

Symbolic Computing Algorithm for One-Point Iteration Formulae of Integral Orders for the Solution of $y(x) = 0$

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ABSTRACT. Symbolic computing algorithm for a class of one-point iteration formulae of integral orders was established to solve $y(x) = 0$. Mathematica software was used to obtain literal analytical expressions of iterative schemes of some orders together with their error formulae as illustration of the symbolic implementation of the algorithm. A numerical application is also given.

Introduction

One of the most basic problems of numerical approximation is the root-finding problem, which involves finding a root x of an equation of the form

$$y(x) = 0 \quad (1.1)$$

This is one of the oldest known approximation problems, towards its research continues in this area at present time.

This problem is usually solved by iterative methods, which in turn need (a) an iterative scheme, and (b) starting value. In fact, these two points are not separated from each other, but there is a full agreement that, even accurate iterative schemes are extremely sensitive to starting value. Moreover, in many cases the starting value leads to drastic situations between divergent and very slow convergent solutions.

Since 1990^[1], in the field of numerical analysis, very powerful techniques have been devoted up to now; to solve transcendental equation(s) without any prior knowledge of the starting value. These techniques are known today as continuation methods, in which one defines a homotopy or deformation and at-

tempt to trace an implicitly parameterized curve(s), so that the problem reduces to a parametric optimization which could be solved by embedding methods.

What concerns the present paper is the second point that is of the iterative scheme. Before constructing an iterative scheme for solving Equation (1.1), some basic assumptions and definitions are to be recalled as follows.

Assumptions

Let $y(x) = 0$ has a solution $x = \alpha$. We assume that there is an interval I containing the point $x = \alpha$ such that

- $y'(x) \neq 0$, $\forall x \in I$
- y is differentiable in I as many as we need,
- All the arguments of $y(x)$ lie in I .

Definitions

- The error in the n^{th} iteration is defined as

$$\varepsilon_n = \alpha - x_n$$

- If the sequence $\{x_n\}$, converges to $x = \alpha$, then

$$\lim_{n \rightarrow \infty} x_n = a$$

- If there exists a real number p with $p \geq 1$ such that

$$\lim_{i \rightarrow \infty} \frac{|x_{i+1} - \alpha|}{|x_i - \alpha|^p} = \lim_{i \rightarrow \infty} \frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} = K \quad \text{with } K \neq 0$$

we say that the convergence of the method is of order p at α . The constant K is called the asymptotic error constant and it depends on $y(x)$, for $p = 1$ the convergence is linear, for $p = 2$, is quadratic, for $p = 3, 4, 5$ is cubic, quartic and quintic respectively.

- One-point iteration formulae are those which use information at only one point.

In the present paper we shall consider stationary one-point iteration formulae which have the form

$$x_{i+1} = F(x_i) \quad \text{with } i = 0, 1, \dots \quad (1.2)$$

Theorem

It could be shown that the order of any one-point iteration function $F(x)$ is a positive integer^[2]. More specifically $F(x)$ has order p if and only if $F(\alpha) = \alpha$;

and

$$F^{(j)}(\alpha) \equiv 0, \quad 1 \leq j < p \quad \text{and} \quad F^{(p)}(\alpha) \neq 0.$$

An iterative scheme for solving $y(x) = 0$ includes derivatives of y as much as the order of the scheme (Equation 2.14). In fact, as it is clear from the above definitions, the higher the order of an iterative scheme, the higher its accuracy and rate of convergence will be. Consequently, higher accuracy needs iterative schemes containing higher derivatives. In the near past, this was very difficult or even impossible to include in an iterative scheme more than the third derivative. Today, due to the existing symbolic software such as Mathematica, one can generate derivatives of any order as desired whatever the function is.

Due to the importance of root finding problem with high accuracy and the existing symbols used for manipulating digital computer programs I tried to establish symbolic computing algorithm for a class of one-point iteration formulae of integral orders to solve $y(x) = 0$.

Formulation of the Method

This section is devoted for developing a class of one-point iteration formulae referred to in the first section. To do so, we shall make use of series reversion algorithm.

Series Reversion Algorithm

Consider the functional equation

$$\eta = \zeta + \beta \phi(\eta); \quad |\beta| < 1, \quad (2.1)$$

Then according to Lagrange expansion theorem^[3], we have

$$\eta = \zeta + \sum_{n=1}^{\infty} \frac{\beta^n}{n!} \frac{d^{n-1}}{d\zeta^{n-1}} [\phi(\zeta)]^n \quad (2.2)$$

Let $y(x)$ be a function which can be expanded in a Taylor series in the neighborhood of $x = x_0$, as

$$y(x) = y_0 + \sum_{j=1}^{\infty} \frac{B_j}{j!} (x - x_0)^j \quad (2.3)$$

where

$$B_j = \frac{d^j y(x)}{dx^j} \Big|_{x=x_0}$$

In the following, we assume that $B_1 \neq 0$ and write Equation (2.3) in the form

$$x = x_0 + (y - y_0) \phi(x), \quad (2.4)$$

where $\phi(x)$ is defined by

$$\phi(x) = \frac{1}{B_1 + \sum_{j=1}^{\infty} \frac{B_{j+1}}{(j+1)!} (x - x_0)^j} \quad (2.5)$$

Since Equation (2.4) is of the form (2.1), we can express x as a power series in $\alpha = y - y_0$ to get

$$x(y) = x_0 + \sum_{n=1}^{\infty} \frac{C_n}{n!} (y - y_0)^n \quad (2.6)$$

where

$$C_n = \frac{d^{n-1}}{dx^{n-1}} [\phi(x)]_{x=x_0}^n \quad (2.7)$$

And $\phi(x)$ is defined by Equation (2.5). The series for $x(y)$ is said to be the reverse of the series for $y(x)$.

Battin^[4] developed an elegant algorithm to express n of the coefficients $C_1, C_2, \dots$ of the reversed series in terms of the coefficients $B_1, B_2, \dots$ of the original series. The basic equations of this algorithm are

$$D_0^1 \Big|_{x=x_0} = \phi^{(0)}(x_0) \equiv \phi_0^{(0)} = \frac{1}{B_1} \quad (2.8.1)$$

$$\frac{d^k \phi(x)}{dx^k} \equiv \phi_0^{(k)} = -\frac{1}{B_1} \sum_{i=1}^k \frac{1}{i+1} \binom{k}{i} B_{i+1} \phi_0^{(k-i)}; \quad k=1, 2, \dots, n-1 \quad (2.8.2)$$

$$D_k^n = \frac{d^k}{dx^k} [\phi(x)]^n = n \sum_{j=0}^{k-1} \binom{k-1}{j} D_j^{n-1} \phi_0^{(k-j)} \quad (2.8.3)$$

$$C_n = D_{n-1}^n, \quad x = x_0 \quad (2.8.4)$$

One-Point Iteration Formulae

Consider the equation

$$y(x) = 0 \quad (2.9)$$

Recalling Equation (2.6) with remainder as

$$x(y) = x_0 + \sum_{j=1}^{m+1} \frac{(y - y_0)^j}{j!} C_j(y_0) + \frac{(y - y_0)^{m+2}}{(m+2)!} C_{m+2}(\zeta) \quad (2.10)$$

Where ζ is between y and y_0 . If α is a root of Equation (2.9), then Equation (2.10) with $x = \alpha$ and $y = 0$ gives

$$\alpha = x_0 + \sum_{j=1}^{m+1} (-1)^j \frac{y_0^j}{j!} C_j(y_0) + \frac{(-1)^{m+2} y_0^{m+2}}{(m+2)!} C_{m+2}(\zeta) \quad (2.11)$$

Therefore, if y_0 is a relatively small quantity (or, equivalently, x_0 is reasonably close to α) and the coefficients $C_1, C_2, \dots$, are “well behaved”, then Equation (2.11) will provide an approximation to the true value of the root α .

Equation (2.11) can be used recursively in the form

$$\alpha = x_i + \sum_{j=1}^{m+1} \frac{(-1)^j}{j!} y_i^j C_j(y_i) + \frac{(-1)^{m+2} y_i^{m+2}}{(m+2)!} C_{m+2}(\zeta) \quad (2.12)$$

from which it is deduced the iterative formula

$$x_{i+1} = x_i + \sum_{j=1}^{m+1} \frac{(-1)^j}{j!} y_i^j C_j(y_i) \quad \text{with } i=0,1,\dots \quad (2.13)$$

Subtracting this Equation (2.13) from Equation (2.12), we get

$$\varepsilon_{i+1} = \frac{(-1)^{m+2}}{(m+2)!} y_i^{m+2} C_{m+2}(\zeta)$$

Since

$$y_i = y(x_i) = y(\alpha - \varepsilon_i) = y(\alpha) - \varepsilon_i y'(\zeta_1) = -\varepsilon_i y'(\zeta_1)$$

where ζ_1 between x_i and α , we have

$$\varepsilon_{i+1} = \frac{1}{(m+2)!} \{ [y'(\zeta_1)]^{m+2} C_{m+2}(\zeta) \} \varepsilon_i^{m+2} \quad (2.14)$$

Equations (2.13) and (2.14) are what we required to set up for the one-point iteration formulae for solving $y(x) = 0$.

Note that, if the root is simple, the term in brackets in Equation (2.14) is bounded in some neighborhood of α . Therefore the order of the iterative formula (2.13) is $m + 2$, and if the starting value is sufficiently good, the iterative method will converge.

Although any of the formulae of a given order is directly obtained as particular case of Equation (2.13), the applications of the corresponding error formulae (Equation 2.14) need some explanations as illustrated by the following example:

For the case $m = 0$ we get from Equations (2.13), (2.14) and (2.8) that

$$x_{i+1} = x_i - \frac{y_i}{y'_i} \quad (2.15)$$

and

$$\varepsilon_{i+1} = - \frac{y''(\zeta)[y'(\zeta_1)]^2}{2[y'(\zeta)]^3} \varepsilon_i^2 \quad (2.16)$$

Now, using Taylor series expansion of $y(\alpha)$ about x_i as

$$y(\alpha) = y(x_i + \varepsilon_i) = y(x_i) + \varepsilon_i y'(x_i) + \frac{1}{2} \varepsilon_i^2 y''(\zeta) = 0$$

dividing by y'_i and then using Equation (2.15) we obtain

$$\varepsilon_{i+1} = - \frac{1}{2} \varepsilon_i^2 \frac{y''(\zeta)}{y'_i} \quad (2.17)$$

From Equation (2.17) it is clear that terms in $y'(x)$ in Equation (2.16) can be cancelled as if $\zeta = \zeta_1 = x_i$. So the error for the familiar Newton-Raphson iterative method (Equation 2.15) is that given by Equation (2.17). This result equally applied well for any iterative formula of order p (say) by putting in the corresponding error formula the x_i for the arguments of $y^{(i)}$, for every $j = 0, 1, 2, \dots, p-1$ and putting ζ as the argument of $y^{(p)}$.

For the case $m = 1$ we get

$$x_{i+1} = x_i - y_i c_1 + \frac{1}{2} y_i^2 c_2 = x_i - \frac{y_i}{y'_i(1)} \left[1 + \frac{1}{2} \frac{y_i y_i^{(2)}}{(y'_i(1))^2} \right] \quad (2.18)$$

and according to the above note we have for the error term the expression;

$$\varepsilon_{i+1} = \left\{ \frac{1}{2} \left(\frac{y_i^{(2)}}{y'_i(1)} \right)^2 - \frac{y_i^3(\zeta)}{6 y_i^{(1)}} \right\} \varepsilon_i^3 \quad (2.19)$$

Let us write Equation (2.18) as

$$x_{i+1} = x_i - \frac{y_i}{y'_i(1)} \frac{1}{\left[1 + \frac{1}{2} \frac{y_i y_i^{(2)}}{(y'_i(1))^2} \right]^{-1}} \approx x_i - \frac{y_i}{y'_i(1) - \frac{1}{2} y_i y_i^{(2)} / y'_i(1)}$$

which is the known Halley's formulae^[2].

Finally, it should be added that, an iterative formula of an integral order p (say) could be written in a suitable recurrence formula as

$$x_{i+1} = x_i + \delta_{i,n} \quad ; \quad i = 0, 1, \dots, n = 2, 3, \dots, p \quad (2.20)$$

where

$$\delta_{i,n} = \delta_{i,n-1} + T_n \quad ; \quad n \geq 3 \quad (2.21)$$

$$\delta_{i,2} = -\frac{y_i}{y_i'} \quad , \quad (2.22)$$

$$T_n = \frac{(-1)^{n-1}}{(n-1)!} y_i^{n-1} C_{n-1} \quad (2.23)$$

Literal analytical expressions of T_n for iterative formulae of orders $n = 3, 4, 8$ are listed in Appendix 1, while in Appendix 2, the corresponding error formulae of Equation (2.14) are given.

Numerical Application

To illustrate the above formulations we consider the numerical solution of the fundamental equation of space dynamics known as Kepler's equation

$$y(E) = E - e \sin E - M = 0 \quad (3.1)$$

This equation allows us to determine the relation of the time and angular displacement within an elliptic orbit.

Kepler's equation is transcendental equation in the eccentric anomaly E , assuming that the mean anomaly M and the eccentricity ($0 \leq e < 1$) are given. It is known that E considered as a function of M is convex up and monotonically increasing, also Kepler's Equation (3.1) always has one and only one solution.

In the present section we shall refine the initial guess

$$E_0 = M \quad (3.2)$$

for the root of Kepler's Equation (3.1) by using two iteration schemes $p = 2, 3$ of Equation (2.20). These schemes are given explicitly by Equations (2.15) (2.18) respectively. Now using Equation (3.1) in these two equations, we get for the second order method ($p = 2$) the scheme

$$E_{i+1} = E_i + \frac{M - M_i}{1 - e \cos E_i} \quad , \quad i = 0, 1, 2, \dots \quad (3.3)$$

while for the third order method ($p = 3$) we get the scheme

$$E_{i+1} = E_i + \frac{M - M_i}{1 - e \cos E_i} - \frac{e \sin E_i}{2(1 - e \cos E_i)} \left(\frac{M - M_i}{1 - e \cos E_i} \right)^2 \quad , \quad i = 0, 1, \dots \quad (3.4)$$

where

$$M_i = E_i - e \sin E_i \quad (3.5)$$

The numerical results of the two schemes for the solution of Equation (3.1) are listed in Table 1 for different values of M and e .

TABLE 1. Solution of Kepler's equation using the second and third order methods.

Case 1: $M = 30^0$, $e = 0.5$

I	$E^0(p = 2)$	$E^0(p = 3)$	$y[E^{(2)}]$	$y[E^{(3)}]$
1	55.2632553252501500	52.8074571433474900	0.461158903131190	−.0003715901360579
2	52.8565495128700100	52.8270871676466000	.0005577128438041	−.0000000000039588
3	52.8270914920880000	52.8270871678557400	.0000000818563447	.0000000000000010
4	52.8270871678558300	52.8270871678557400	.0000000000000017	.0000000000000010
5	52.8270871678557400	52.8270871678557400	.0000000000000010	.0000000000000010

Case 2: $M = 30^0$, $e = 0.7$

I	$E^0(p = 2)$	$E^0(p = 3)$	$y[E^{(2)}]$	$y[E^{(3)}]$
1	80.9254149780700800	60.8100244089464500	.2098638174093505	−.0908684461105215
2	68.2001384880060800	66.8629123182475500	.0196163951869200	−.0003756129964443
3	66.9011901515155400	66.8880363324020100	.0001966542633666	−.000000000239091
4	66.8880376742279000	66.8880363340012400	.0000000200368666	.0000000000000000
5	66.8880363340012600	66.8880363340012400	.0000000000000002	.0000000000000000

Case 3: $M = 60^0$, $e = 0.2$

I	$E^0(p = 2)$	$E^0(p = 3)$	$y[E^{(2)}]$	$y[E^{(3)}]$
1	71.0265779084358500	70.8223820212425900	.0028703990281810	−.0000127753679361
2	70.8233597398489400	70.8232868147900900	.0000010296762848	.0000000000000000
3	70.8232868147994700	70.8232868147900900	.0000000000001324	.0000000000000000
4	70.8232868147900900	70.8232868147900900	.0000000000000000	.0000000000000000
5	70.8232868147900900	70.8232868147900900	.0000000000000000	.0000000000000000

Case 4: $M = 60^0$, $e = 0.5$

I	$E^0(p = 2)$	$E^0(p = 3)$	$y[E^{(2)}]$	$y[E^{(3)}]$
1	93.0797337253075300	87.5664447710896100	.0500894099201412	−0.121093750671416
2	88.7235343898815400	88.6397695857317100	.0009444606755318	−0.000005413162159
3	88.6398484716030000	88.6398175679023400	.0000003487914860	.0000000000000000
4	88.6398175679065600	88.6398175679023400	.0000000000000476	.0000000000000000
5	88.6398175679023400	88.6398175679023400	.0000000000000000	.0000000000000000

Case 5: $M = 60^0$, $e = 0.8$

I	$E^0(p=2)$	$E^0(p=3)$	$y[E^{(2)}]$	$y[E^{(3)}]$
1	126.1594674506151000	82.0531558168716800	.2084570192812542	– .2140287676793484
2	106.3554996746224000	102.9440588572355000	.0187586609088529	– .0139188676582960
3	104.4181843862673000	104.3969198965817000	.0002014942840806	– .0000021941298076
4	104.3971514524429000	104.3971489574781000	.0000000238987831	.0000000000000000
5	104.3971489574781000	104.3971489574781000	.0000000000000004	.0000000000000000

Case 6: $M = 75^0$, $e = 0.2$

I	$E^0(p=2)$	$E^0(p=3)$	$y[E^{(2)}]$	$y[E^{(3)}]$
1	55.2632553252501500	52.8074571433474900	.0461158903131190	– .0003715901360579
2	52.8565495128700100	52.827087167646600	.0005577128438041	– .0000000000039588
3	52.8270914920880000	52.8270871678557400	.0000000818563447	.0000000000000000
4	52.8270871678557400	52.8270871678557400	.0000000000000017	.0000000000000000
5	52.8270871678557400	52.8270871678557400	.0000000000000000	.0000000000000000

Finally, it should be noted that, in going from a program using the second order method ($p = 2$) to one using the third order method ($p = 3$), only one additional instruction is needed. Also from Table 1, it is clear that, once the convergence is well established, the number of correct figures is doubled for successive iterates in the second order method, and tripartite in the third order method.

All of the above mentioned facts are of course valid for higher order methods $p \geq 3$. So one may now find a solution of $y(x) = 0$ of ten or even more digit accuracy within at most two iterations by using higher order method.

References

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Appendix 1

Literal analytical expressions of T_n for iterative schemes of orders $n = 3, 4, \dots, 8$

$$\begin{aligned}
 T_3 &= -\frac{y_i^2 y_i^{(2)}}{2(y_i^{(0)})^3}, \\
 T_4 &= +\frac{y_i^3 y_i^{(3)}}{6(y_i^{(0)})^4} - \frac{y_i^3 (y_i^{(2)})^2}{2(y_i^{(2)})^5}, \\
 T_5 &= -\frac{y_i^4 y_i^{(4)}}{24(y_i^{(0)})^5} + \frac{5}{12} - \frac{y_i^4 y_i^{(2)} y_i^{(3)}}{(y_i^{(0)})^6} - \frac{5}{8} \frac{y_i^4 (y_i^{(2)})^3}{(y_i^{(0)})^7}, \\
 T_6 &= +\frac{y_i^5 y_i^{(5)}}{120(y_i^{(0)})^6} - \frac{5}{8} \frac{y_i^4 (y_i^{(2)})^3}{(y_i^{(0)})^7} - \frac{y_i^5 y_i^{(2)} y_i^{(4)}}{8(y_i^{(0)})^7} - \frac{y_i^5 (y_i^{(3)})^2}{12(y_i^{(0)})^7} + \frac{7}{8} \frac{y_i^5 (y_i^{(2)})^2 y_i^{(3)}}{(y_i^{(0)})^8} - \\
 &\quad \frac{7}{8} \frac{y_i^5 (y_i^{(2)})^4}{(y_i^{(0)})^9}, \\
 T_7 &= -\frac{y_i^7 y_i^{(5)}}{270(y_i^{(0)})^7} + \frac{7}{144} \frac{y_i^6 y_i^{(3)} y_i^{(4)}}{(y_i^{(0)})^8} + \frac{7}{240} \frac{y_i^6 y_i^{(2)} y_i^{(5)}}{(y_i^{(0)})^8} - \frac{7}{18} \frac{y_i^6 y_i^{(2)} (y_i^{(3)})^2}{(y_i^{(0)})^9} - \\
 &\quad \frac{7}{24} \frac{y_i^6 (y_i^{(2)})^2 y_i^{(4)}}{(y_i^{(0)})^9} + \frac{7}{4} \frac{y_i^6 (y_i^{(2)})^3 y_i^{(3)}}{(y_i^{(0)})^{10}} - \frac{21}{16} \frac{y_i^6 (y_i^{(2)})^5}{(y_i^{(0)})^{11}}, \\
 T_8 &= +\frac{1}{5040} \frac{y_i^7 y_i^{(7)}}{(y_i^{(0)})^8} - \frac{1}{90} \frac{y_i^7 y_i^{(3)} y_i^{(5)}}{(y_i^{(0)})^9} - \frac{1}{144} \frac{y_i^7 (y_i^{(4)})^2}{(y_i^{(0)})^9} - \frac{1}{180} \frac{y_i^7 y_i^{(2)} y_i^{(6)}}{(y_i^{(0)})^9} + \\
 &\quad \frac{1}{4} \frac{y_i^7 y_i^{(2)} y_i^{(3)} y_i^{(4)}}{(y_i^{(0)})^{10}} + \frac{1}{18} \frac{y_i^7 (y_i^{(3)})^3}{(y_i^{(0)})^{10}} + \frac{3}{40} \frac{y_i^7 (y_i^{(2)})^2 y_i^{(5)}}{(y_i^{(0)})^{10}} - \\
 &\quad \frac{5}{4} \frac{y_i^7 (y_i^{(2)})^2 (y_i^{(3)})^3}{(y_i^{(0)})^{11}} - \frac{5}{8} \frac{y_i^7 (y_i^{(2)})^3 y_i^{(4)}}{(y_i^{(0)})^{11}} + \frac{55}{16} \frac{y_i^7 (y_i^{(2)})^4 y_i^{(3)}}{(y_i^{(0)})^{12}} - \frac{33}{16} \frac{y_i^6 (y_i^{(2)})^6}{(y_i^{(0)})^{13}}
 \end{aligned}$$

Appendix 2

Literal analytical expressions of the error formulae of iterative schemes of orders $n = 2, 3, \dots, 8$

Order: 2

$$\varepsilon_{i+1} = -\frac{1}{2} \frac{y_i^{(2)}(\zeta)}{y_i^{(0)}} \varepsilon_i^2$$

Order: 3

$$\varepsilon_{i+1} = \left\{ \frac{1}{2} \left(\frac{y_i^{(2)}}{y_i^{(0)}} \right)^2 - \frac{y_i^{(3)}(\zeta)}{6y_i^{(0)}} \right\} \varepsilon_i^3$$

Order: 4

$$\varepsilon_{i+1} = - \left\{ \frac{5}{8} \left(\frac{y_i^{(2)}}{y_i^{(1)}} \right)^3 - \frac{5}{12} \frac{y_i^{(2)} y_i^{(3)}}{(y_i^{(1)})^2} + \frac{y_i^{(4)}(\zeta)}{24 y_i^{(1)}} \right\} \varepsilon_i^4$$

Order: 5

$$\varepsilon_{i+1} = \left\{ \frac{7}{8} \left(\frac{y_i^{(2)}}{y_i^{(1)}} \right)^4 - \frac{7}{8} \frac{(y_i^{(2)})^2 y_i^{(3)}}{(y_i^{(1)})^3} + \frac{1}{12} \left(\frac{y_i^{(3)}}{y_i^{(1)}} \right)^2 + \frac{y_i^{(2)} y_i^{(4)}}{8 (y_i^{(1)})^2} - \frac{y_i^{(5)}(\zeta)}{120 y_i^{(1)}} \right\} \varepsilon_i^5$$

Order: 6

$$\varepsilon_{i+1} = - \left\{ \frac{21}{61} \left(\frac{y_i^{(2)}}{y_i^{(1)}} \right)^5 - \frac{7}{4} \frac{(y_i^{(2)})^3 y_i^{(3)}}{(y_i^{(1)})^4} + \frac{7}{18} \frac{y_i^{(2)} (y_i^{(3)})^2}{(y_i^{(1)})^3} + \frac{7}{24} \frac{(y_i^{(2)})^2 y_i^{(4)}}{(y_i^{(1)})^3} + \right. \\ \left. \frac{7}{144} \frac{y_i^{(3)} y_i^{(4)}}{(y_i^{(1)})^2} - \frac{7}{240} \frac{y_i^{(2)} y_i^{(5)}}{(y_i^{(1)})^2} + \frac{y_i^{(6)}(\zeta)}{720 y_i^{(1)}} \right\}$$

Order: 7

$$\varepsilon_{i+1} = \left\{ \frac{33}{16} \left(\frac{y_i^{(2)}}{y_i^{(1)}} \right)^6 - \frac{55}{16} \frac{(y_i^{(2)})^4 y_i^{(3)}}{(y_i^{(1)})^5} + \frac{5}{4} \frac{(y_i^{(2)})^2 (y_i^{(3)})^2}{(y_i^{(1)})^4} + \frac{5}{8} \frac{(y_i^{(2)})^3 y_i^{(4)}}{(y_i^{(1)})^4} - \right. \\ \left. \frac{1}{18} \left(\frac{y_i^{(3)}}{y_i^{(1)}} \right)^3 - \frac{1}{4} \frac{y_i^{(2)} y_i^{(3)} y_i^{(4)}}{(y_i^{(1)})^3} - \frac{3}{40} \frac{(y_i^{(2)})^2 y_i^{(5)}}{(y_i^{(1)})^3} + \frac{1}{144} \left(\frac{y_i^{(4)}}{y_i^{(1)}} \right)^2 + \right. \\ \left. \frac{1}{90} \frac{y_i^{(3)} y_i^{(5)}}{(y_i^{(1)})^2} + \frac{1}{180} \frac{y_i^{(2)} y_i^{(6)}}{(y_i^{(1)})^2} - \frac{1}{5040} \frac{y_i^{(7)}(\zeta)}{y_i^{(1)}} \right\} \varepsilon_i^7$$

Order: 8

$$\varepsilon_{i+1} = - \left\{ \frac{429}{128} \left(\frac{y_i^{(2)}}{y_i^{(1)}} \right)^7 - \frac{429}{64} \frac{(y_i^{(2)})^5 y_i^{(3)}}{(y_i^{(1)})^6} + \frac{55}{16} \frac{(y_i^{(2)})^3 (y_i^{(3)})^2}{(y_i^{(1)})^5} + \frac{165}{128} \frac{(y_i^{(2)})^4 y_i^{(4)}}{(y_i^{(1)})^5} - \right. \\ \frac{55}{144} \frac{y_i^{(2)} (y_i^{(3)})^3}{(y_i^{(1)})^4} - \frac{55}{64} \frac{(y_i^{(2)})^2 y_i^{(3)} y_i^{(4)}}{(y_i^{(1)})^4} - \frac{11}{64} \frac{(y_i^{(2)})^3 y_i^{(5)}}{(y_i^{(1)})^4} + \frac{5}{96} \frac{(y_i^{(3)})^2 y_i^{(4)}}{(y_i^{(1)})^3} + \\ \frac{5}{128} \frac{y_i^{(2)} (y_i^{(4)})^2}{(y_i^{(1)})^3} + \frac{1}{16} \frac{y_i^{(2)} y_i^{(3)} y_i^{(5)}}{(y_i^{(1)})^3} + \frac{1}{64} \frac{(y_i^{(2)})^2 y_i^{(6)}}{(y_i^{(1)})^3} - \frac{1}{320} \frac{y_i^{(4)} y_i^{(5)}}{(y_i^{(1)})^2} - \frac{1}{480} \\ \left. \frac{y_i^{(3)} y_i^{(6)}}{(y_i^{(1)})^2} - \frac{1}{1120} \frac{y_i^{(2)} y_i^{(7)}}{(y_i^{(1)})^2} + \frac{1}{40320} \frac{y_i^{(8)}(\zeta)}{y_i^{(1)}} \right\} \varepsilon_i^{(8)}$$

تكوين حسابي رمزي لصيغ تكرارية ذات النقطة الواحدة من الرتب

$$y(x)=0 \text{ الصحيحة حل}$$

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قسم العلوم والرياضيات - كلية إعداد المعلمين

مكة المكرمة - المملكة العربية السعودية

المستخلص . تم في هذا البحث تشييد تكوين حسابي رمزي لصيغ تكرارية ذات النقطة الواحدة من الرتب الصحيحة ، وذلك لحل المعادلة $y(x)=0$. واستخدمت برامج الحاسب الآلي المعروفة باسم "Mathematica" للحصول على التعبير التحليلي الحرفي لبعض الصيغ التكرارية من رتب مختلفة ، وأيضاً التعبير التحليلي الحرفي للأخطاء المصاحبة ، وذلك لتوضيح التطويع الرمزي للتكوين الحسابي . واشتمل البحث أيضاً على تطبيق عددي .