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k-out of-n System with Dependent Failures and Standby Support

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ABSTRACT. In this paper, we have studied a k-out of-n system with dependent failures and standbys. The failure of any working unit affects the system reliability and the standby support increases the reliability of the system. To improve the grade of service we have studied two standby support options for degraded systems. The expressions for reliability and expected operational time are being derived. To validate the analytical results, illustrative examples with numerical results are also facilitated.

1. Introduction

Reliability criteria for modern systems, includes the stipulations of system load, its performance, the subsystems, and supporting systems. Due to colossal dimensions of these systems, it is also difficult to do reliability computation as a whole. On the other hand, every subsystem has its own characteristics with different failure patterns, reliability indices and analysis methods. The redundancy in a system is usually employed to design highly reliable systems. Generally, it is assumed that the failure of subsystems in k-out of-n systems and in parallel ones, does not affect the functional ones. Nevertheless, in practice, this is not true; failure affects the efficiency of the functional units also. It increases stresses on the surviving ones, and the resulting effect on the functional ones is some increase in their failure rates. Thus, dependence occurs and we found that due to dependence the failure rates of subsystems degrade. Many authors have discussed failure rates of systems depending on workload. The system reliability of shared load was investigated by Pham [7], Shao and Lamberson [8] and Moustafa [4,5]. Pham et al., [6] studied a multistage degraded system and calculated various measures of reliability. Littlewood [3] and

Hughes[1] studied correlated failure rate models in which the failure rate of a unit depends on performance of other working unit. Iyer et al., [2] analysed the model in which the performance of a system depends on workload of system units.

Taking into consideration the numerous system-requirements, we have tried to establish some interesting results for k-out of-n system, which can be useful to enhance reliability and options. The organization of the paper is as follows: In section 2, assumptions related to the system modeling are made and the approaches of the introduction of standbys are discussed. The basic equation to calculate the system reliability has been provided in section 3. In section 4, we have established results to calculate the expected operational time. In Section 5, a numerical illustration has been given. In section 6, conclusion and remarks regarding further extensions have been accommodated.

2. The Standby Support System

The system consists of a main unit, n- subsystems, and s-standbys. The system works if at least k or more subsystems and main units work. The other system characterizations are as follows: The failure of the main unit, which supervises the system, causes the total system failure and has the constant failure rate λ_p. The system is modeled by using a continuous-time discrete-state Markov process. State i is the state of the system indicating that exactly i subsystems are failed (i=1,2, ..., n+s-k) at time t. The state transition rate from state i to state i+1 is given by Λ_i. The system is imperfect; the failure of fault coverage is constant and equal to λ_c. Each subsystem either is in working or failed mode. The failure rates of all working subsystems are constant and same, and depend on the number of working units having failure rate λ_j (j=k, k+1,..., n). The failure rate of j^{th} (j=k, k+1,…,n) working units λ_j obeys $\lambda_n \le \lambda_{n-1} \le ... \le \lambda_k$. The standbys have constant failure rates β.

To overcome models limitations such as cost constraints, standbys support, availability of the standby, which may or may not be of equivalent measures for the operating subsystems, two standby support models are considered to measure the system reliability.

2.1 *Model 1*

A system, consisting of n subsystems is considered and s standbys are provided to support the system in the beginning. When any of the operating subsystems fails, it is replaced by the standby available. If all the standbys are consumed i.e., none is available for replacement, the system works as degraded system until only k subsystems work. The availability of less than k successful subsystems in operation makes the system fail.

Consider the failure rate of each standby equivalent to the failure rate of the n-th working subsystem i.e. λ_n. The state transition rate of the system is given by

$$\Lambda_i = \begin{cases} n\lambda_n + (s-i)\beta & 0 \le i \le s \\ (n+s-i)\lambda_{n+s-i} & s+1 \le i \le s+n-k \end{cases} \qquad ...(1)$$

2.2 *Model 2*

The system initially consists of n subsystems and s standbys. The system works as a degraded system until k subsystems function and standbys remain idle. At the failure of kth, subsystem in operation period s standbys becomes active and starts supporting the system. This model finds an application when there is an easy availability of spare part support.

Taking failure rate of each standby equivalent to the failure probability of the k th working subsystem, i.e. λ_k. The state transition rate of the system is given by

$$\Lambda_i = \begin{cases} (n-i)\lambda_{n-i} & 0 \le i \le n-k \\ k\lambda_k + (s+n-k-i)\beta & n-k+1 \le i \le n+s-k \end{cases} \qquad ...(2)$$

In particular when s=0 *i.e.* for the system without standby support the state transition rate of the above two models reduces to

$$\Lambda_i = (n-i)\lambda_{n-i} \qquad \ldots(3)$$

3. The system reliability measures

To calculate the system reliability measures the system can be modeled by using a continuous-time discrete-state Markov process.
The equations governing the states of the system are given by

$$\frac{dp_0(t)}{dt} = -\left(\Lambda_0 + \lambda_p + \lambda_c\right)p_0(t) \qquad ...(4)$$

and $$\frac{dp_i(t)}{dt} = -(\Lambda_i + \lambda_p) p_i(t) + \Lambda_{i-1} p_{i-1}(t) \quad ...(5)$$

for $i=1,2,\ldots,n-k+s$

Taking the Laplace transform of (4) and (5), with initial conditions $p_0(0) = 1$, $p_i(0) = 0$, we have

$$\overline{p_0}(u) = \frac{1}{u + \Lambda_0 + \lambda_p + \lambda_c} \quad ...(6)$$

$$\overline{p_i}(u) = \frac{\Lambda_{i-1}}{u + \Lambda_i + \lambda_p} \overline{p_{i-1}}(u) \quad ...(7)$$

solving *(7)* recursively, we obtain

$$\overline{p_i}(u) = \frac{\prod_{j=0}^{i-1} \Lambda_{j-1}}{(u + \Lambda_0 + \lambda_p + \lambda_c) \prod_{j=1}^{i} \left(u + \Lambda_j + \lambda_p\right)} \quad ...(8)$$

Taking the Inverse Laplace transform of (6) and (8), we obtain

$$p_0(t) = \exp(-(\Lambda_0 + \lambda_p + \lambda_c)t) \quad ...(9)$$

$$p_i(t) = \prod_{j=0}^{i-1} \Lambda_j \left\{ \frac{\exp\left(-\left(\Lambda_0 + \lambda_p + \lambda_c\right) t\right)}{\prod_{j=1}^{i} \left(\Lambda_j - \Lambda_0 - \lambda_c\right)} + \sum_{j=1}^{i} \frac{\exp\left(-\left(\Lambda_j + \lambda_p\right) t\right)}{\left(\Lambda_0 - \Lambda_j - \lambda_c\right) \prod_{\substack{p=1 \\ p \neq j}}^{i} \left(\Lambda_p - \Lambda_j\right)} \right\}$$

...(10)

The system reliability for the given system can be obtained as

$$R(t) = \sum_{i=0}^{n-k+s} p_i(t)$$

$$= \exp(-(\Lambda_0 + \lambda_p + \lambda_c)t)$$

$$+ \sum_{i=1}^{n-k+s} \left(\prod_{j=0}^{i-1} \Lambda_j \right) \left(\frac{\exp\left(-\left(\Lambda_0 + \lambda_p + \lambda_c\right)t\right)}{\prod_{j=1}^{i} \left(\Lambda_j - \Lambda_0 - \lambda_c\right)} + \sum_{j=1}^{i} \frac{\exp\left(-\left(\Lambda_j + \lambda_p\right)t\right)}{\left(\Lambda_0 - \Lambda_j + \lambda_c\right) \prod_{\substack{p=1 \\ p \neq j}}^{i} \left(\Lambda_p - \Lambda_j\right)} \right)$$

…(11)

4. Expected operational time

The expected time spent in state i, during time (0,t).

$$E_i(t) = \int_0^t p_i(x)dx$$

$$= \begin{cases} \dfrac{1 - \exp^{-(\Lambda_0 + \lambda\ \ + \lambda_c)t}}{\Lambda_0 + \lambda_p + \lambda_c} & i = 0 \\ \prod_{j=0}^{i-1} \Lambda_j \left\{ \begin{aligned} & \frac{1 - \exp\left(-\left(\Lambda_0 + \lambda_p + \lambda_c\right)t\right)}{\left(\Lambda_0 + \lambda_p + \lambda_c\right) \prod_{j=1}^{i} \left(\Lambda_j - \Lambda_0 - \lambda_c\right)} \\ & + \sum_{j=1}^{i} \frac{\exp\left(-\left(\Lambda_j + \lambda_p\right)t\right)}{\left(\Lambda_j + \lambda_p\right)\left(\Lambda_0 - \Lambda_j + \lambda_c\right) \prod_{\substack{p=1 \\ p \neq j}}^{i} \left(\Lambda_p - \Lambda_j\right)} \end{aligned} \right\} & i = 1,2,\ldots,n-k+s \end{cases}$$

…(12)

The expected operational time (EOT) of a system in time (0,t) is,

$$EOT = \sum_{i=0}^{n-k+s} \int_0^t p_i(x)dx$$

$$= \frac{1-\exp^{-(\Lambda_0+\lambda_p+\lambda_c)t}}{\Lambda_0+\lambda_p+\lambda_c} + \sum_{i=1}^{n-k+s}\left[\prod_{j=0}^{i-1}\Lambda_j\left\{\begin{array}{l}\dfrac{1-\exp(-(\Lambda_0+\lambda_p+\lambda_c)t)}{(\Lambda_0+\lambda_p+\lambda_c)\prod_{j=1}^{i}(\Lambda_j-\Lambda_0-\lambda_c)} \\ +\sum_{j=1}^{i}\dfrac{\exp(-(\Lambda_j+\lambda_p)t)}{(\Lambda_0+\lambda_p)(\Lambda_0-\Lambda_j+\lambda_c)\prod_{\substack{p=1\\p\neq j}}^{i}(\Lambda_p-\Lambda_j)}\end{array}\right\}\right] \quad \ldots(13)$$

5. Numerical illustration

Numerical illustrations have been made to calculate the reliability, the state probabilities $p_i(t)$ and Expected operational time (EOT).

In Tables 1-2 by fixing n=7, k=3 and s=2, $\lambda_p = 0.003, \lambda_c = 0.007, \beta = .009$ $\lambda_7 = 0.01, \lambda_6 = 0.04, \lambda_5 = 0.07, \lambda_4 = 0.1, \lambda_3 = 0.13$, we have provided state probabilities for models 1 and 2 respectively. Comparison of reliabilities and EOT for models 1, 2 and no standby support model is made in Table (3).

TABLE 1: State probabilities $p_i(t)$ for standby support model 1

t	$p_0(t)$	$p_1(t)$	$p_2(t)$	$p_3(t)$	$p_4(t)$	$p_5(t)$	$p_6(t)$
5	0.612626	0.280631	0.057021	0.005517	0.001334	0.000386	0.000112
10	0.375311	0.358163	0.149760	0.024528	0.009789	0.004772	0.002440
15	0.229925	0.343019	0.221399	0.046850	0.023655	0.014745	0.009948
20	0.140858	0.292169	0.258779	0.063846	0.036924	0.026526	0.021090
25	0.086294	0.233427	0.266015	0.072672	0.045847	0.036053	0.031876
30	0.052866	0.179131	0.252176	0.074043	0.049582	0.041454	0.039411
35	0.032387	0.133716	0.226104	0.070031	0.048951	0.042750	0.042798
40	0.019841	0.097829	0.194661	0.062814	0.045337	0.040888	0.042519
45	0.012155	0.070493	0.162495	0.054155	0.040065	0.037029	0.039630
50	0.007447	0.050194	0.132398	0.045288	0.034165	0.032185	0.035225

TABLE 2: State probabilities $p_i(t)$ for standby support model 2

t	$p_0(t)$	$p_1(t)$	$p_2(t)$	$p_3(t)$	$p_4(t)$	$p_5(t)$	$p_6(t)$
5	0.612626	0.184343	0.087481	0.042962	0.019696	0.007481	0.000001
10	0.375311	0.162922	0.116881	0.092206	0.073197	0.050016	0.029564
15	0.229925	0.113366	0.094957	0.091041	0.092988	0.085468	0.070250
20	0.140858	0.073127	0.065474	0.068753	0.080029	0.086997	0.087324
25	0.086294	0.045796	0.042242	0.046326	0.057757	0.069167	0.078613
30	0.052866	0.028326	0.026480	0.029644	0.038286	0.048421	0.059393
35	0.032387	0.017427	0.016389	0.018524	0.024349	0.031718	0.040700
40	0.019841	0.010696	0.010086	0.011450	0.015179	0.020079	0.026428
45	0.012155	0.006558	0.006192	0.007043	0.009374	0.012495	0.016672
50	0.007447	0.004019	0.003797	0.004322	0.005764	0.007711	0.010361

TABLE 3: Comparison between reliabilities and EOT for model 1 & 2

t	Model 1	Model 2	No Standby support	Model 1	Model 2	No Standby support	t
10	0.924763	0.900098	0.844735	9.592439	9.516733	4.878931	10
20	0.840193	0.602562	0.501318	18.463829	17.235891	13.217659	20
30	0.688663	0.283416	0.243053	26.159641	21.586346	18.279684	30
40	0.503889	0.113760	0.111132	32.125542	23.373304	20.764330	40
50	0.336901	0.043420	0.050118	36.300350	24.185949	21.904411	50
60	0.210725	0.016358	0.022536	39.002159	24.470270	22.418961	60
70	0.125463	0.006144	0.010128	40.652267	24.549475	22.650356	70
80	0.071974	0.002306	0.004551	41.617420	24.579378	22.754353	80
90	0.040123	0.000866	0.002045	42.163567	24.589371	22.801092	90
100	0.021869	0.000325	0.000919	42.464954	24.593981	22.822086	100

Taking, $\lambda_p = 0.001, \lambda_c = 0.005, \beta = 0.009, \lambda_{10} = .01, \lambda_9 = .03, \lambda_8 = .05, \lambda_7 = .09$ the effect of varying parameters such as n, k and s on the reliability and expected operational time are displayed in Figures 1-4.

For model 1 and 2, figures 1 and 2 respectively show the effect of increased s, n, k on system reliability. We choose n=10,k=6 in figures 1a and 2a; k=6,s=2 in figure 1b and 2b;and n=10,s=2 in figures 1c and 2c for model 1 and 2 respectively. It is observed that with the increase in s and n, the system reliability increases.

In figure 3a and 4a for n=10, k=6; in figure 3b and 4b for s=2,k=6; and in figure 3c and 4c for n=10, s=2, the effect of varying s, n, k in expected operational time are displayed for model 1 and 2 respectively. With the increase in time the EOT first increases rapidly and then it becomes constant. With the increase in n and s the EOT of the system increases.

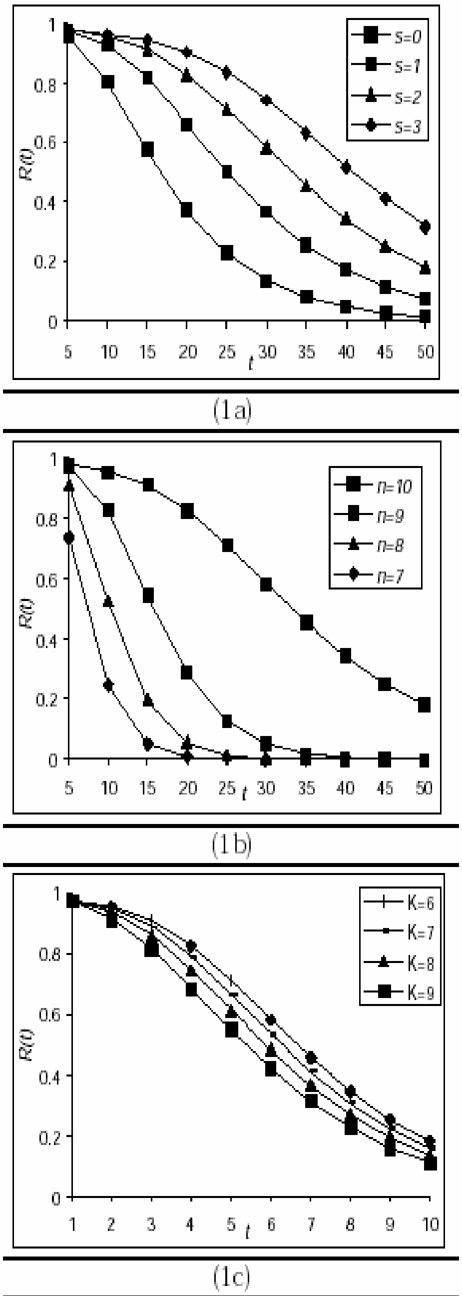

Figure 1: Reliability for standby model 1 by varying (a) s (b) n (c) k

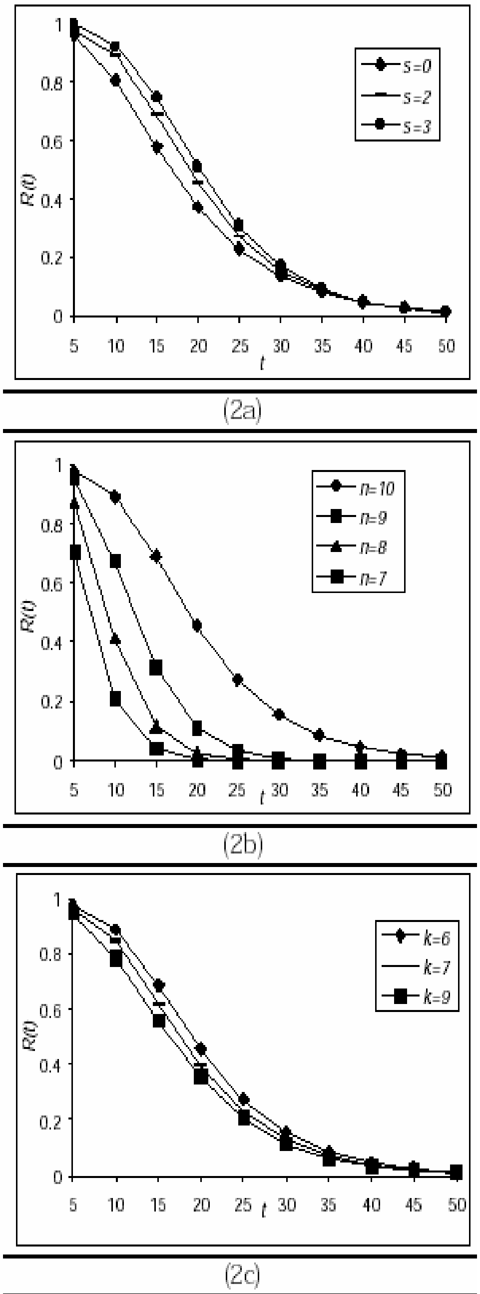

Figure 2: Reliability for standby model 2 by varying (a) s (b) n (c) k

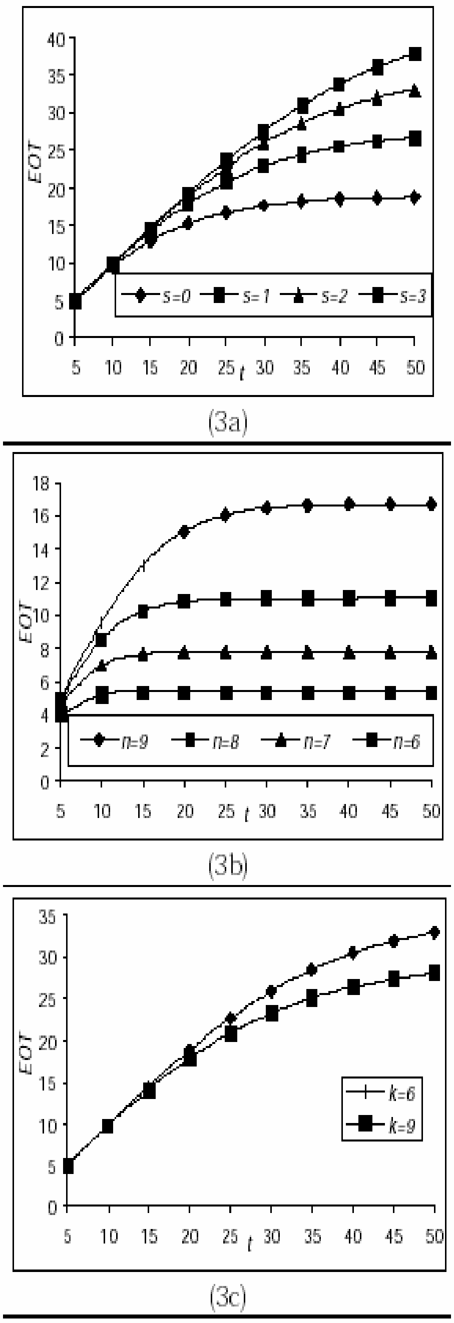

Figure 3: Expected operational time for standby model 1 by varying (a) s (b) n (c) k

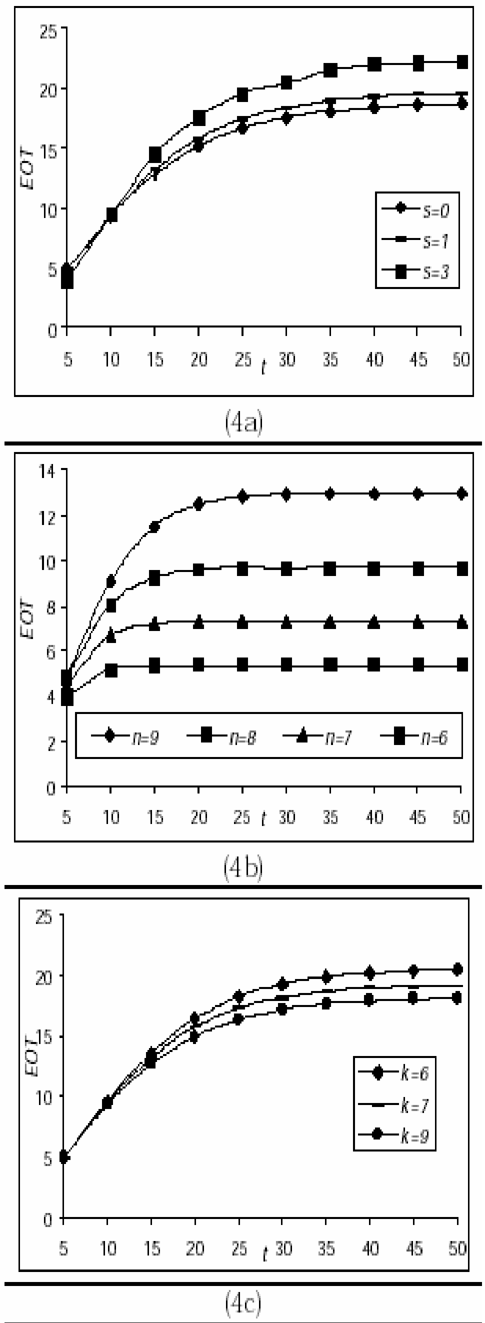

Figure 4: Expected operational time for standby model 2 by varying (a) s (b) n (c) k

6. Conclusions

To get the state probability for the k-out of - n system with dependent failures and standby support a Markov model has been used and the system reliability and expected operational times are obtained for the system with standby support. The incorporation of standby support may be helpful in improving the reliability and system performance to some extent. The sensitivity analysis provided gives insight in the choice of the right system design associated with a given technology.

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نظام "ك" من "ن" مع حالات فشل ودعم احتياطي يعتمد بعضها على بعض ومدعم بوحدات احتياطية

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المستخلص . درسنـا في ورقـة البحث هـذه نظامـا وافرا جزئيا (ك من بين ن) يتعرض لحالات فشل يعتمد بعضها على بعض وبه وحدات احتياطية . إن فشل أية وحدة عاملة يؤثر في معولية النظام . كما أن دعم النظام بوحدات احتياطيـة يزيد من هذه المعولية ، واستهدافا لتحسين درجة جودة الخدمة قمنا بدراسة خيارين للدعم الاحتياطي للنظام المضمحلة ، حيث قمنا باشتقاق تعبيرات للمعولية ولزمن التشغيل المتوقع . وللتحقق من صحة النتائج التحليلية ، وفرنا أمثلة توضيحية تتضمن نتائج عددية .