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Two-Servers Heterogeneous Overflow Queues

M.O. ABOU-EL-ATA and S.A. IBRAHIM
Faculty of Engineering, Banha University,
Shoubra, Cairo, Egypt

ABSTRACT. In this research, we treat the system of the overflow queue with a primary truncated two-channels queue: $M/M/2/k$ and a secondary infinite-channels queue: $M/M/\infty$. The two servers have different rates μ_1, μ_2 which are different from the infinite servers with identical rate μ_3. The discipline considered is a modification of the usual one F I F O.

1. Introduction

The system of overflow queue had been studied by many researchers. The pioneer work was that of Kosten[1] who studied the system $M/M/C/C$ as a primary queue and $M/M/\infty$ as a secondary queue. In that work, he derived the probabilities in very complicated formulas. The most important work is that of Herzog and Kuhn[2] followed by Rath and Sheng[3]. They studied the factorial moments of the overflow queue and gave an explicit formula for $M_{(1)}$ and implicit for $M_{(2)}$ when $\mu = 1$.

The aim of this paper is to derive the factorial moments in an explicit form. In fact, we treat the system of overflow queues in which the primary queue is a truncated two channels in the heterogeneous case with a modified discipline. We also consider $\mu \neq 1$, (*i.e.* in terms of), and deduce some special cases. This work is an extension to Abou-El-Ata and Alseedy[4] and could be applied to communication and telephone system.

2. Analysis of the Problem

Consider the system of overflow queue with a truncated primary queue: $M/M/2/k$ and the server's rates: μ_1, μ_2, also an infinite servers queue: $M/M/\infty$ each with rate

μ_3. Assume that the arrival rate of the units to the system is λ. Also consider the units to be served according to Krishnamoorthi's[5] discipline as follows :

(i) If the two-channels are free, the head unit of the queue goes to the 1*st* channel with prob. π_1 or to the 2*nd* channel with prob. π_2, $(\pi_1 + \pi_2 = 1)$.

(ii) If one channel is free, the head unit goes directly to it.

(iii) If the two channels are busy, the units wait in their order until any channel becomes vacant.

Let us define $P_{n,m}$, the probability that there are n units in the primary and m units in the secondary queues. Also let $P_{i,j,m}$ be defined by:

$P_{i,j,m}$ = Prob. that there are i units in the 1*st* channel, j units in the 2*nd* channel, and m units in the secondary queue, $i, j = 0,1$, $m = 0\,(1)\ldots$.

i.e. $P_{0,m} = P_{0,0,m}$, $P_{1,m} = P_{1,0,m} + P_{0,1,m}$, and; $P_{2,m} = P_{1,1,m}$.

As in the usual δ-technique, the steady-state difference equations could be deduced as follows :

$$(m\mu_3 + \lambda)\, P_{0,0,m} = (m+1)\,\mu_3\, P_{0,0,m+1} + \mu_1\, P_{1,0,m} + \mu_2\, P_{0,1,m}\ , \quad n = 0 \tag{1}$$

$$\left.\begin{aligned}(m\mu_3 + \mu_1 + \lambda)\, P_{1,0,m} &= (m+1)\,\mu_3\, P_{1,0,m+1} + \mu_2\, P_{1,1,m} + \lambda\pi_1\, P_{0,0,m}\\ (m\mu_3 + \mu_2 + \lambda)\, P_{0,1,m} &= (m+1)\,\mu_3\, P_{0,1,m+1} + \mu_1\, P_{1,1,m} + \lambda\pi_2\, P_{0,0,m}\end{aligned}\right\}\ n = \tag{2}$$

$$(m\mu_3 + \mu + \lambda)\, P_{n,m} = (m+1)\,\mu_3\, P_{n,m+1} + \mu\, P_{n+1,m} + \lambda P_{n-1,m}\ , \quad 2 \leq n < k \tag{3}$$

$$(m\mu_3 + \mu + \lambda)\, P_{k,m} = (m+1)\,\mu_3\, P_{k,m+1} + \lambda P_{k,m-1} + \lambda P_{k-1,m}\ , \quad n = k \tag{4}$$

where
$$\mu = \mu_1 + \mu_2, \quad \text{and,} \qquad P_{k,-1} = 0 \tag{5}$$

Define the conditional ℓth factorial moments

$$M_{(\ell)}(n) = \sum_{m=0}^{\infty} m_{(\ell)}\, P_{n,m} \tag{6}$$

where $m_{(\ell)} = m(m-1)\ldots(m-\ell+1)$, $\ell \geqslant \quad m_{(0)} = 1$

Thus, the ℓth factorial moment is

$$M_{(\ell)} = \sum_{n=0}^{k} M_{(\ell)}(n) = \sum_{n=0}^{k}\ \sum_{m=}^{\infty} m_{(\ell)}\, P_{n,m} \tag{7}$$

Also the factorial moment generating function is

$$M(n;t) = \sum_{\ell=0}^{\infty} M_{(\ell)}(n)\, \frac{t^{\ell}}{\ell!}\ \sum_{m=0}^{\infty} (1+t)^m\, P_{n,m} \tag{8}$$

Now, from relations (1) → (8) we can easily deduce the moment-difference equations as follows :

$$(\ell\mu_3 + \lambda)\, M_{(\ell)}(0) \qquad \mu_1\, M_{(\ell)}(1,0) + \mu_2\, M_{(\ell)}(0,1)\ , \quad n = 0 \tag{9}$$

$$\left.\begin{aligned}(\ell\mu_3 + \mu_1 + \lambda)\, M_{(\ell)}(1,0) &= \mu_2\, M_{(\ell)}(2) + \lambda\, \pi_1\, M_{(\ell)}(0)\\ (\ell\mu_3 + \mu_2 + \lambda)\, M_{(\ell)}(0,1) &= \mu_1\, M_{(\ell)}(2) + \lambda\, \pi_2\, M_{(\ell)}(0)\end{aligned}\right\} = \qquad (10)$$

$$(\ell\mu_3 + \mu + \lambda)\, M_{(\ell)}(n) = \mu\, M_{(\ell)}(n+1) + \lambda\, M_{(\ell)}(n-1) \quad , 2 \leqslant n < k \qquad (11)$$

$$(\ell\mu_3 + \mu)\, M_{(\ell)}(k) = \lambda\ell\, M_{(\ell-1)}(k1) + \lambda\, M_{(\ell)}(k-1)\,,\ n = k \qquad (12)$$

where μ is given in relation (5).

Summing up equations (9) → (12) over $n = 0\,(1)\,k$ and using relation (7) we have :

$$M_{(\ell)} = \sum_{n=0}^{k} M_{(\ell)}(n) = \rho_3\, M_{(\ell-1)}(k)\ ,\ \rho_i = \frac{\lambda}{\mu_i}\ ,\ i = 1,2,3 \qquad (13)$$

To calculate the moments, we have to calculate first of all $M_{(\ell)}(k)$, $\ell = 0, 1, \dots$.

2.1 The First Factorial Moment

To calculate $M_{(1)}$, the 1*st* factorial moment we have to derive first of all $M_{(0)}(k)$. From relations (9) and (10) with $\ell = 0$ we obtain :

$$M_{(0)}(2) = \theta\, M_{(0)}(1) \qquad (14)$$

where $\theta = \dfrac{\rho_1\, \rho_2}{\rho_1 + \rho_2}$, $\rho_i = \dfrac{\lambda}{\mu_i}$ $i = 1, 2, 3$ (15)

$$\text{and}\quad \left.\begin{aligned} M_{(0)}(1,0) &= \frac{\theta}{[illegible]}\, M_{(0)}(1) + \frac{\pi_1\, \rho_1}{1 + \rho_1}\, M_{(0)}(0)\\ M_{(0)}(0,1) &= \frac{[illegible]}{[illegible]}\, M_{(0)}(1) + \frac{\pi_2\, \rho_2}{1 + \rho_2}\, M_{(0)}(0)\end{aligned}\right\}$$

But, from $P_{1,m} = P_{1,0,m} + P_{0,1,m}$ we can deduce that

$$M_{(\ell)}(0,1) + M_{(\ell)}(1,0) = M_{(\ell)}(1) \qquad (17)$$

Then, from (16) we could easily get :

$$M_{(0)}(1) = \gamma\, M_{(0)}(0) \qquad (18)$$

where $\gamma = \dfrac{\rho_1\pi_1 + \rho_2\pi_2 + \rho_1\rho_2}{1 + 2\,\theta}$ (19)

Therefore, from (14) and (18) we have

$$M_{(0)}(2) = \gamma\theta\, M_{(0)}(0)$$

From equations (9), (10) with $n = 0, 1$, (11) with $n = 2(1)\,k - 1$ $\ell = 0$, and (20) we get :

$$M_{(0)}(3) = \theta\, M_{(0)}(2) - \gamma\, \theta^2\, M_{(0)}(0)$$

Thus by mathematical induction we have :

$$M_{(0)}(n) = \gamma\theta^{n-1}\, M_{(0)}(0) \quad , \quad 1 \leqslant n \leqslant k$$

Using relations (6), (7) with $\ell = 0$, (22) and the boundary condition

$$M_{(0)} = \sum_{n=0}^{k} \sum_{m=0}^{\infty} P_{n,m} = 1, \text{ we get :}$$

$$M_{(0)}(0) = 1 + \gamma \left(\frac{1 - \theta^k}{1 - \theta} \right)$$

Then, from relation (22) with $n = k$ and (23) we obtain

$$M_{(0)}(k) = \gamma\, \theta^{k-1} \; 1 + \gamma \; \frac{1 - \theta^k}{1 - \theta})]^{-1}$$

Therefore, from relations (13) with $\ell = 1$ and (24) it could be easily deduced that

$$M_{(1)} = \rho_3 \gamma\, \theta^{k-1} \left[1 + \gamma \left(\frac{1 - \theta^k}{1 - \theta} \right) \right]^{-1} \tag{25}$$

where θ is given in relation (15).

2.2 The Second Factorial Moment

To calculate $M_{(2)}$, the second factorial moment, we have to compute first of all $M_{(1)}(k)$, and thus from relations (9), (10) with $\ell = 1$ we get :

$$M_{(1)}(2) = \frac{\theta}{\rho_3} \left[M_{(1)}(0) + (1 + \rho_3) M_{(1)}(1) \right.$$

$$\left.\begin{aligned} \text{and } M_{(1)}(1, 0) &= \nu_1 \; \pi_1 + \frac{\theta}{\rho_2 \rho_3}) \; M_{(1)}(0) + \nu_1 \theta \left(\frac{1 + \rho_3}{\rho_2 \rho_3} \right) M_{(1)}(1) \\ M_{(1)}(0, 1) &= \nu_2 \left(\pi_2 + \frac{\theta}{\rho_1 \rho_3} \right) M_{(1)}(0) + \nu_2 \theta \; \frac{1 + \rho_3}{\rho_1 \rho_3}) \; M_{(1)}(1) \end{aligned} \right\}$$

where θ is given in relation (15), and ;

$$\nu_1 = \frac{\rho_1 \rho_3}{\rho_1 + \rho_3 + \rho_1 \rho_3}, \quad \nu_2 = \frac{\rho_2 \rho_3}{\rho_2 + \rho_3 + \rho_2 \rho_3}$$

Thus, from both relations of (27) we can easily obtain

$$M_{(1)}(1) = \delta\, M_{(1)}(0)$$

$$\text{where } \delta = \frac{\nu_1 \pi_1 + \nu_2 \pi_2 + \nu}{1 - (1 + \rho_3)\, \nu}, \quad \nu = \left(\frac{\nu_1}{\rho_2 \rho_3} + \frac{\nu_2}{\rho_1 \rho_3} \right) \theta$$

Using relations (26) and (29) we have

$$M_{(1)}(2) = \Delta\, M_{(1)}(0) \tag{31}$$

where $\Delta = \frac{\theta}{\rho_3} \quad 1 + (1 + \rho_3)\,\delta$

From equation (11) for $n = 2\,(1)\,k-1 \quad , \ell - 1$ we can deduce $M_{(1)}\,(n+2)$ recurrsively as follows :

$$M_{(1)}\,(n+2) = M_{(1)}\,(2) \sum_{i=0}^{n-1} \frac{(-n+i)_i}{} \theta^{n-i}\, \phi^{n-2i}$$

$$M_{(1)}\,(1) \sum_{0}^{n-} \frac{(-n+1+i)_i}{i\,!} \theta^{n-i}\, \phi^{n-1-2i}$$

where $n = 1(1)\ k-2 \quad$, θ is given in relation (15) and

$$\phi = \quad + \sum_{=}^{3} \frac{}{\rho_i}$$

Using relations (33) with $n = k-2 \quad$, (29) and (31) we obtain

$$M_{(1)}\,(k) = (\Delta u - \delta v)\, M_{(1)}\,(0) \quad , \quad k \geqslant 3$$

where :

$$\left.\begin{aligned} u &= \sum^{k-3} \frac{(-k+2+i)_i}{i\,!} \theta^{k-2-i}\, \phi^{k-2-2i} \\ v &= \sum_{i=0}^{k-3} \frac{(-k+3+i)_i}{i\,!} \theta^{k-2-i}\, \phi^{k-3-2i} \end{aligned}\right\}$$

to find $M_{(1)}\,(0)$ use relations (7) with $\ell = 1$, (29), (31) and (33) we get:

$$M_{(1)}\,(0) = M_{(1)}\,[\,1 + \delta\,(\,1-x\,) + \Delta\,(\,1+y\,)\,]^{-1}$$

where :

$$\left.\begin{aligned} x &= \sum_{n=1}^{k-2} \sum_{i=0}^{n-1} \frac{(-n+1+i)_i}{i\,!} \theta^{n-i}\, \phi^{n-1-2i} \\ y &= \sum_{n=1}^{k-2} \sum_{i=0}^{n-1} \frac{(-n+i)_i}{i\,!} \theta^{n-i}\, \phi^{n-2i} \end{aligned}\right\}$$

Then from relations (35) and (37) we can deduce

$$M_{(1)}\,(k) = \frac{(\,\Delta u - \delta v\,)\, M_{(1)}}{\ + \delta(1-x) + \Delta(\ \ + y)} \quad , \quad k \geqslant 3$$

Therefore $M_{(2)}$ can be deduced from relations (13) with $\ell = 2$ and (39) as follows

$$M_{(2)} = \frac{\rho_3\,(\Delta u - \delta v)\,M_{(1)}}{1 + \delta(1-x) + \Delta(1+y)} \quad , \quad k \geqslant 3 \tag{40}$$

where u, v are given in relation (36) and x, y in relation (38). Also the variance is :
variance $= M_{(2)} + M_{(1)} - M^2_{(1)}$.

If $k = 2$, it is clear that $u = v = x = y = 0$, thus use relation (31) to get $M_{(1)}\,(2)$.

2.3 Particular Cases

Model I

Let $k = 2 \Rightarrow (M/M/2/2, M/M/\infty)$ and put $\rho_i = \rho$, $i = 1, 2, 3$ then from relation (25) we get :

$$M_{(1)} = \frac{\rho^3}{\rho^2 + 2\rho + 2}\ , \ \rho = \frac{\lambda}{\mu} \tag{41}$$

Since $k = 2$, then from relation (38) we have $x = y = 0$, and thus relation (37) becomes :

$$M_{(1)}\,(0) = M_{(1)}\,[\,1 + \delta + \Delta\,]^{-1}$$

where $\Delta = \frac{1}{2}\,(\rho^2 + 2\rho + 2)\,,\ \delta = \rho + 1$

$$M_{(1)}(0) = \frac{2\,\rho^3}{(\rho^2 + 2\,\rho + 2)\,(\rho^2 + 4\,\rho + 6)}$$

$$M_{(1)}\ (2) = \Delta M_{(1)}\,(0) = \frac{\rho^3}{\rho^2 + 4\rho + 6}$$

Using relation (13) with $\ell = 2$, thus we get

$$M_{(2)} = \rho M_{(1)}\,(2) = \frac{\rho^4}{\rho^2 + 4\rho + 6} \tag{42}$$

which are the same results as in Riordan[6] with $c = k = 2$.

Model II

Let $k = 3 \Rightarrow (M/M/2/3, M/M/\infty)$ and put $\rho_i = \rho$, $i =$ 2, 3 then from relation (25) we obtain :

$$M_{(1)} = \frac{\rho^4}{\rho^3 + 2\rho^2 + 4\rho + 4} \tag{43}$$

From relations (30), (32), (36) and (38) we get :

$\delta = \rho + 1,\ \Delta = \frac{1}{2}\,(\rho^2 + 2\rho + 2),\ u = y = \frac{1}{2}\,(\rho + 3),\ v = x = \frac{1}{2}\,\rho.$

Thus from relation (40) we have

$$M_{(2)} = \frac{\rho^5 (\rho^3 + 3\rho^2 + 6\rho + 6)}{(\rho^3 + 2\rho^2 + 4\rho + 4)\ (\rho^3 + 5\rho^2 + 14\rho + 18)} \tag{44}$$

Model III

Let $k = 4 \Rightarrow (M/M/2/4, M/M/\infty)$ and put $\rho_i = \rho$, $i = 1, 2, 3$ thus from relation (25) we have :

$$M_{(1)} = \frac{\rho^5}{\rho^4 + 2\rho^3 + 4\rho^2 + 8\rho + 8}$$

Using relations (30), (32), (36) and (38) we can have

$$\delta = \quad + \rho \quad \Delta = 1/2\,(\rho^2 + 2\rho + 2),\ u = \frac{\rho^2 + 4\rho + 9}{4}\ ,\ v = \frac{\rho^2 + 3\rho}{4}$$

$$x = \frac{\rho^2 + 5\rho}{4}\ ,\ y = \frac{\rho^2 + 6\rho + 15}{4}$$

Therefore, from relation (40) we get

$$M_{(2)} = \frac{\rho^5 (\rho^4 + 4\rho^3 + 11\rho^2 + 20\rho + 18)}{(\rho^4 + 2\rho^3 + 4\rho^2 + 8\rho + 8)\ (\rho^4 + 6\rho^3 + 21\rho^2 + 48\rho + 54)}$$

Note: For other cases of k such as $k = 5$, 10 see the table give below.

TABLE The factorial moments $M_{(1)}$, $M_{(2)}$ of the overflow queue.

Moment k	$M_{(1)}$	$M_{(2)}$
$k = 2$	$\frac{\rho^3}{\rho^2 + 2\rho + 2}$	$\frac{\rho^4}{\rho^2 + 4\rho + 6}$
$k = 3$	$\frac{\rho^4}{\rho^3 + 2\rho^2 + 4\rho + 4}$	$\frac{\rho(\rho^3 + 3\rho^2 + 6\rho + 6)\, M_{(1)}}{\rho^3 + 5\rho^2 + 14\rho + 18}$
$k = 4$	$\frac{\rho^5}{\rho^4 + 2\rho^3 + 4\rho^2 + 8\rho + 8}$	$\frac{\rho(\rho^4 + 4\rho^3 + 11\rho^2 + 20\rho + 18)\, M_{(1)}}{\rho^4 + 6\rho^3 + 21\rho^2 + 48\rho + 54}$
$k = 5$	$\frac{\rho^6}{\rho^5 + 2\rho^4 + 4\rho^3 + 8\rho^2 + 16\rho + 16}$	$\frac{\rho(\rho^5 + 5\rho^4 + 17\rho^3 + 41\rho^2 + 66\rho + 54)\, M_{(1)}}{\rho^5 + 7\rho^4 + 29\rho^3 + 83\rho^2 + 162\rho + 162}$
$k = 10$	$\frac{\rho^{11}}{\rho^{10} + 2\rho^9 + 4\rho^8 + 8\rho^7 + 16\rho^6 + 32\rho^5 + 64\rho^4 + 128\rho^3 + 256\rho^2 + 512\rho + 512}$	$\frac{A\, M_{(1)}}{B}$

where

$$A = \rho(\rho^{10} + 10\rho^9 + 62\rho^8 + 286\rho^7 + 1046\rho^6 + 3110\rho^5 + 7544\rho^4 + 14706\rho^3 + 22133\rho^2 + 23328\rho + 13122)$$

$$B = \rho^{10} + 12\rho^9 + 84\rho^8 + 428\rho^7 + 1712\rho^6 + 5544\rho^5 + 14646\rho^4 + 31212\rho^3 + 51759\rho^2 + 61236\rho + 39366$$

Also we draw some curves of $M_{(1)}$ for the different models given above when $k = 2$, 3, 4, 5, 10.

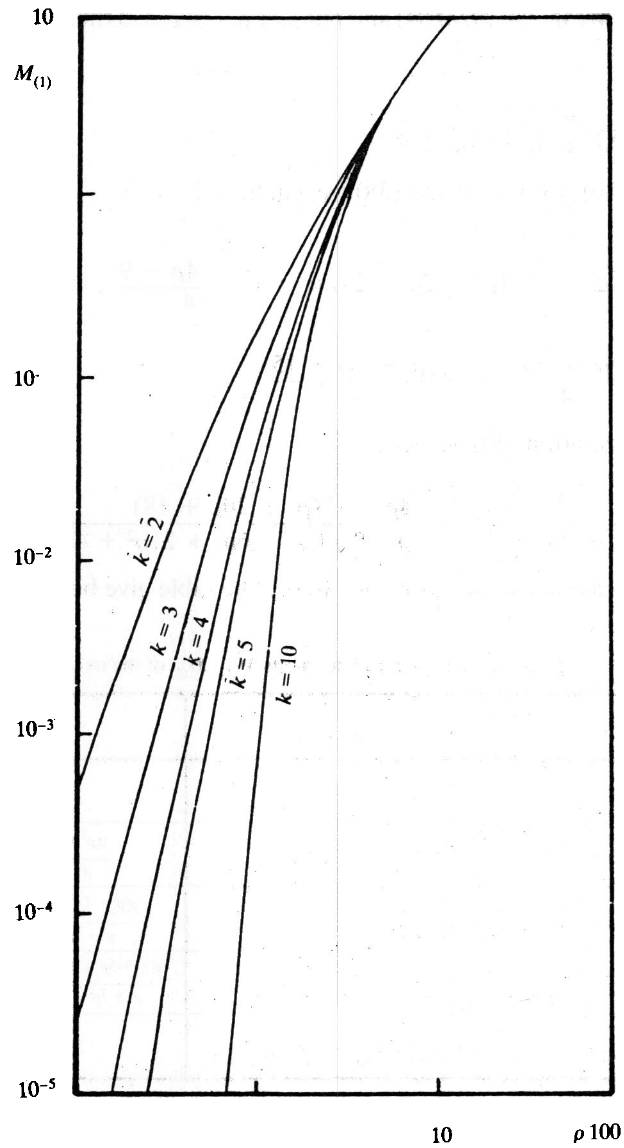

FIG. 1. The graphs of $M_{(1)}$ is plotted against ρ for different values of k.

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الطوابير الفائضة بخادمين غير متجانسين

متولي عثمان أبو العطا و شوقي أحمد إبراهيم
كلية الهندسة ، جامعة بنها ، القاهرة ، مصر

المستخلص . يعالج هذا البحث نظام الصفوف الفائض ذا صفٌّ أولى مبتور بخادمين لهما معدلين مختلفين ، وصفٌّ ثانوي لا نهائي في عدد الخدم ومعدلهمْ متساوٍ ويختلف عن معدلي الخادمين في الصف الأولى . نظام الخدمة تعديل لنظام الأولوية FIFO .