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A Canonical Matrix Representation of 2-D Linear Discrete Systems

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Abstract. In this paper, a matrix form analoguous to the companion matrix which is often encountered in the theory of one dimensional (1-D) linear control systems is suggested for a class of polynomials in two indeterminates and real coefficients, here referred to as two dimensional (2-D) polynomials. These polynomials arise in the context of the theory of 2-D linear discrete control systems. Necessary and sufficient conditions are also presented under which a matrix is equivalent to this companion form. Examples are used to illustrate the ideas developed in this paper.

1. Introduction

Canonical forms play an important role in the modern theory of linear control systems. One particular form that has proved to be very useful for 1-D linear systems is the so-called companion matrix which is associated with its characteristic polynomial. Barnett[3] showed that many of the concepts encountered in 1-D linear systems theory can be nicely linked via the companion matrix.

It is therefore worthwhile to seek a form of matrix which is associated with 2-D polynomials and which can play a role similar to that of its 1-D counterpart.

In this paper, a matrix form which can be regarded as a 2-D companion form is presented. The characteristic polynomial of the matrix is in the form which arises from 2-D linear first order discrete equations e.g., those describing 2-D image processing systems as suggested by Roesser[5]. The condition of equivalence to the Smith form, as given by Frost and Boudellioua[2], is used to obtain necessary and sufficient conditions for the equivalence of a matrix to the 2-D companion form.

2. Statement of the Problem

Let

$$d(z_1,z_2)=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} P(j,i)\, z_1^{n_1-i}\; z_2^{n_2-j} \tag{2.1}$$

be a polynomial in the indeterminates z_1 and z_2 and with real coefficients $P(j, i)$. The leading monomial of $d(z_1, z_2)$ has a coefficient equal to 1, i.e., $P(0, 0) = 1$ and has degress in z_1 and z_2 greater or equal to those of the remaining monomials of $d(z_1, z_2)$.

The problem is to find a $(n_1 + n_2) \times (n_1 + n_2)$ matrix (henceforth referred to as a companion matrix) in the block form:

$$F \equiv \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} \tag{2.2}$$

where F_1 is $n_1 \times n_1$, F_2 is $n_1 \times n_2$, F_3 is $n_2 \times n_1$ and F_4 is $n_2 \times n_2$ which has a form which is similar to the 1-D companion matrix and such that the determinant of the characteristic matrix:

$$zI_{n_1+n_2} - F \equiv \begin{pmatrix} z_1 I_{n_1} - F_1 & -F_2 \\ -F_3 & z_2 I_{n_2} - F_4 \end{pmatrix} \tag{2.3}$$

is given by the polynomial $d(z_1, z_2)$. Furthermore, it is required to determine the necessary and sufficient conditions for any matrix A, in the general block form of equation (2.2) to be equivalent to the companion form F.

The matrix F often presented in the literature as a 2-D companion form (see for example[6] is one in which F_1 and F_2 are in companion form but F_2 and F_3 have no special forms. In the following, a 2-D companion form is presented in which F_1 and F_4 are in companion form and moreover, F_2 is such that the overall matrix F, like its 1-D counterpart, has all the elements above the diagonal equal to zero except for the elements on the super diagonal which are equal to 1.

3. A Companion Form for 2-D Polynomials

Proposition A.

Given a 2-D polynomial $d(z_1, z_2)$ given by equation (2.1), then a 2-D companion matrix associated with $d(z_1, z_2)$ is given by:

$$F \equiv \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ f_1(n_1,1) & f_1(n_1,2) & \cdots & f_1(n_1,n_1) & 1 & 0 & \cdots & 0 \\ f_3(1,1) & f_3(n_1,2) & \cdots & f_3(1,n_1) & 0 & 1 & \cdots & 0 \\ f_3(2,1) & f_3(2,2) & \cdots & f_3(2,n_1) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ f_3(n_2-1,1) & f_3(n_2-1,2) & \cdots & f_3(n_3-1,n_1) & 0 & 0 & \cdots & 1 \\ f_3(n_2,1) & f_3(n_2,2) & \cdots & f_3(n_2,n_1) & f_4(n_2,1) & f_4(n_2,2) & \cdots & f_4(n_2,n_2) \end{pmatrix} \tag{3.1}$$

where F_1 and F_4 are $n_1 \times n_1$, $n_2 \times n_2$ matrices in companion forms respectively with last rows given respectively by:

$$f_1(n_1, i) = - P(0, n_1 - i + 1), i = 1, 2, ... , n_1,$$

and

$$f_4(n_2, j) = - P(n_2 - j + 1, 0) , j = 1,2, ... , n_2 \tag{3.2}$$

where

$$det(z_1 I_{n_1} - F_1) = z_1^{n_1} + P(0, 1) z_1^{n_1-1} + P(0, 2)z_1^{n_1-2} + \dots + P(0, n_1)$$

and

$$det(z_2 I_{n_2} - F_4) = z_2^{n_2} + P(1, 0) z_2^{n_2-1} + P(2, 0) z_2^{n_2-2} + ... + P(n_2, 0)$$

The matrix F_2 is $n_1 \times n_2$ and has all its columns zero except for the first one which is given by E_{n_1} i.e., the first column of the identity I_{n_1}. The elements of F_3 are uniquely and recursively determined from the following equation:

$$f_3(i,j) = P(i,0)P(0,n_1 - j + 1) - P(i,n_1 - j + 1) - \sum_{k=1}^{i-1} P(i,-k,0) f_3(k,j) \tag{3.3}$$

$$i = 1,2, \ ..., \ n_2, \qquad j = 1,2, \ ... \ ,n_1.$$

Furthermore if $d(z_1, z_2)$ is separable i.e., can be written as a product of two 1-D polynomials, then the matrix F_3 is taken to be the null matrix.

The proof of the proposition follows in a simple way by expanding the determinant of the matrix $zI_{n_1+n_2} - F$ and equating the result with the polynomial $d(z_1, z_2)$. A detailed proof is setout in[1]. Obviously, a similar form to F can be obtained based on the matrices F_1, F_2, F_3 and F_4 is such a way that the overall matrix obtained is the transposed matrix of F.

Example 1.

Let

$$d(z_1, z_2) = (z_2^2 + 2) z_1^2 + (z_2^2 + 3z_2 - 1) z_1 + z_2^2 + 2z_2 + 2.$$

Here we have,

$$n_1 = n_2 = 2,$$

$$P(0, 0) = 1, P(1,0) = 0, P (2, 0) = 2,$$

$$P(0, 1) = 1, P(1, 1) = 3, P(2, 1) = - 1,$$

$$P(0, 2) = 1, P(1, 2) = 2, P(2, 2) = 2.$$

It follows that:

$$f_1(2, 1) = - P(0, 2) = - 1, f_1(2, 2) = - P(0, 1) = - 1,$$

$$f_4(2, 1) = - P(2, 0) = - 2, f_4(2, 2) = - P(1, 0) = 0,$$

$$f_3(1, 1) = P(1, 0)\, P(0, 2) - P(1, 2) = 0 \times 1 - (2) = -2,$$

$$f_3(1, 2) = P(1, 0)\, P(0, 1) - P(1, 1) = 0 \times 1 - (3) = -3,$$

$$f_3(2, 1) = P(2, 0)\, P(0, 2) - P(2, 2) = 2 \times 1 - 2 = 0,$$

$$f_3(2, 2) = P(2, 0)\, P(0, 1) - P(2, 1) - P(1, 0)\, f_3(1, 2) = 2 \times 1 - (-1) - 0 \times (-3) = 3.$$

Therefore,

$F_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $F_3 = \begin{pmatrix} -2 & -3 \\ 0 & 3 \end{pmatrix}$, and $F_4 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$ and the overall matrix F is given by the following companion matrix:

4. Algebraic Equivalence

In 1-D systems theory, two $n \times n$ matrices A and B are algebraically equivalent (similar) if and only if their corresponding characteristic matrices $SI_n - A$ and $sI_n - B$ are equivalent i.e., there exist unimodular $n \times n$ matrices over the ring $R[s]$, $M(s)$ and $N(s)$ such that:

$$sI_n - B \;=\; M(s)\;(sI_n - A)\;N(s) \tag{4.1}$$

In fact, it can be shown (see for example[7]) that when it exists, this transformation can be reduced to a similarity transformation i.e.,

$$sI_n - B = M_0\,(sI_n - A)\,M_0^{-1} \tag{4.2}$$

In 2-D systems theory, however, this result is not true i.e., two matrices $zI_{n_1+n_2} - A$ and $zI_{n_1+n_2} - B$ in the form given by equation (2.3) may be equivalent over the ring $R[z_1, z_2]$ without implying that the matrices A and B being similar. In fact the similarity transformation used in the literature e.g.[5] and [7] which is a block diagonal transformation is only a special case of the general equivalence. In the following, a more general notion of algebraic equivalence is used.

Definition 1.

Two matrices A and B in the form of equation (2.2) are algebraically equivalent if their corresponding characteristic matrices $zI_{n_1+n_2} - A$ and $zI_{n_1+n_2} - B$ are equivalent over the ring $R[z_1 , z_2]$, i.e., there exist $(n_1 + n_2) \times (n_1 + n_2)$ unimodular matrices over $R[z_1, z_2]$, $M(z_1, z_2)$ and $N(z_1, z_2)$ such that:

$$zI_{n_1+n_2} - B \;=\; M(z_1 , z_2)\;(zI_{n_1+n_2} - A)\;N(z_1 , z_2) \tag{4.3}$$

Using this definition of algebraic equivalence, we now present a theorem that gives necessary and sufficient conditions under which a matrix A in the form of equation (2.2) is algebraically equivalent to the companion matrix F given by equation (3.1).

Theorem 1

A matrix A in the form of equation (2.2) is equivalent to the companion matrix F given by equation (3.1) if and only if its characteristic matrix $zI_{n_1+n_2} - A$ is equivalent over $R[z_1 , z_2]$ to the Smith form:

$$S(z_1, z_2) = \begin{pmatrix} I_{n_1+n_2-1} & 0 \\ 0 & det(zI_{n_1+n_2} - A \end{pmatrix} \quad (4.4)$$

Proof:

Necessity: Suppose that the matrix A is equivalent to the companion form F, then it is clear from the form of the matrix $zI_{n_1+n_2} - F$ that it can be brought by elementary row and column operations to the Smith form $S(z_1, z_2)$ given in equation (4.4). In follows that the matrix $zI_{n_1+n_2} - A$ is equivalent to the Smith form $S(z_1, z_2)$.

Sufficiency: Suppose that the matrix $zI_{n_1+n_2} - A$ is equivalent to the Smith form $S(z_1, z_2)$. By Proposition 1, there exists a companion form F associated with the characteristic polynomial given by $det(zI_{n_1+n_2} - A)$. Now since both $zI_{n_1+n_2} - A$ and $zI_{n_1+n_2} - F$ are equivalent to the same Smith form $S(z_1, z_2)$, they are equivalent to each other i.e., the matrices A and F are algebraically equivalent.

Theorem 2.

A matrix A in the form of equation (2.2) is equivalent to the companion matrix F given by equation (3.1) if and only if there exists $(n_1 + n_2)$ column vector $b(z_1, z_2)$ which has no zeros such that the matrix:

$$(zI_{n_1+n_2} - A \quad b(z_1, z_2))$$

has no zeros.

The definition of a zero of a matrix over $R[z_1, z_2]$ is the value of the complex pair (z_1, z_2) such that the matrix is rank deficient, see for example[4].

Proof:

Necessity: Suppose that the matrix A is equivalent to the companion form F, then there exist $(n_1 + n_2) \times (n_1 + n_2)$ unimodular matrices over $R[z_1, z_2]$, $M(z_1, z_2)$ and $N(z_1, z_2)$ such that:

$$zI_{n_1+n_2} - A = M(z_1, z_2)\ (zI_{n_1+n_2} - F)\ N(z_1, z_2) \quad (4.5)$$

it follows that:

$$M(z_1, z_2)\ (zI_{n_1+n_2} - F \quad E_{n_1+n_2})\ N(z_1, z_2) = (zI_{n_1+n_2} - A \quad b(z_1, z_2)) \quad (4.6)$$

It is clear that the matrix $(zI_{n_1+n_2} - F \quad E_{n_1+n_2})$ has no zeros since it has one highest order minor equal to 1. Therefore the matrix $(zI_{n_1+n_2} - A \quad b(z_1, z_2))$ has also no zeros. It remains to prove that the vector $b(z_1, z_2)$ has no zeros. This follows from the fact that $b(z_1, z_2) = M(z_1, z_2)\ E_{n_1+n_2}$.

Sufficiency: Suppose that there exists a $(n_1 + n_2)$ column vector $b(z_1, z_2)$ which has no zeros such that the matrix $(zI_{n_1+n_2} - A \quad b(z_1, z_2))$ has also no zeros. Then, since the vector $b(z_1, z_2)$ has no zeros, there exists a $(n_1 + n_2) \times (n_1 + n_2)$ unimodular matrix $M_1(z_1, z_2)$ over $R[z_1, z_2]$ such that:

$$M_1(z_1, z_2)\ b(z_1, z_2) = E_{n_1+n_2} \quad (4.7)$$

i.e.,

$$M_1(z_1,z_2)\ (zI_{n_1+n_2} - A\ \ b(z_1,z_2)) = \begin{pmatrix} T_1(z_1,z_2) & 0 \\ T_2(z_1,z_2) & 1 \end{pmatrix} \tag{4.8}$$

where $T_1(z_1, z_2)$, $T_2(z_1, z_2)$ are $(n_1 + n_2 - 1) \times (n_1 + n_2)$ and $1 \times (n_1 + n_2)$ polynomial matrices respectively. Now since the matrix on the RHS of equation (4.8) has no zeros, the matrix $T_1(z_1, z_2)$ must also have no zeros. Therefore there exists a unimodular $(n_1 + n_2) \times (n_1 + n_2)$ matrix $N(z_1, z_2)$ such that:

$$T_1(z_1, z_2)\, N(z_1, z_2) = (I_{n_1+n_2}\ \ 0)$$

i.e.,

$$M_1(z_1,z_2)\ (zI_{n_1+n_2} - A\ \ b(z_1,z_2)) = \begin{pmatrix} N(z_1,z_2) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{n_1+n_2-1} & 0 & 0 \\ T_3(z_1,z_2) & t_4(z_1,z_2) & 1 \end{pmatrix} \tag{4.9}$$

Premultiplying the matrix on the RHS of equation (4.9) by the $(n_1 + n_2) \times (n_1 + n_2)$ unimodular matrix:

$$M_2(z_1,z_2) = \begin{pmatrix} I_{n_1+n_2} & 0 & 0 \\ -T_3(z_1,z_2) & t_4(z_1,z_2) & 1 \end{pmatrix}$$

yields the matrix:

$$\begin{pmatrix} I_{n_1+n_2} & 0 & 0 \\ 0 & t_4(z_1,z_2) & 1 \end{pmatrix} \tag{4.10}$$

where $t_4(z_1, z_2) = \lambda.det(zI_{n_1+n_2} - A)$, $\lambda \in R^*$. It follows that the matrices $zI_{n_1+n_2} - A$ and $S(z_1, z_2)$ are related by the following unimodular transformation:

$$M_2(z_1,z_2)\ \ M_1(z_1,z_2) \begin{pmatrix} I_{n_1+n_2-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} (zI_{n_1+z_2} - A)\ \ N(z_1,z_2) = S(z_1,z_2) \tag{4.11}$$

i.e., the matrix is $zI_{n_1+n_2} - A$ equivalent to its Smith form $S(z_1, z_2)$. Therefore, by Theorem 1, the matrix A is algebraically equivalent to the companion form F. This completes the proof.

Example 2.

Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 6 & 1 & -2 \\ 2 & 1 & 2 \end{pmatrix}$, then it can be easily verified that the vector $b = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ satisfies the conditions in Theorem 2. Furthermore here we have $det(zI_3 - A) = (z_1^2 - z_1 - 6)(z_2 - 2)$ i.e., the determinant is separable. In fact by premultiplying the matrix $zI_3 - A$ by the unimodular matrix:

$$\begin{pmatrix} -3 & 1 & -1 \\ 3z_1-6 & -z_1+2 & z_1-1 \\ -3z_1z_2+6z_1+12z_2-34 & z_1z_2-2z_1-4z_2+3 & -5z_1+15 \end{pmatrix}$$

and postmultiplying it by the unimodular matrix:

$$\frac{1}{25}\begin{pmatrix} z_1^2z_2-7z_1^2-2z_1z_2+14z_1-8z_2+41 & -5 & z_2-7 \\ z_1^2z_2-7z_1^2-2z_1z_2+14z_1-8z_2+36 & -15 & -3z_2+21 \\ 5z_1^2-10z_1-40 & 0 & -5 \end{pmatrix},$$

yields the characteristic matrix:

$$zI_3-F\equiv\begin{pmatrix} z_1 & -1 & 0 \\ -6 & z_1-1 & -1 \\ 0 & 0 & z_2-2 \end{pmatrix}$$

corresponding to the companion form:

$$F\equiv\begin{pmatrix} 0 & 1 & 0 \\ 6 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Notice that because the determinant of the matrix $zI_3 - A$ is separable, the matrix F_3 is zero.

Conclusion

In this paper, a canonical form which may be considered as a 2-D companion matrix is presented. By introducing a more general notion of equivalence, some of the conditions of equivalence to the companion form existing in 1-D systems theory are extended to the 2-D case. This work can be taken further by considering the usefulness of this 2-D companion matrix in the solution of problems such as realisation, controllability, observability, pole assignability, etc. of 2-D linear systems.

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تمثيـل الأنظمـة المتقطعـة ثنائيـة البعـد بواسطـة مصفوفـة في الشـكل القـانوني

محمد الصالح بودليوة
الكلية التنقية بالأحساء ، الهفوف – المملكة العربية السعودية

المستخلص . في هذا البحث نقترح شكلاً لمصفوفة تعتبر إمتداداً للمصفوفة المرافقة الشائعة في نظرية أنظمة التحكم الخطية وحيدة البعد . شكل المصفوفة هذا يرافق قسمًا من كثيرات الحدود ذات متغيرين ومعاملات حقيقية . نطلق في هذا البحث تسمية كثيرات الحدود ثنائية البعد على هذا القسم من كثيرات الحدود التي تستعمل في نظرية أنظمة التحكم الرقمية ثنائية البعد . كما يتم تقديم الشروط الضرورية والكافية لكون مصفوفة ما متكافئة مع المصفوفة المرافقة المقترحة . كما تحتوي ورقة البحث أمثلة توضح الأفكار التي طرحت .