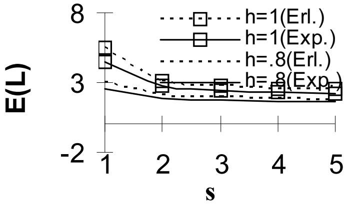
(a)



(b)

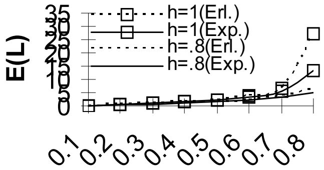


(c)

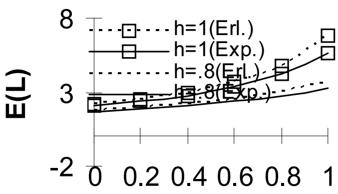


(d)

Fig. 1. Effect of (a) λ, (b) α, (c) θ & (d) s on E(L) on E(L) for different sets of h for Erlangian model.

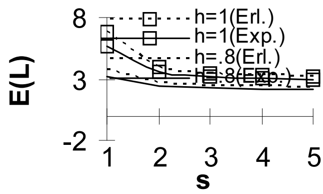


(a)



(b)

(c)



(d)

Fig. 2. Effect of (a) λ, (b) α, (c) θ & (d) s for different sets of h for exponential model.

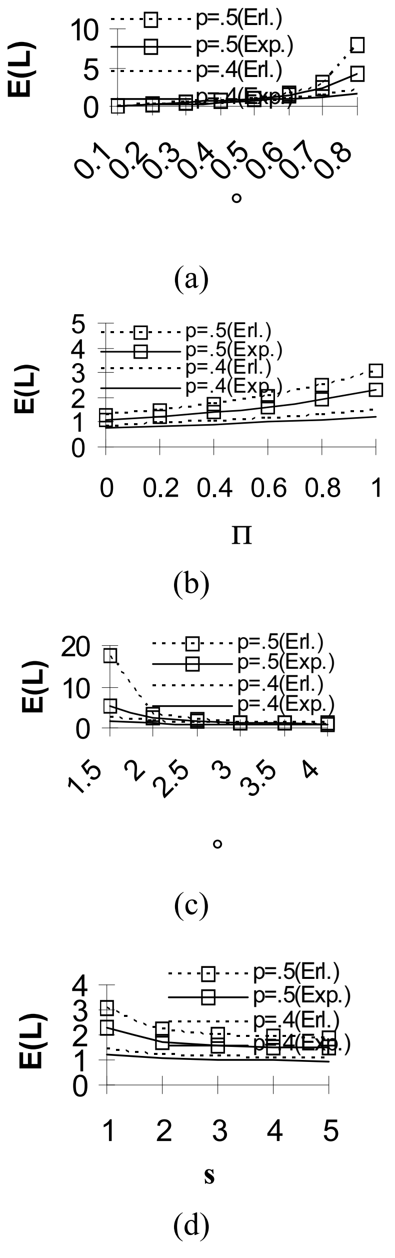


Fig. 3. Effect of (a) λ, (b) α, (c) θ & (d) s on E(L) for different sets of p for Erlangian model.

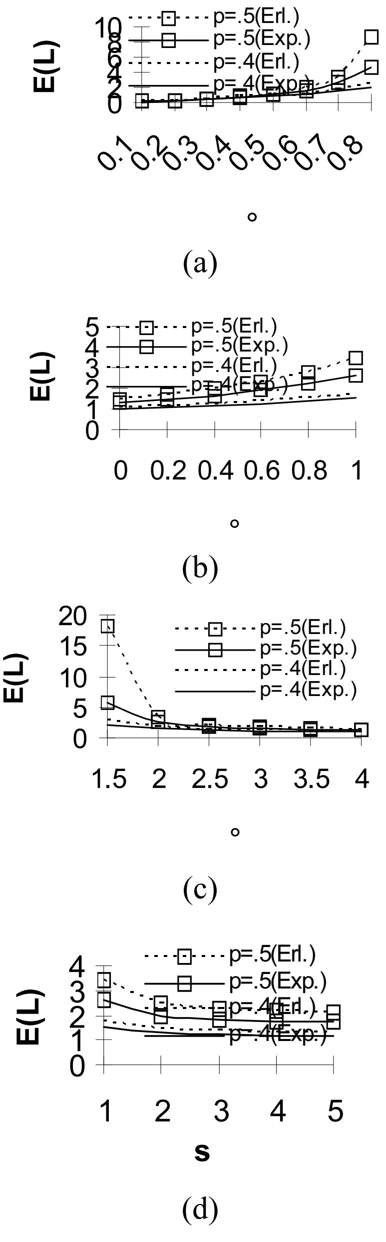


Fig. 4. Effect of (a) λ, (b) α, (c) θ & (d) s on E(L) for different sets of p for exponential model.

From all tables and graphs, we conclude that:

- With the increase in failure rate (α) of the server, arrival rate (λ), feedback parameter (p) and joining probability (h) of the jobs, we notice that the availability (AV) decreases but failure frequency (FR) of the server increases. But on increasing service rate (μ), repair rate (γ) and retrial rate (θ), the availability (AV) increases but the failure frequency (FR) of the server decreases. The patterns of the graphs are in agreement with physical situations.

- As we expect, the expected number of jobs in the orbit E(L) increases as arrival rate (λ), feedback parameter (p) and joining probability (h) of the jobs and failure rate (α) of the server increase, but there is a decreasing trend in the number of jobs in the orbit with the increase in retrial rate (θ) and setup rate (s).

- Balking behaviour of jobs significantly affect the expected number of jobs in the orbit as decreasing trend is quite visible; this may be due to fact that effective arrival rate decreases in such a case.

9. Conclusion

In the present study, a single unreliable server M/G/1 queueing system with modified vacation, repeated attempts and discouragement is considered. To obtain analytical expressions for various performance indices of interest, we employ the generating function approach. An important feature of the model with general retrial policy studied, is that we have obtained the analytical solutions in closed form. The modified vacation policy concept is introduced for utilizing the idle time of the server. The provision of K-phase optional repair makes our model more versatile in real congestion situations as broken down server may need different phases of repair, some of which may be optional.

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Appendix A

Proof of Theorem 1

Solving eqs. (16), (18) & (19), we get

$$A(z,u) = A(z,0)\exp\{-\lambda u\}\overline{A}(u) \tag{A.1}$$

$$S(z,u,v) = S(z,u,0)\exp\{-\lambda h(1-z)v\}\overline{S}(v) \tag{A.2}$$

$$R^{(k)}(z,u,v) = R^{(k)}(z,u,0)\exp\{-\lambda h(1-z)v\}\overline{G}^{(k)}(v) \tag{A.3}$$

Now for k≥2, using eq. (25), eq. (A.3) becomes

$$R^{(k)}(z,u,v) = \int_0^\infty r_{k-1} R^{(k-1)}(z,u,v)\overline{g}^{(k-1)}(v)\exp\{-\lambda h(1-z)v\}\overline{G}^{(k)}(v)dv \tag{A.4}$$

For k=1, using Eq. (24), Eq. (A.3) yields

$$R^{(1)}(z,u,v) = \int_0^\infty S(z,u,v)\overline{s}(v)\exp\{-\lambda h(1-z)v\}\overline{G}^{(1)}(v)\,dv \tag{A.5}$$

Using (A.2) and (23), eq. (A.5) provides

$$R^{(1)}(z,u,v) = \alpha B(z,u)s^*\{\lambda h(1-z)\}\exp\{-\lambda h(1-z)v\}\overline{G}^{(1)}(v) \tag{A.6}$$

Similarly, we obtain

$$R^{(k)}(z,u,v) = \alpha B(z,u)s^*\{\lambda h(1-z)\}\prod_{n=1}^{k-1}[r_n g^{(n)*}\{\lambda h(1-z)\}]\exp\{-\lambda h(1-z)v\}\overline{G}^{(k)}(v)\,,\ (2\le k\le K) \tag{A.7}$$

On solving Eq. (17) and using Eq. (21) & (A.7), we get

$$B(z,u) = B(z,0)\exp\{-H(z)u\}\overline{B}(u)$$

$$= [A(z,0)\{\frac{a^*(\lambda)+z(1-a^*(\lambda))}{z}\} + \lambda A_0]\exp\{-H(z)u\}\overline{B}(u) \tag{A.8}$$

Solution of eq. (6) at n=0, gives

$$V_0^j(u) = V_0^j(0)\exp\{-\lambda hu\}\overline{V}(u)\ ,\ (1\le j\le\ J) \tag{A.9}$$

$$\int_0^\infty V_0^j(u)\overline{v}(u)du = \int_0^\infty V_0^j(0)\overline{v}(u)\exp\{-\lambda hu\}\overline{V}(u)du = V_0^j(0)v^*(\lambda h)$$

$$\text{For } j=J\ ,\quad V_0^J(0) = \frac{\lambda A_0}{v^*(\lambda h)} \tag{A.10}$$

From eqs (A.10) and (14), we have

$$V_0^j(0) = \frac{\lambda A_0}{[v^*(\lambda h)]^{J-j+1}}, \qquad (1 \le j \le J-1) \tag{A.11}$$

$$V^j(z,0) = \frac{\lambda A_0}{[v^*(\lambda h)]^{J-j+1}}, \qquad (1 \le j \le J) \tag{A.12}$$

Using eq. (A.9), we obtain

$$\int_0^\infty V_0^j(u)du = \int_0^\infty V_0^j(0)\exp\{-\lambda hu\}\overline{V}(u)du \tag{A.13}$$

$$V_0^j = \frac{A_0}{[v^*(\lambda h)]^{J-j+1}} \times \frac{[1 - v^*(\lambda h)]}{h}, \qquad (1 \le j \le J) \tag{A.14}$$

$$V_0 = \frac{A_0[1 - \{v^*(\lambda h)\}^J]}{h\{v^*(\lambda h)\}^J} \tag{A.15}$$

$$V^j(z,u) = V^j(z,0)\exp\{-\lambda h(1-z)u\}\overline{V}(u) \tag{A.16}$$

From eq. (21), we have

$$A(z,0) = \lambda A_0(N(z)-1) + (q+pz)B(z,0)b^*\{H(z)\} \tag{A.17}$$

From eqs (A.8) and (A.17), we get

$$B(z,0) = \frac{\lambda A_0[\{N(z)-1\}\{a^*(\lambda) + z(1-a^*(\lambda)\} + z]}{z - (q+pz)\{a^*(\lambda) + z(1-a^*(\lambda))\}b^*(H(z))} \tag{A.18}$$

$$A(z,0) = \frac{\lambda A_0 z\{N(z) - 1 + (q+pz)b^*(H(z))\}}{z - (q+pz)\{a^*(\lambda) + z(1-a^*(\lambda))\}b^*(H(z))} \tag{A.19}$$

Proof of Theorem 2

For the prove of this theorem, we use

$$A(z) = \int_0^\infty A(z,u)\,du \;; \qquad B(z) = \int_0^\infty B(z,u)\,du \;; \; S(z) = \int_0^\infty\int_0^\infty S(z,u,v)\,du\,dv \;;$$

$$R^{(k)}(z) = \int_0^\infty\int_0^\infty R^{(k)}(z,u,v)\,du\,dv \;\; (k=1,2,\ldots,K); \quad V^j(z) = \int_0^\infty V^j(z,u)\,du \;\; (j=1,2,\ldots,J)$$

Using eq. (26) at z=1, we can find normalizing constant A_0.

Appendix B

Proof of Theorem 3

On solving Eq. (73), (74) and (75), we get

$$A^*(s,z,u) = A^*(s,z,0)e^{-(s+\lambda)u}\overline{A}(u) \tag{B.1}$$

$$B^*(s,z,u) = B^*(s,z,0)e^{-\{s+\alpha+\lambda h(1-z)\}u}\overline{B}(u) \tag{B.2}$$

$$V^{j*}(s,z,u) = V^{j*}(s,z,0)e^{-\{s+\lambda h(1-z)\}u}\overline{V}(u) \tag{B.3}$$

From eq. (67) at n=0, we obtain

$$V_0^j(s,u) = V_0^j(s,0)\exp\{-(s+\lambda h)u\}\overline{V}(u), \qquad (1\le j \le J) \tag{B.4}$$

$$\text{at } j=J \qquad V_0^{J*}(s,0) = \frac{[(s+\lambda)A_0^*(s)-1]}{[v^*(s+\lambda h)]^{J-j+1}}, \qquad (1\le j\le J) \tag{B.5}$$

$$V^{j*}(z,s,0) = \frac{[(s+\lambda)A_0^*(s)-1]}{[v^*(s+\lambda h)]^{J-j+1}}$$

$$\text{Now } \int_0^\infty A^*(s,z,u)\overline{a}(u)du = A^*(s,z,0)a^*(s+\lambda) \tag{B.6}$$

$$\int_0^\infty A^*(s,z,u)du = A^*(s,z,0)\frac{1-a^*(s+\lambda)}{(s+\lambda)} \tag{B.7}$$

$$\begin{aligned} A^*(s,z,0) &= (q+pz)B^*(s,z,0)b^*\{s+\alpha+\lambda h(1-z)\} - (s+\lambda)A_0^*(s)+1 \\ &- \sum_{j=1}^{J}\frac{[(s+\lambda)A_0^*(s)-1]}{[v^*(s+\lambda h)]^{J-j+1}} + \sum_{j=1}^{J} V^*\{s+\lambda h(1-z)\}\frac{[(s+\lambda)A_0^*(s)-1]}{[v^*(s+\lambda h)]^{J-j+1}} \end{aligned} \tag{B.8}$$

Using eq. (B.5), Eq. (76) yields

$$A^*(s,z,0) = (q+pz)B^*(s,z,0)b^*\{s+\alpha+\lambda h(1-z)\} + \{(s+\lambda)A_0^*(s)-1\}\{N(z,s)-1\} \tag{B.9}$$

Using (B.6) & (B.7), again Eq. (77) becomes

$$B^*(s,z,0) = \frac{[(s+\lambda)a^*(s+\lambda)+\lambda z(1-a^*(s+\lambda)][\{N(z,s)-1\}\{(s+\lambda\}A_0^*(s)-1]+z(s+\lambda)\lambda A_0^*(s)}{z(s+\lambda)-(q+pz)\{(s+\lambda)a^*(s+\lambda)+\lambda z[1-a^*(s+\lambda)]\}b^*(s+\alpha+\lambda h(1-z))} \tag{B.10}$$

$$A^*(s,z,0) = \frac{z(s+\lambda)[\{N(z,s)-1\}\{(s+\lambda\}A_0^*(s)-1]+z(s+\lambda)\lambda A_0^*(s)(q+pz)b^*(s+\alpha+\lambda h(1-z))}{z(s+\lambda)-(q+pz)\{(s+\lambda)a^*(s+\lambda)+\lambda z[1-a^*(s+\lambda)]\}b^*(s+\alpha+\lambda h(1-z))} \tag{B.11}$$

$$A^*(z,s)=\frac{z(s+\lambda)[\{N(z,s)-1\}\{(s+\lambda\}A_0^*(s)-1]+z(s+\lambda)\lambda A_0^*(s)(q+pz)b^*(s+\alpha+\lambda h(1-z))}{z(s+\lambda)-(q+pz)\{(s+\lambda)a^*(s+\lambda)+\lambda z[1-a^*(s+\lambda)]\}b^*(s+\alpha+\lambda h(1-z))}\times\frac{[1-a^*(s+\lambda)}{(s+\lambda)} \tag{B.12}$$

$$B^*(z,s)=\frac{[\{N(z,s)-1\}\{(s+\lambda)A_0^*(s)-1\}\{(s+\lambda)a^*(s+\lambda)+\lambda z(1-a^*(s+\lambda))\}+z(s+\lambda)\lambda A_0^*(s)][1-b^*(s+\alpha+\lambda h(1-z))]}{[s+\alpha+\lambda h(1-z)][z(s+\lambda)-(q+pz)\{(s+\lambda)a^*(s+\lambda)+\lambda z[1-a^*(s+\lambda)]\}b^*(s+\alpha+\lambda h(1-z))]} \tag{B.13}$$

$$V^{j*}(z,s)=\frac{[(s+\lambda)A_0^*(s)-1][1-V^*(s+\lambda h(1-z))]}{[s+\lambda h(1-z)][V^*(s+\lambda h)]^{J-j+1}} \tag{B.14}$$

We have $P^*(z,s)=A^*(z,s)+B^*(z,s)+\sum_{j=1}^{J} V^{j*}(z,s)$ (B.15)

$$=\frac{\begin{array}{l}\{N(z,s)-1\}\{(s+\lambda\}A_0^*(s)-1\}z\chi\phi_z\psi_z+\{(s+\lambda)a^*(s+\lambda)+\lambda z\chi\}\{N(z,s)-1\}\{(s+\lambda\}A_0^*(s)-1\}\phi_z\{1-b^*\{\psi_z\}\}\\ +(q+pz)b^*\{\psi_z\}z\lambda A_0^*(s)\chi\phi_z\psi_z+z\lambda A_0^*(s)(s+\lambda)\phi_z\{1-b^*\{\psi_z\}\}+\{(s+\lambda\}A_0^*(s)-1\}N(z,s)\psi_z\end{array}}{\phi_z\psi_z[z(s+\lambda)-(q+pz)\{(s+\lambda)a^*(s+\lambda)+\lambda z\chi\}b^*\{\psi_z\}]} \tag{B.16}$$

where $\psi_z=[s+\alpha+\lambda h(1-z)]$, $\phi_z=[s+\lambda h(1-z)]$, $\chi=[1-a^*(s+\lambda)]$

$$N(z,s)=\frac{1-[v^*(s+\lambda h)]^J}{[v^*(s+\lambda h)]^J[1-v^*(s+\lambda h)]}[v^*\{s+\lambda h(1-z)\}-1]$$

By Rouche's theorem, the denominator of the above Eq. (B.16) has one zero $\omega(s)$ inside the unit circle $|z|=1$ for Re(s)>0, and it is also the zero point for the numerator of above Eq. (B.16).

This is sufficient to determine the only unknown $A_0^*(s)$ appearing in the numerator.

$$A_0^*(s)=\frac{\{N(\omega(s),s)-1\}\phi_s[\omega(s)\chi\psi_s+\{(s+\lambda)a^*(s+\lambda)+\lambda\omega(s)\chi\}\{1-b^*(\psi_s)\}]+\psi_s N(\omega(s),s)}{\{N(\omega(s),s)-1\}(s+\lambda)\ \ \phi_s[\omega(s)\chi\psi_s+\{(s+\lambda)a^*(s+\lambda)+\lambda\omega(s)\chi\}\{1-b^*(\psi_s)\}]+\nabla} \tag{B.17}$$

where

$$\nabla=(q+p\omega(s))b^*(\psi_s)\omega(s)\lambda\psi_s\phi_s\chi+\omega(s)(s+\lambda)\lambda\phi_s\{1-b^*(\psi_s)\}+(s+\lambda)\psi_s N(\omega(s),s)$$

$\psi_s=[s+\alpha+\lambda h(1-\omega(s))]$, $\phi_s=[s+\lambda h(1-\omega(s))]$, $\chi=[1-a^*(s+\lambda)]$,

$$N(\omega(s),s)=\frac{1-[v^*(s+\lambda h)]^J}{[v^*(s+\lambda h)]^J[1-v^*(s+\lambda h)]}[v^*\{s+\lambda h(1-\omega(s))\}-1]$$

where $\omega(s)$ is the root of the equation

$$\phi_z\psi_z[z(s+\lambda)-(q+pz)\{(s+\lambda)a^*(s+\lambda)+\lambda z\chi\}b^*\{\psi_z\}]=0 \tag{B.18}$$

Using above results we obtain reliability of the server as

$$R^*(s) = A_0^*(s) + \underset{z\to 1}{\mathrm{Lt}}[A^*(z,s) + B^*(z,s) + \sum_{j=1}^{J} V^{J*}(z,s)] \tag{B.19}$$

منظومة صفوف الخوادم عديمة الاعتمادية M/G/1 ذات التغذية العكسية البرنويلية والمحاولات المتكررة مع الفراغ القابل للتغير وأطوار الإصلاح المتعددة وإمكانية المغادرة

مادهي جان، و شارو بهرجافا

قسم الرياضيات ، معهد العلوم الأساسية ، خانداري ،

أكرا – الهند

المستخلص. يتناول البحث منظومة صفوف الخوادم العامة ذات الفراغ القابل للتغير وأطوار الإصلاح المتعددة، حيث تصل المهام بطريقة بواسون، وعندما يكون الخادم مشغولاً أو في حالة بدء التشغيل أو الفراغ فإن المهام إما أن تستمر في المدار أو تخرج من المنظومة، وتقوم المهام الموجودة في المدار بطلب الخدمة بعد وقت عشوائي، وتتم الخدمة بأسلوب من يأتي أولاً يخدم أولاً، وبعد الحصول على خدمة ناجحة فإنه يمكن للمهمة أن تلتحق بذيل الصف أو أن تغادر المنظومة تماماً. وقد تمت دراسة إمكانية انهيار الخادم مع إمكانية أطوارالإصلاح المتعددة لإعادته إلى حالة ماقبل الانهيار، حيث يكون الطور الأول للإصلاح أساسياً بينما تكون بقية الأطوار اختيارية، ويحتاج المصلح إلى وقت معين ليبدأ الطور الأول من الإصلاح يعرف بوقت الإعداد، ويكون الوقت اللازم للإصلاح ووقت الإعداد مستقلان ويتبعان توزيعاً عاماً. أما عندما يكون المدار خالياً فإن الخادم ينتقل إلى عدد من حالات الفراغ المتعاقبة، إلى أن تسجل إحدى المهام –على الأقل– وجودها

في المدار. وقد تم الحصول على الدالة المولدة للاحتمالات المستقرة زمانياً في فترات عشوائية باستخدام طريقة المتغير الإضافي، كما تمت مناقشة الحالات التي سبقت دراستها كحالات خاصة من النموذج المقترح عن طريق اختيار قيم المتغيرات، وتم كذلك استنتاج معايير أداء الصفوف ومعايير الاعتمادية اللازمة لدراسة سلوك المنظومة، كما تمت دراسة تأثير المتغيرات المختلفة على أداء المنظومة عددياً.